Axiom E6 is needed in Proposition 6.9 of “Game-Theoretic Foundations for Probability and Finance”

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Abstract. In the winter of 2020-2021, when Covid-19 turned our lives around, Glenn Shafer gave a series of online lectures on “Game-theoretic foundations for statistical testing and imprecise probabilities” at the SIPTA school on imprecise probabilities. During those lectures, he mentioned a small open problem that intrigued him: in Proposition 6.9 of his new book with Vladimir Vovk, “Game-Theoretic Foundations for Probability and Finance”, is Axiom E6\([0,\infty]\) needed or can it be removed? It intrigued me as well. In this short note, I provide a counterexample that demonstrates it cannot be removed.

Let \(\mathcal{Y} := \{H, T\}\) and let \([0, \infty]^\mathcal{Y}\) be the set of all maps from \(\mathcal{Y}\) to \([0, \infty]\)—so all nonnegative extended real functions on \(\mathcal{Y}\). Consider the operator \(\mathbb{E}\) on \([0, \infty]^\mathcal{Y}\) defined by

\[
\mathbb{E}(f) := \min \{\alpha \in [0, \infty] : f(H) \leq \alpha, f(T) \leq 2\alpha\}
\]

for all \(f \in [0, \infty]^\mathcal{Y}\).

This operator does not satisfy E6\([0,\infty]\) in [1, Proposition 6.9]. For example, if we let \(1_T \in [0, \infty]^\mathcal{Y}\) denote the indicator of \(T\), defined by \(1_T(H) := 0\) and \(1_T(T) := 1\), then \(\mathbb{E}(1_T) = \frac{1}{2}\) and \(\mathbb{E}(1_T + 1) = 1\), hence \(\mathbb{E}(1_T + 1) \neq \mathbb{E}(1_T) + 1\). So \(\mathbb{E}\) fails E6\([0,\infty]\) for \(f = 1_T\) and \(c = 1\). It therefore follows from [1, Proposition 6.9] that \(\mathbb{E}\) is not a \([0, \infty]-upper\) expectation on \(\mathcal{Y}\).

On the other hand, it is fairly straightforward to show that \(\mathbb{E}\) satisfies Axioms E1\([0,\infty]\) to E5\([0,\infty]\) in [1, Proposition 6.9]. Hence, Axiom E6\([0,\infty]\) cannot be removed from [1, Proposition 6.9]. Interestingly though, as can be seen from [1, Proposition 6.8] and the definition of an upper expectation on p.113 of [1], this is not true if the role of \([0, \infty]\) is replaced by \(\mathbb{R}\) or \(\overline{\mathbb{R}}\). So the restriction to nonnegative functions seems to have a fundamental effect on the axioms that are needed to characterise (restrictions of) upper expectations.

References