Buy low, sell high

Wouter M. Koolen Vladimir Vovk

Royal Holloway University of London
CWI Centrum Wiskunde & Informatica
QUT Queensland University of Technology
Berkeley University of California

GTP 2014 Guanajuato Mexico
Saturday 15th November, 2014

Road map

Motivation

"buy low, sell high" as a porcelain-tile pseudo wisdom

Motivation

"buy low, sell high" as a porcelain-tile pseudo wisdom

we take it seriously

Motivation

"buy low, sell high" as a porcelain-tile pseudo wisdom

we take it seriously

online learning style

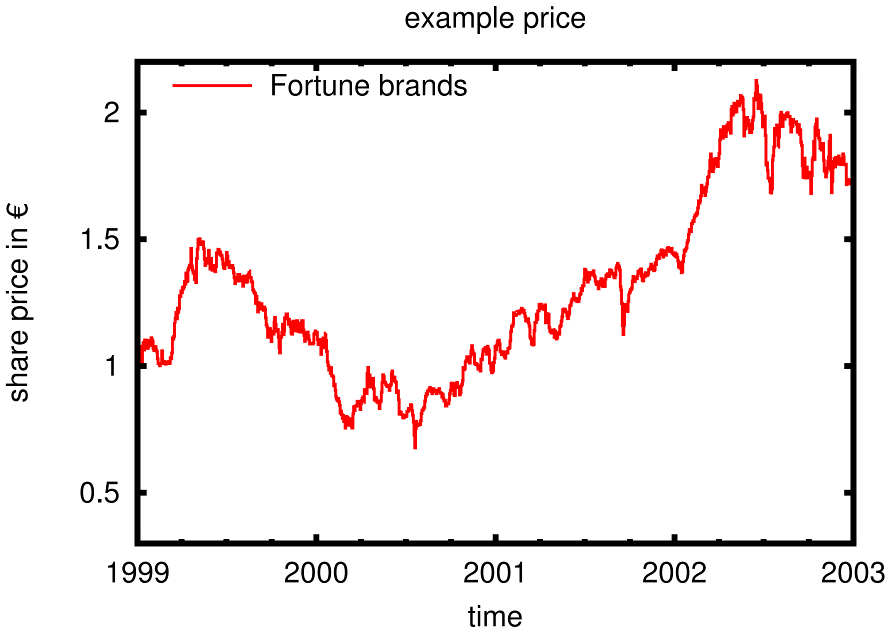
Motivation

"buy low, sell high" as a porcelain-tile pseudo wisdom

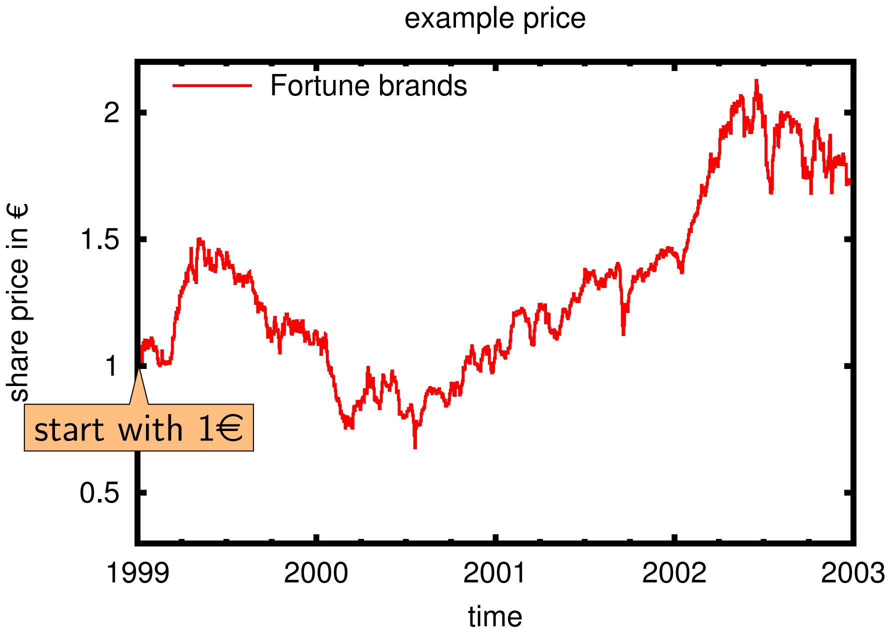
we take it seriously

online learning style

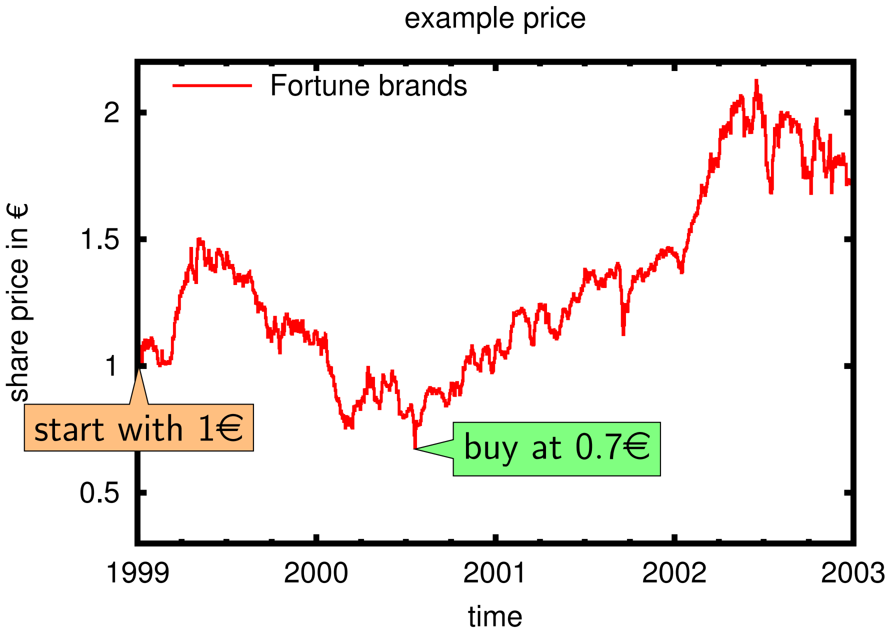
and uncover its surprisingly intricate theory

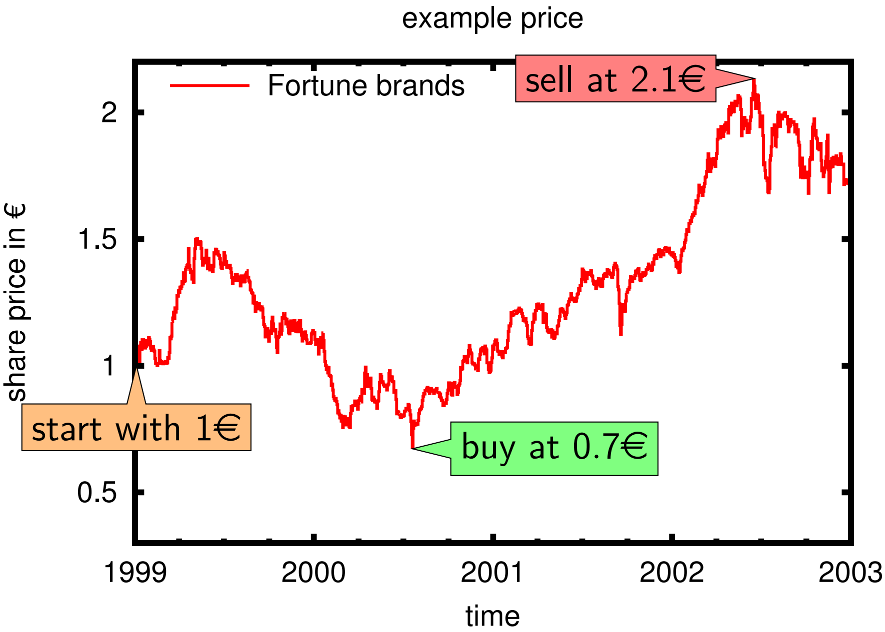


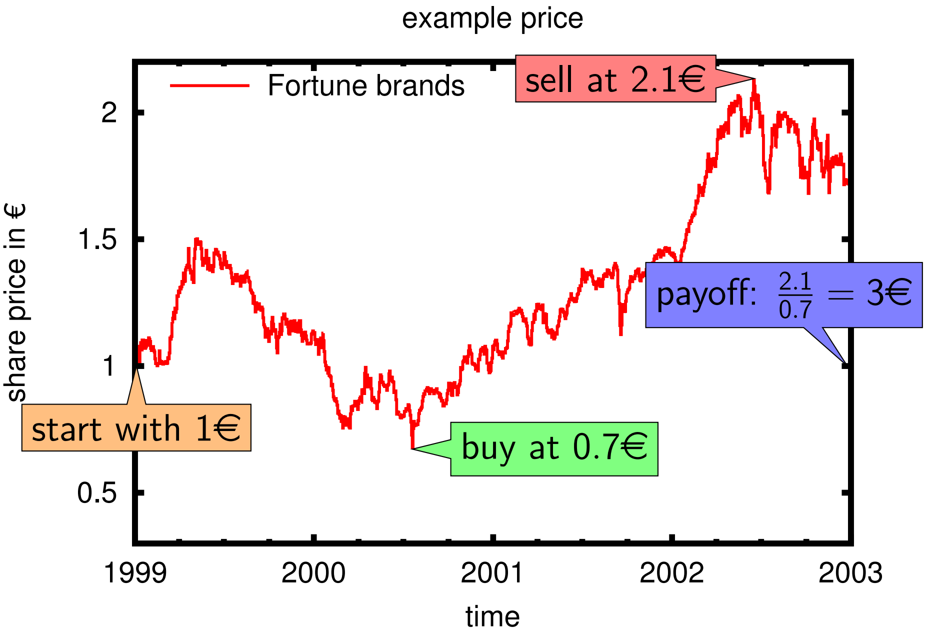
example price
Fortune brands
share price in €
2
1.5
1
0.5
1999
2000
2001
2002
2003
time



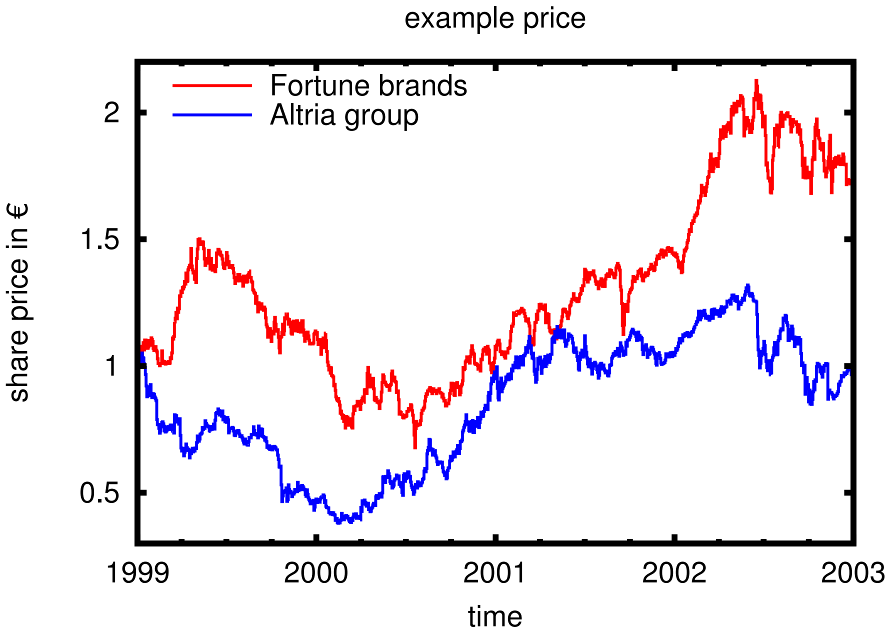
example price
Fortune brands
share price in €
2
1.5
1
0.5
1999
2000
2001
2002
2003
time
start with 1€


example price
Fortune brands
share price in €
2
1.5
1
0.5
1999
2000
2001
2002
2003
time
start with 1€
buy at 0.7€


example price
Fortune brands
sell at 2.1€
start with 1€
buy at 0.7€
share price in €
2
1.5
1
0.5
1999
2000
2001
2002
2003
time



example price
Fortune brands
sell at 2.1€
payoff: $\frac{2.1}{0.7} = 3$€
start with 1€
buy at 0.7€
share price in €
2
1.5
1
0.5
1999
2000
2001
2002
2003
time



example price
Fortune brands
Altria group
share price in €
2
1.5
1
0.5
1999
2000
2001
2002
2003
time

At a glance

- "Buy low, sell high" a desirable target
- Simple: just need low and high trading price
- Problem: Good trading prices *depend on the data*.

At a glance

- "Buy low, sell high" a desirable target
- Simple: just need low and high trading price
- Problem: Good trading prices *depend on the data*.

- Idea: Could we learn best low/high prices?
- I.e. by online trading guarantee payoff as if we knew them?

At a glance

- "Buy low, sell high" a desirable target
- Simple: just need low and high trading price
- Problem: Good trading prices *depend on the data*.

- Idea: Could we learn best low/high prices?
- I.e. by online trading guarantee payoff as if we knew them?

Answer: a firm and crisp almost

At a glance

- "Buy low, sell high" a desirable target
- Simple: just need low and high trading price
- Problem: Good trading prices *depend on the data*.

- Idea: Could we learn best low/high prices?
- I.e. by online trading guarantee payoff as if we knew them?

Answer: a firm and crisp almost

Our work: complete characterisation of that "almost".

Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

A position

- $S_t < 0$ is called short
- $S_t > 0$ is called long
- $S_t > K_{t-1}/\omega_{t-1}$ is called leveraged

Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

A position

- $S_t < 0$ is called short
- $S_t > 0$ is called long
- $S_t > K_{t-1}/\omega_{t-1}$ is called leveraged

Bankrupt when capital $K_t < 0$ is negative.

Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

A position

- $S_t < 0$ is called short
- $S_t > 0$ is called long
- $S_t > K_{t-1}/\omega_{t-1}$ is called leveraged

Bankrupt when capital $K_t < 0$ is negative.

No assumptions about price-generating process. Full information

Goal

We want to become rich when the share price exhibits a large upcrossing ($[a, b]$ is upcrossed when the price drops below a before it exceeds b)

Goal

We want to become rich when the share price exhibits a large upcrossing ($[a, b]$ is upcrossed when the price drops below a before it exceeds b)

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function G, and guarantees to

> keep our capital above $G(a, b)$ for all upcrossed intervals $[a, b]$

Goal

We want to become rich when the share price exhibits a large upcrossing ($[a, b]$ is upcrossed when the price drops below a before it exceeds b)

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function G, and guarantees to

keep our capital above $G(a, b)$ for all upcrossed intervals $[a, b]$

Ideally, $G(a, b)$ is close to b/a.

Goal

We want to become rich when the share price exhibits a large upcrossing ($[a, b]$ is upcrossed when the price drops below a before it exceeds b)

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function G, and guarantees to

keep our capital above $G(a, b)$ for all upcrossed intervals $[a, b]$

Ideally, $G(a, b)$ is close to b/a.

We would like to find out:

- Is guaranteeing G possible?
- Can more than G be guaranteed?
- Can we reverse engineer a strategy for G?

Road map

Sell high only

We want to become rich when the share price is *ever high*

Sell high only

We want to become rich when the share price is *ever high*

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function F, and guarantees to

keep our capital above $F(y)$ for all exceeded price levels y

Sell high only

We want to become rich when the share price is *ever high*

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function F, and guarantees to

keep our capital above $F(y)$ for all exceeded price levels y

Ideally, $F(y)$ is close to y.

Sell high only

We want to become rich when the share price is *ever high*

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function F, and guarantees to

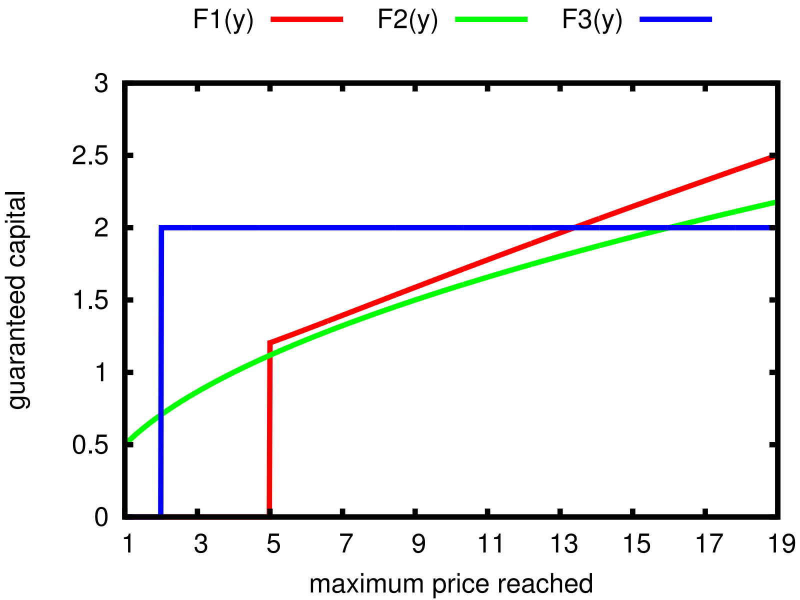
keep our capital above $F(y)$ for all exceeded price levels y

Ideally, $F(y)$ is close to y.

We would like to find out:

- Is guaranteeing F possible?
- Can more than F be guaranteed?
- Can we reverse engineer a strategy for F?

Example guarantees F



Adjuster

A strategy prescribes position S_t based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

Definition

A function $F : [1, \infty) \to [0, \infty)$ is called an adjuster if there is a strategy that guarantees

$$K_t \geq F\left(\max_{0 \leq s \leq t} \omega_s\right).$$

An adjuster F is admissible if it is not strictly dominated.

Threshold adjusters

Fix a price level $u \geq 1$. The threshold adjuster

$$F_u(y) \;:=\; u\mathbf{1}_{\{y \geq u\}}$$

Threshold adjusters

Fix a price level $u \geq 1$. The threshold adjuster

$$F_u(y) \;:=\; u\mathbf{1}_{\{y \geq u\}}$$

is witnessed by the threshold strategy S_u that

- takes position 1 until the price first exceeds level u.
- takes position 0 thereafter

The GUT of Adjusters

Consider a right-continuous and increasing *candidate guarantee* F.

Theorem (Characterisation)

F is an **adjuster** *iff*

$$\int_1^\infty \frac{F(y)}{y^2}\,\mathrm{d}y \;\le\; 1.$$

Moreover, F is **admissible** *iff this holds with equality.*

The GUT of Adjusters

Consider a right-continuous and increasing *candidate guarantee* F.

Theorem (Characterisation)

F is an **adjuster** *iff*

$$\int_1^\infty \frac{F(y)}{y^2}\,\mathrm{d}y \;\leq\; 1.$$

Moreover, F is **admissible** *iff this holds with equality.*

Theorem (Representation)

F is an **adjuster** *iff there is a probability measure P on $[1,\infty)$ such that*

$$F(y) \;\leq\; \int F_u(y)\,\mathrm{d}P(u),$$

again with equality iff F is admissible.

Road map

Adjuster

Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

Adjuster

Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

Definition

A strategy prescribes position S_t based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

Adjuster

Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

Definition

A strategy prescribes position S_t based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

Definition

A function $G : (0, 1] \times [0, \infty) \to [0, \infty)$ is called an adjuster if there is a strategy that guarantees

$$K_t \;\geq\; G\,(a, b)$$

for each $[a, b]$ upcrossed by $\omega_0, \ldots, \omega_t$.
An adjuster G is admissible if it is not strictly dominated.

Adjuster

Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

Definition

A strategy prescribes position S_t based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

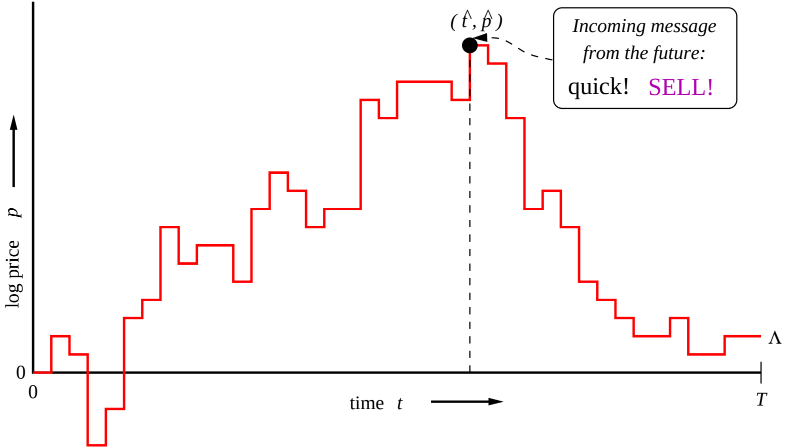
Definition

A function $G : (0, 1] \times [0, \infty) \to [0, \infty)$ is called an adjuster if there is a strategy that guarantees

$$K_t \;\geq\; G\,(a, b)$$

for each $[a, b]$ upcrossed by $\omega_0, \ldots, \omega_t$.
An adjuster G is admissible if it is not strictly dominated.

Sneak peak: Ideal $G(a, b) = b/a$ is not an adjuster. But we can get close.



$(\hat{t}, \hat{p})$
Incoming message from the future:
quick! SELL!
log price p
0
0
time t
T
Λ

Sequential Threshold strategies

- More of the same
- Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha,\beta}(a, b) \; = \; \frac{\beta}{\alpha} \mathbf{1}_{\{a \leq \alpha\}} \mathbf{1}_{\{b \geq \beta\}}$$

is witnessed by the threshold strategy $S_{\alpha,\beta}$ that

  - takes position 0 until the price drops below α
  - takes position $1/\alpha$ until the price rises above β
  - takes position 0 thereafter

Sequential Threshold strategies

- More of the same
- Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha,\beta}(a, b) \;=\; \frac{\beta}{\alpha}\mathbf{1}_{\{a \leq \alpha\}}\mathbf{1}_{\{b \geq \beta\}}$$

  is witnessed by the threshold strategy $S_{\alpha,\beta}$ that
  - takes position 0 until the price drops below α
  - takes position $1/\alpha$ until the price rises above β
  - takes position 0 thereafter
- Optimal strategies allocate their 1$ to threshold strategies according to some probability measure $P(\alpha, \beta)$, and hence achieve

$$G_P(a, b) \;=\; \int G_{\alpha,\beta}(a, b)\,\mathrm{d}P(\alpha, \beta).$$

Sequential Threshold strategies: Fallacy

- More of the same
- Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha,\beta}(a, b) \;=\; \frac{\beta}{\alpha} \mathbf{1}_{\{a \le \alpha\}} \mathbf{1}_{\{b \ge \beta\}}$$

  is witnessed by the threshold strategy $S_{\alpha,\beta}$ that
  - takes position 0 until the price drops below α
  - takes position $1/\alpha$ until the price rises above β
  - takes position 0 thereafter
- Optimal strategies allocate their 1\$ to threshold strategies according to some probability measure $P(\alpha, \beta)$, and hence achieve

$$G_P(a, b) \;=\; \int G_{\alpha,\beta}(a, b) \, \mathrm{d}P(\alpha, \beta).$$

G_P is typically strictly dominated

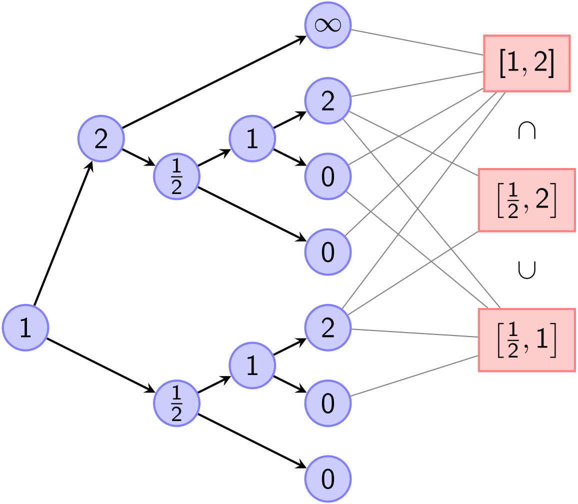
Road map

Mixtures of thresholds are generally dominated

$$G(a,b) := \frac{1}{2}G_{1,2}(a,b)+\frac{1}{2}G_{\frac{1}{2},1}(a,b) = \mathbf{1}_{\{a \leq 1 \text{ and } b \geq 2\}}+\mathbf{1}_{\left\{a \leq \frac{1}{2} \text{ and } b \geq 1\right\}}.$$

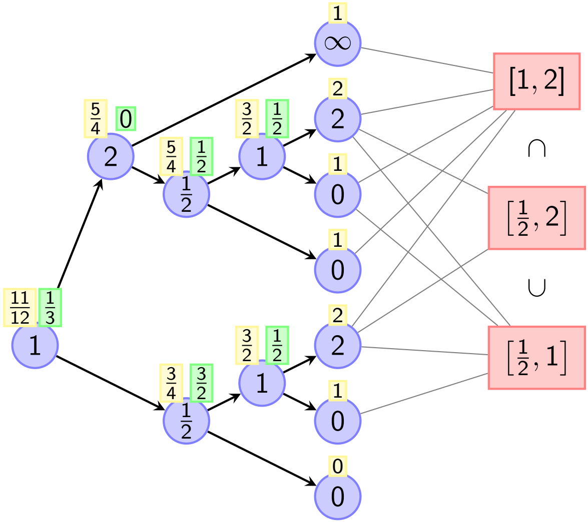
Mixtures of thresholds are generally dominated

$$G(a,b) := \frac{1}{2}G_{1,2}(a,b) + \frac{1}{2}G_{\frac{1}{2},1}(a,b) = \mathbf{1}_{\{a \leq 1 \text{ and } b \geq 2\}} + \mathbf{1}_{\left\{a \leq \frac{1}{2} \text{ and } b \geq 1\right\}}.$$



Mixtures of thresholds are generally dominated

$$G(a,b) := \frac{1}{2} G_{1,2}(a,b) + \frac{1}{2} G_{\frac{1}{2},1}(a,b) = \mathbf{1}_{\{a \le 1 \text{ and } b \ge 2\}} + \mathbf{1}_{\{a \le \frac{1}{2} \text{ and } b \ge 1\}}.$$



Road map

The GUT of Adjusters

Let G be left/right continuous and de/increasing.

Theorem (Characterisation)

G is an **adjuster** *iff*

$$\int_0^\infty 1 - \exp\left(-\int_{G(a,b)\geq h} \frac{\mathrm{d}a\,\mathrm{d}b}{(b-a)^2}\right) \mathrm{d}h \;\leq\; 1.$$

Moreover, G is **admissible** *iff this holds with equality and G is saturated.*

The GUT of Adjusters

Let G be left/right continuous and de/increasing.

Theorem (Characterisation)

G is an **adjuster** *iff*

$$\int_0^\infty 1 - \exp\left(- \int_{G(a,b) \geq h} \frac{\mathrm{d}a \, \mathrm{d}b}{(b-a)^2} \right) \mathrm{d}h \; \leq \; 1.$$

Moreover, G is **admissible** *iff this holds with equality and G is saturated.*

- Lower bound from option pricing
- Upper bound from explicitly constructed strategy
- Temporal reasoning evaporated.

Road map

Simple adjusters

Corollary (Sell high Dawid, De Rooij, Grünwald, Koolen, Shafer, Shen, Vereshchagin, Vovk (2011)**)**

Let $G(a, b) := F(b \vee 1)$. *G is an adjuster iff*

$$\int_1^\infty \frac{F(y)}{y^2} \, \mathrm{d}y \;\leq\; 1.$$

Simple adjusters

Corollary (Sell high Dawid, De Rooij, Grünwald, Koolen, Shafer, Shen, Vereshchagin, Vovk (2011))

Let $G(a,b) := F(b \vee 1)$*. G is an adjuster iff*

$$\int_1^\infty \frac{F(y)}{y^2}\,\mathrm{d}y \;\leq\; 1.$$

Corollary (Length)

Let $G(a,b) := F(b-a)$*. G is an adjuster iff*

$$\int_0^\infty F(y)\frac{e^{-1/y}}{y^2}\,\mathrm{d}y \;\leq\; 1.$$

Simple adjusters

Corollary (Sell high Dawid, De Rooij, Grünwald, Koolen, Shafer, Shen, Vereshchagin, Vovk (2011)**)**

Let $G(a, b) := F(b \vee 1)$. *G is an adjuster iff*

$$\int_1^\infty \frac{F(y)}{y^2}\,\mathrm{d}y \;\leq\; 1.$$

Corollary (Length)

Let $G(a, b) := F(b - a)$. *G is an adjuster iff*

$$\int_0^\infty F(y)\frac{e^{-1/y}}{y^2}\,\mathrm{d}y \;\leq\; 1.$$

Corollary (Ratio)

Let $G(a, b) := F(b/a)$ *for some unbounded* F. *Then* G *is not an adjuster.*

Our favourite adjuster

Let $0 \leq q < p < 1$. Then

$$G(a, b) \;:=\; \underbrace{\frac{(b-a)^p}{a^q}}_{\approx b/a} \; \underbrace{\frac{\left(\frac{p-q}{p}\right)^p}{\Gamma(1-p)}}_{\text{normalisation}}$$

is an admissible adjuster.

Our favourite adjuster

Let $0 \leq q < p < 1$. Then

$$G(a,b) \;:=\; \underbrace{\frac{(b-a)^p}{a^q}}_{\approx b/a} \; \underbrace{\frac{\left(\frac{p-q}{p}\right)^p}{\Gamma(1-p)}}_{\text{normalisation}}$$

is an admissible adjuster.

Strategy: In situation ω with minimum price m take position

$$S(\omega) \;=\; \frac{(p-q)}{m^{1-p+q}} \, \Phi\left(\frac{m^{\frac{p-q}{p}}}{\big(X_G(\omega)\Gamma(1-p)\big)^{1/p}}\right)$$

where Φ is the CDF of the Gamma distribution.

Road map

What just happened

- We took "buy low, sell high" as the learning target
- We consider *parametrised* payoff guarantees
- We classified candidate guarantees using a simple formula
  - (≤ 1) Attainable adjuster
  - ($= 1$) Admissible adjuster
  - (> 1) Not an adjuster
- Looked at some interesting example adjusters

Open problems

- Sell high, buy low, then sell high again.
- . . .

Thank you!