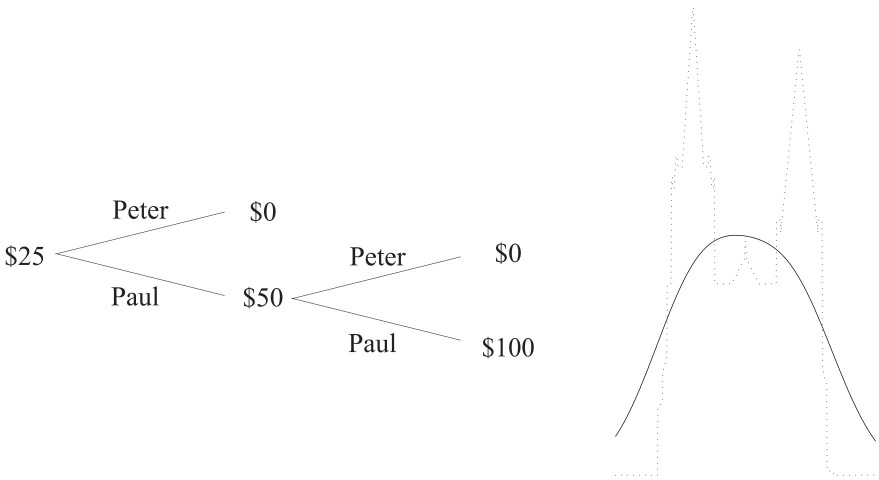
Continuous-time trading and the emergence of randomness

Vladimir Vovk



The Game-Theoretic Probability and Finance Project

Working Paper #24

First posted December 8, 2007. Last revised June 9, 2008.

Project web site:
http://www.probabilityandfinance.com

Abstract

A new definition of events of game-theoretic probability zero in continuous time is proposed and used to prove results suggesting that trading in financial markets results in the emergence of properties usually associated with randomness. This paper concentrates on "qualitative" results, stated in terms of order (or order topology) rather than in terms of the precise values taken by a price process. No stochastic assumptions are made, and the only assumption is that the price process is continuous.

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1 Introduction

This paper proposes (in Section 2) a new definition of continuous-time events of zero game-theoretic probability. The applications (Sections 3 and 4) are to an idealized securities market, with a security price modelled as a continuous process. We show that the price path will, almost surely, satisfy various properties usually associated with randomness. The phrase "almost surely" refers to the fact that a speculator can become arbitrarily rich risking only 1 monetary unit if the price path does not behave this way; therefore, if we believe that the market is to some degree efficient, we expect that those properties will be satisfied.

We consider some of the standard properties of sample paths of Brownian motion usually found in probability textbooks (such as [7], Section 2.9). This paper is inspired by [15], which in turn develops some ideas in [18]; both those papers attempt to formalize the "$\sqrt{\mathrm{d}t}$ effect" (the fact that a typical change in the value of a non-degenerate diffusion process over time period $\mathrm{d}t$ has order of magnitude $\sqrt{\mathrm{d}t}$). We, however, concentrate on those properties that depend only on the ordering of the security prices at different times, rather than on the actual values of the prices. Among such properties are, for example, the absence of isolated zeroes of the price path and the absence of points of strict increase or decrease. The difference of the game-theoretic treatment from the standard results is that we do not assume *a priori* any stochastic picture; we start instead from a simple trading protocol without making any probabilistic assumptions.

This paper is part of the recent revival of interest in game-theoretic probability (whose idea goes back to Ville [16]; more recent publications include [3, 12, 14, 9, 6, 8]). The treatment of continuous time in [12] and [18] uses non-standard analysis; an important contribution of [15] is to avoid non-standard analysis (which is both unfamiliar to many readers and somewhat awkward in certain respects) in studying the $\sqrt{\mathrm{d}t}$ effect. This paper also avoids non-standard analysis. Its main result is Theorem 2 (all other results will be fairly obvious to readers familiar with game-theoretic probability). Our key tool will be "high-frequency limit order strategies", introduced in game-theoretic probability by [15].

The words "positive", "negative", "increasing", "decreasing", "before", and "after" will be used in the wide sense of the inequalities $\le$ or $\ge$, as appropriate; we will add qualifiers "strict" or "strictly" when meaning the narrow sense of $<$ or $>$. We will also be using the usual notation $u \vee v := \max(u, v)$, $u \wedge v := \min(u, v)$, and $u^+ := u \vee 0$.

2 Null, almost certain, and completely uncertain events

Continuous time will be represented by the semi-infinite interval $[0, \infty)$. We consider a perfect-information game between two players called Reality and

Sceptic.[1] Reality outputs a continuous function $\omega : [0,\infty) \to \mathbb{R}$, interpreted as the price path of a financial asset (although we do not insist on ω taking positive values), and Sceptic tries to profit by trading in ω. First Sceptic presents his trading strategy and then Reality chooses ω. We start by formalizing this picture.

Let Ω be the set of all continuous functions $\omega : [0,\infty) \to \mathbb{R}$. For each $t \in [0,\infty)$, $\mathcal{F}_t$ is defined to be the smallest σ-algebra that makes all functions $\omega \mapsto \omega(s)$, $s \in [0,t]$, measurable. A *process* S is a family of functions $S_t : \Omega \to [-\infty,\infty]$, $t \in [0,\infty)$, each S_t being $\mathcal{F}_t$-measurable (we drop "stochastic" since no probability measure on Ω is given, and drop "adapted" for brevity). An *event* is an element of the σ-algebra $\mathcal{F}_\infty := \sigma(\cup_{t\in[0,\infty)}\mathcal{F}_t)$. Stopping times $\tau : \Omega \to [0,\infty]$ w.r.t. to the filtration $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual; $\omega(\tau(\omega))$ and $S_{\tau(\omega)}(\omega)$ will be simplified to $\omega(\tau)$ and $S_\tau(\omega)$, respectively.

The class of allowed strategies for Sceptic is defined in two steps. An *elementary trading strategy* G consists of: (a) an increasing infinite sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ such that $\lim_{n\to\infty}\tau_n(\omega) = \infty$ for each $\omega \in \Omega$; (b) for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable function h_n. (It is possible that $\tau_n = \infty$ from some n on, which recovers the case of finite sequences.) To such G and an *initial capital* $c \in \mathbb{R}$ corresponds the *elementary capital process*

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega)\big(\omega(\tau_{n+1}\wedge t) - \omega(\tau_n \wedge t)\big), \quad t \in [0,\infty); \tag{1}$$

the value $h_n(\omega)$ will be called the *portfolio* chosen at time τ_n, and $\mathcal{K}_t^{G,c}(\omega)$ will sometimes be referred to as Sceptic's capital at time t. Notice that the sum of finitely many elementary capital processes is again an elementary capital process.

A *positive capital process* is any process S that can be represented in the form

$$S_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n,c_n}(\omega), \tag{2}$$

where the elementary capital processes $\mathcal{K}_t^{G_n,c_n}(\omega)$ are required to be positive, for all t and ω, and the positive series $\sum_{n=1}^{\infty} c_n$ is required to converge (intuitively, the total capital invested has to be finite). The sum (2) is always positive, but we allow it to take value $+\infty$. Since $\mathcal{K}_0^{G_n,c_n}(\omega) = c_n$ does not depend on ω, $S_0(\omega)$ also does not depend on ω and will sometimes be abbreviated to S_0.

The *upper probability* of a set $E \subseteq \Omega$ is defined as

$$\overline{\mathbb{P}}(E) := \inf\big\{S_0 \bigm| \forall\omega\in\Omega : \liminf_{t\to\infty} S_t(\omega) \ge \mathbb{I}_E(\omega)\big\}, \tag{3}$$

where S ranges over the positive capital processes and $\mathbb{I}_E$ stands for the indicator of E. Notice that $\overline{\mathbb{P}}(\Omega) = 1$ (in the terminology of [12], our game protocol is

[1] Other names for these players, used in [12], are Market and Speculator, respectively.

"coherent"): indeed, $\overline{\mathbb{P}}(\Omega) < 1$ would mean that some positive capital process strictly increases between time 0 and ∞ for all $\omega \in \Omega$, and this is clearly impossible for constant ω.

We say that $E \subseteq \Omega$ is *null* if $\overline{\mathbb{P}}(E) = 0$, and we say that E is *completely uncertain* if $\overline{\mathbb{P}}(E) = 1$ and $\overline{\mathbb{P}}(\Omega \setminus E) = 1$. A property of $\omega \in \Omega$ will be said to hold *almost surely* (a.s.) if the set of ω where it fails is null. Correspondingly, a set $E \subseteq \Omega$ is *almost certain* if $\overline{\mathbb{P}}(\Omega \setminus E) = 0$.

Remark 1. The interpretation of almost certain events given in Section 1 was that we expect such events to happen in markets that are efficient to some degree; similarly, we do not expect null events to happen (provided such an event is singled out in advance). However, some qualifications are needed, since our definition of upper probability involves Sceptic's capital at infinity, which may be infinite without necessarily contradicting market efficiency. That interpretation is, e.g., valid for events $E \in \mathcal{F}_T$ that happen or fail to happen before a finite horizon T: say, if E is null, Sceptic can become arbitrarily rich by time T if E happens.

The definition (3) enjoys a certain degree of robustness:

Lemma 1. *We will obtain an equivalent definition replacing the* $\liminf_{t\to\infty}$ *in (3) by* $\sup_{t\in[0,\infty)}$ *(and, therefore, by* $\limsup_{t\to\infty}$*).*

Proof. Suppose $\overline{\mathbb{P}}(E) < c < 1$ in the sense of the definition with sup and select a positive capital process S witnessing this, i.e., satisfying $S_0 < c$ and

$$\forall \omega \in \Omega : \sup_{t\in[0,\infty)} S_t(\omega) \geq \mathbb{I}_E(\omega).$$

For any $\epsilon > 0$, we can multiply S by $1 + \epsilon$ and stop it when it hits 1; this will give a positive capital process witnessing $\overline{\mathbb{P}}(E) < (1 + \epsilon)c$ in the sense of the definition with lim inf. □

Upper probability also enjoys the following useful property of σ-subadditivity (obviously containing the property of finite subadditivity as a special case):

Lemma 2. *For any sequence of subsets* $E_1, E_2, \dots$ *of* Ω,

$$\overline{\mathbb{P}}\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \overline{\mathbb{P}}(E_n).$$

In particular, a countable union of null sets is null.

Proof. This follows immediately from the countability of a countable union of countable sets (of elementary capital processes). □

The definition of a null set can be restated as follows.

Lemma 3. *A set* $E \subseteq \Omega$ *is null if and only if there exists a positive capital process* S *with* $S_0 = 1$ *such that* $\lim_{t\to\infty} S_t(\omega) = \infty$ *for all* $\omega \in E$.

Proof. Suppose $\overline{\mathbb{P}}(E) = 0$. For each $n \in \{1, 2, \dots\}$, let S^n be a positive capital process with $S^n_0 = 2^{-n}$ and $\liminf_{t\to\infty} S^n_t \geq 1$. It suffices to set $S := \sum_{n=1}^{\infty} S^n$. □

3 Level sets of the price process

Our first theorem is a simple game-theoretic counterpart of a standard measure-theoretic fact (usually stated in the case of Brownian motion).

Theorem 1. *Let $b \in \mathbb{R}$. Almost surely, the level set*

$$\mathcal{L}_\omega(b) := \{t \in [0,\infty) \mid \omega(t) = b\}$$

has no isolated points in $[0,\infty)$.

Proof. If $\mathcal{L}_\omega(b)$ has an isolated point, there are rational numbers $a \geq 0$ and $D \neq 0$ such that strictly after the time $\inf\{t \mid t \geq a, \omega(t) = b\}$ ω does not take value b before hitting the value $b + D$ (this is true even if 0 is the only isolated point of $\mathcal{L}_\omega(b)$). Suppose, for concreteness, that D is positive (the case of negative D is treated analogously). This event, which we denote $E_{a,D}$, is null: there is a positive capital process that starts from ϵ (arbitrarily small positive number) and takes value $D + \epsilon$ when $E_{a,D}$ happens (choose portfolio 1 at the time $\inf\{t \mid t \geq a, \omega(t) = b\}$ and then choose portfolio 0 when the set $\{b - \epsilon, b + D\}$ is hit). Since each event $E_{a,D}$ is null, it remains to apply Lemma 2. □

Remark 2. As discussed in Remark 1, almost certain events in $\mathcal{F}_T$ are expected to happen in markets that are efficient to some degree. The almost certain properties E of sample paths that we establish in this paper do not belong to any $\mathcal{F}_T$, $T < \infty$, but it remains true that we expect them to happen in such markets. Each of these properties E is "falsifiable" in the following sense: there exists a stopping time τ, called a *rejection time* for E, such that $E = \{\omega \mid \tau(\omega) = \infty\}$. Moreover, it is possible to choose a rejection time τ for E such that for any monotonically increasing (however fast) function $f : [0,\infty) \to [0,\infty)$ there exists a positive capital process S with $S_0 = 1$ such that $S_\tau(\omega) \geq f(\tau(\omega))$ for all $\omega \in \Omega$ with $\tau(\omega) < \infty$. For example, the proof of Theorem 1 shows that Sceptic can become arbitrarily rich immediately after an isolated point in $\mathcal{L}_\omega(b)$ is observed.

Corollary 1. *For each $b \in \mathbb{R}$, it is almost certain that the set $\mathcal{L}_\omega(b)$ is perfect, and so either is empty or has the cardinality of continuum.*

Proof. Since ω is continuous, the set $\mathcal{L}_\omega(b)$ is closed and so, by Theorem 1, perfect. Non-empty perfect sets in $\mathbb{R}$ always have the cardinality of continuum (see, e.g., [1], Theorem 4.26). □

The following lemma, which uses some standard notions of measure-theoretic probability, will allow us to show that many events of interest to us are completely uncertain.

Lemma 4. *Suppose $P(E) = 1$, where E is an event and P is a probability measure on $(\Omega, \mathcal{F}_\infty)$ which makes the process $S_t(\omega) := \omega(t)$ a martingale w.r. to the filtration $(\mathcal{F}_t)$. Then $\overline{\mathbb{P}}(E) = 1$.*

Proof. It suffices to prove that (1) is a local martingale under P. Indeed, in this case $\overline{\mathbb{P}}(E) < 1$ in conjunction with the maximal inequality for positive supermartingales would contradict the assumption that $P(E) = 1$. It can be checked using the optional sampling theorem that each addend in (1) is a martingale, and so each partial sum in (1) is a martingale and (1) itself is a local martingale.

In the rest of this proof I will check, for the sake of the readers with little experience in measure-theoretic probability (like myself), that each addend

$$h_n(\omega)\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\big) \tag{4}$$

in (1) is indeed a martingale. (See [17] for a marginally simpler argument.) For each $t \in [0, \infty)$, (4) is integrable by the boundedness of h_n and the optional sampling theorem (see, e.g., [11], Theorem II.3.2). We only need to prove, for $0 < s < t$, that (omitting, until the end of the proof, the argument ω and "a.s.")

$$\mathbb{E}\left(h_n\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\big) \mid \mathcal{F}_s\right) = h_n\big(\omega(\tau_{n+1} \wedge s) - \omega(\tau_n \wedge s)\big). \tag{5}$$

We will check this equality on three $\mathcal{F}_s$-measurable events separately:

$\{\tau_{n+1} \le s\}$: Both sides of the equality

$$\mathbb{E}\left(h_n\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\big) \mathbb{I}_{\{\tau_{n+1} \le s\}} \mid \mathcal{F}_s\right) \\ = h_n\big(\omega(\tau_{n+1} \wedge s) - \omega(\tau_n \wedge s)\big) \mathbb{I}_{\{\tau_{n+1} \le s\}}$$

are equal to the $\mathcal{F}_s$-measurable function $h_n\big(\omega(\tau_{n+1}) - \omega(\tau_n)\big) \mathbb{I}_{\{\tau_{n+1} \le s\}}$ (its $\mathcal{F}_s$-measurability follows, e.g., from Lemma 1.2.15 in [7] and the monotone-class theorem).

$\{\tau_n \le s < \tau_{n+1}\}$: We need to check

$$\mathbb{E}\left(h_n\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n)\big) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}} \mid \mathcal{F}_s\right) \\ = h_n\big(\omega(s) - \omega(\tau_n)\big) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}}.$$

Since $h_n \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}}$ is bounded and $\mathcal{F}_s$-measurable, it suffices to check

$$\mathbb{E}\left(\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n)\big) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}} \mid \mathcal{F}_s\right) \\ = \big(\omega(s) - \omega(\tau_n)\big) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}}.$$

Since $\omega(\tau_n) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}}$ is $\mathcal{F}_s$-measurable, it suffices to check

$$\mathbb{E}\left(\omega(\tau_{n+1} \wedge t) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}} \mid \mathcal{F}_s\right) = \omega(s) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}},$$

which is the same thing as

$$\mathbb{E}\left(\omega(s \vee \tau_{n+1} \wedge t) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}} \mid \mathcal{F}_s\right) = \omega(s) \mathbb{I}_{\{\tau_n \le s < \tau_{n+1}\}}$$

($s \vee x \wedge t$ being a shorthand for $(s \vee x) \wedge t$ or, equivalently, $s \vee (x \wedge t)$). The stronger equality

$$\mathbb{E}\left(\omega(s \vee \tau_{n+1} \wedge t) \mid \mathcal{F}_s\right) = \omega(s)$$

follows from the optional sampling theorem.

$\{s < \tau_n\}$**:** We are required to prove

$$\mathbb{E}\left(h_n\big(\omega(\tau_{n+1}\wedge t) - \omega(\tau_n \wedge t)\big)\,\mathbb{I}_{\{s<\tau_n\}} \mid \mathcal{F}_s\right) = 0,$$

but we will prove more:

$$\mathbb{E}\left(h_n\big(\omega(\tau_{n+1}\wedge t) - \omega(\tau_n \wedge t)\big)\,\mathbb{I}_{\{s<\tau_n\}} \mid \mathcal{F}_{s\vee\tau_n\wedge t}\right) = 0.$$

Since the event $\{\tau_n \le t\}$, being equivalent to $\tau_n \le s \vee \tau_n \wedge t$, is $\mathcal{F}_{s\vee\tau_n\wedge t}$-measurable (see [7], Lemma 1.2.16), it is sufficient to prove

$$\mathbb{E}\left(h_n\big(\omega(\tau_{n+1}\wedge t) - \omega(\tau_n \wedge t)\big)\,\mathbb{I}_{\{s<\tau_n\le t\}} \mid \mathcal{F}_{s\vee\tau_n\wedge t}\right) = 0 \tag{6}$$

and

$$\mathbb{E}\left(h_n\big(\omega(\tau_{n+1}\wedge t) - \omega(\tau_n \wedge t)\big)\,\mathbb{I}_{\{t<\tau_n\}} \mid \mathcal{F}_{s\vee\tau_n\wedge t}\right) = 0.$$

The second equality is obvious, so our task has reduced to proving the first, (6). Since $h_n\,\mathbb{I}_{\{\tau_n\le t\}} = h_n\,\mathbb{I}_{\{\tau_n\le s\vee\tau_n\wedge t\}}$ is bounded and $\mathcal{F}_{s\vee\tau_n\wedge t}$-measurable, (6) reduces to

$$\mathbb{E}\left(\big(\omega(\tau_{n+1}\wedge t) - \omega(\tau_n \wedge t)\big)\,\mathbb{I}_{\{s<\tau_n\le t\}} \mid \mathcal{F}_{s\vee\tau_n\wedge t}\right) = 0,$$

which is the same thing as

$$\mathbb{E}\left(\big(\omega(s\vee\tau_{n+1}\wedge t) - \omega(s\vee\tau_n \wedge t)\big)\,\mathbb{I}_{\{s<\tau_n\le t\}} \mid \mathcal{F}_{s\vee\tau_n\wedge t}\right) = 0.$$

The optional sampling theorem now gives

$$\begin{aligned}\mathbb{E}\left(\big(\omega(s\vee\tau_{n+1}\wedge t) - \omega(s\vee\tau_n \wedge t)\big) \mid \mathcal{F}_{s\vee\tau_n\wedge t}\right)&\\ = \omega(s\vee\tau_n\wedge t) - \omega(s\vee\tau_n\wedge t) = 0.& \quad \square\end{aligned}$$

The following proposition shows that two standard properties of typical sample paths of Brownian motion become completely uncertain for continuous price processes.

Proposition 1. *Let $b \in \mathbb{R}$. The following events are completely uncertain:*

(a) the Lebesgue measure of $\mathcal{L}_\omega(b)$ is zero;

(b) the set $\mathcal{L}_\omega(b)$ is unbounded.

Proof. To see that the upper probability of (b) and of the complement of (a) is 1, consider the martingale that is identically equal to b. To see that the upper probability of (a) and of the complement of (b) is 1, consider a constant martingale not equal to b. (Notice that these arguments do not really require Lemma 4.) □

4 Properties related to non-increase

Let us say that $t \in [0, \infty)$ is a *point of semi-strict increase* for ω if there exists $\delta > 0$ such that $\omega(s) \le \omega(t) < \omega(u)$ for all $s \in ((t-\delta)^+, t)$ and $u \in (t, t+\delta)$. Points of semi-strict decrease are defined in the same way except that $\omega(s) \le \omega(t) < \omega(u)$ is replaced by $\omega(s) \ge \omega(t) > \omega(u)$. The following theorem is the game-theoretic counterpart of Dvoretzky, Erdős, and Kakutani's [5] result for Brownian motion (Dubins and Schwarz [4] noticed that it continues to hold for all continuous martingales); its proof can be found in Appendix B.

Theorem 2. *Almost surely, ω has no points of semi-strict increase or decrease.*

We will also state several corollaries of Theorem 2. First, the price process is nowhere monotone (unless constant):

Corollary 2. *Almost surely, ω is monotone in no open interval, unless it is constant in that interval.*

Proof. This is an obvious corollary of Theorem 2, but is also easy to prove directly: each interval of monotonicity where ω is not constant contains a rational time point a after which ω increases (if we assume, for concreteness, that "monotonicity" means "increase") by a rational amount $D > 0$ before hitting the level $\omega(a)$ again; as in the proof of Theorem 1, it is easy to show that this event, denoted $E_{a,D}$, is null, and it remains to apply Lemma 2 to deduce that $\cup_{a,D} E_{a,D}$ is also null. □

Let us say that a closed interval $[t_1, t_2] \subseteq [0, \infty)$ is an *interval of local maximum* for ω if (a) ω is constant on $[t_1, t_2]$ but not constant on any larger interval containing $[t_1, t_2]$, and (b) there exists $\delta > 0$ such that $\omega(s) \le \omega(t)$ for all $s \in ((t_1 - \delta)^+, t_1) \cup (t_2, t_2 + \delta)$ and all $t \in [t_1, t_2]$. In the case where $t_1 = t_2$ we can say "point" instead of "interval". A ray $[t, \infty)$, $t \in [0, \infty)$, is a *ray of local maximum* for ω if (a) ω is constant on $[t, \infty)$ but not constant on any larger ray $[s, \infty)$, $s \in (0, t)$, and (b) there exists $\delta > 0$ such that $\omega(s) \le \omega(t)$ for all $s \in ((t-\delta)^+, t)$. An *interval* or *ray of strict local maximum* is defined in the same way except that $\omega(s) \le \omega(t)$ is replaced by $\omega(s) < \omega(t)$. The definitions of intervals and rays of (strict) local minimum are obtained by obvious modifications; as usual "extremum" means maximum or minimum. We say that $t \in [0, \infty)$ is a *point of constancy* for ω if there exists $\delta > 0$ such that $\omega(s) = \omega(t)$ for all $s \in ((t-\delta)^+, t+\delta)$; points $t \in [0, \infty)$ that are not points of constancy are *points of non-constancy.* (Notice that we do not count points of constancy among points of local extremum.)

Corollary 3. *Almost surely, every interval of local extremum is a point, all points and the ray (if it exists) of local extremum are strict, the set of points of local extremum is countable, and any neighbourhood of any point of non-constancy contains a point of local maximum and a point of local minimum.*

Proof. We will prove only the statements concerning local maxima.

If ω has an interval of local maximum $[t_1, t_2]$ with $t_1 \neq t_2$, t_2 will be a point of semi-strict decrease, and by Theorem 2 it is almost certain that there will be no such points (alternatively, one could use the direct argument given in the proof of Corollary 2). We can see that no such $[t_1, t_2]$ can even be an interval of local maximum "on the right".

Now suppose that there is a point or ray of local maximum that is not strict. In this case there is a quadruple $0 < t_1 < t_2 < t_3 < t_4$ of rational numbers and another rational number $D > 0$ such that $\max_{t \in [t_1, t_2]} \omega(t) = \max_{t \in [t_3, t_4]} \omega(t) > \omega(t_4) + D$. The event that such a set of rational numbers exists is null: proceed as in the proof of Theorem 1.

The set of all points of strict local maximum is countable, as the following standard argument demonstrates: each point of strict local maximum can be surrounded by an open interval with rational end-points in which that point is a strict maximum, and all these open intervals will be different.

Finally, Corollary 2 immediately implies that every neighbourhood of every point of non-constancy contains a point of local maximum. □

This is a simple game-theoretic version of the classical result about the nowhere differentiability of Brownian motion (Paley, Wiener, and Zygmund [10]):

Corollary 4. *Almost surely, ω does not have a non-zero derivative anywhere.*

Proof. A point where a non-zero derivative exists is a point of semi-strict increase or decrease. □

It would be interesting to find stronger versions of the Paley–Wiener–Zygmund result (but see parts (a) and (c) of Proposition 2).

The following proposition demonstrates the necessity of various conditions in Corollaries 2–4.

Proposition 2. *The following events are completely uncertain:*

(a) ω is constant on $[0, \infty)$;

(b) for some $t \in (0, \infty)$, $[t, \infty)$ is the ray of local maximum (or minimum) for ω;

(c) $\omega'(t)$ exists for no $t \in [0, \infty)$.

Proof. We will be using Lemma 4. To see that the upper probability of (c), of the complement of (a), and of the complement of (b) is 1, remember that Brownian motion is a martingale. To see that the upper probability of (a) and of the complement of (c) is 1, consider a constant martingale. To see that the upper probability of (b) is 1, consider the following continuous martingale: start as Brownian motion from 0 and stop when 1 (or -1) is hit. □

5 Conclusion

This paper gives provisional definitions of upper probability and related notions (such as that of null events) for the case of continuous time. It may stay too close to the standard measure-theoretic framework in that the flow of information is modelled as a filtration. In discrete-time game-theoretic probability, as presented in [12], measurability does not play any special role, whereas in measure-theoretic probability measurability has the obvious technical role to play. On one hand, we could drop all conditions of measurability in all the definitions given above (equivalently, replace each σ-algebra that we used by the smallest class of subsets of Ω containing that σ-algebra and closed under arbitrary unions and intersections); it is obvious that all our theorems and corollaries, Proposition 1, and parts of Proposition 2 still hold (and it is an interesting problem to establish whether the remaining parts of Proposition 2 continue to hold). On the other hand, one might want to strengthen the requirement of measurability to that of computability.

Acknowledgments

This work was partially supported by EPSRC (grant EP/F002998/1), MRC (grant G0301107), and the Cyprus Research Promotion Foundation.

Appendix A: A one-sided law of large numbers

In this appendix we establish a result that will be needed in the proof of Theorem 2. This result involves the following perfect-information game protocol depending on two parameters, $N \in \{1, 2, \ldots\}$ (the horizon) and $c > 0$:

Players: Reality, Sceptic
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots, N$:
  Sceptic announces $s_n \geq 0$.
  Reality announces $x_n \in [-c, c]$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + s_n x_n$.
END FOR

This is a one-sided version of the bounded forecasting game in [12], p. 65 (intuitively, the restriction $s_n \geq 0$ means that the expected value of x_n is zero or strictly negative, and $\mathcal{K}_n$ is interpreted as Sceptic's capital). A strategy for Sceptic is *prudent* if it guarantees $\mathcal{K}_n \geq 0$, for all n and regardless of Reality's moves. The definition of upper probability in this simple discrete-time case becomes

$$\overline{\mathbb{P}}(E) := \inf\bigl\{\delta \bigm| \text{Sceptic has a prudent strategy that guarantees } \mathcal{K}_N \geq 1/\delta \text{ when } (x_1, \ldots, x_N) \in E\bigr\},$$

where E is a subset of the *sample space* $[-c, c]^N$.

The following lemma is a simple game-theoretic one-sided weak law of large numbers.

Lemma 5. *Let $\delta_1 > 0$, $\delta_2 > 0$, and $N \ge c^2/\delta_1\delta_2^2$. Then*

$$\overline{\mathbb{P}}\left(\frac{1}{N}\sum_{n=1}^{N} x_n \ge \delta_2\right) \le \delta_1.$$

In the proof of Theorem 2 we will actually need the following more precise version of Lemma 5:

Lemma 6. *Sceptic has a strategy that guarantees that his capital $\mathcal{K}_n$ will satisfy*

$$\mathcal{K}_n \ge \frac{N-n}{N} + \frac{1}{c^2 N}\left(\sum_{j=1}^{n} x_j\right)^{+,2} \tag{7}$$

for $n = 0, 1, \ldots, N$, where $t^{+,2} := (t^+)^2$.

Proof. This proof is based on the idea used in [13] (proof of Lemma 5). When $n = 0$, (7) reduces to $\mathcal{K}_0 \ge 1$, which we know is true. So it suffices to show that if (7) holds for $n < N$, then Sceptic can make sure that the corresponding inequality for $n+1$,

$$\mathcal{K}_{n+1} \ge \frac{N-n-1}{N} + \frac{1}{c^2 N}\left(\sum_{j=1}^{n+1} x_j\right)^{+,2}, \tag{8}$$

also holds. This is how Sceptic chooses his move:

- If $\sum_{j=1}^{n} x_j \ge 0$, then Sceptic sets

$$s_{n+1} := \frac{2}{c^2 N}\sum_{j=1}^{n} x_j \ge 0. \tag{9}$$

  In this case

$$\begin{aligned}
\mathcal{K}_{n+1} &= \mathcal{K}_n + \frac{2}{c^2 N}\left(\sum_{j=1}^{n} x_j\right) x_{n+1} \\
&\ge \frac{N-n}{N} + \frac{1}{c^2 N}\left(\sum_{j=1}^{n} x_j\right)^2 + \frac{2}{c^2 N}\left(\sum_{j=1}^{n} x_j\right) x_{n+1} \\
&= \frac{N-n}{N} + \frac{1}{c^2 N}\left(\sum_{j=1}^{n+1} x_j\right)^2 - \frac{x_{n+1}^2}{c^2 N}
\end{aligned}$$

$$\geq \frac{N-n-1}{N} + \frac{1}{c^2 N}\left(\sum_{j=1}^{n+1} x_j\right)^{+,2}$$

(the last inequality uses $t^2 \geq t^{+,2}$), which coincides with (8).

- If $\sum_{j=1}^{n} x_j < 0$, Sceptic sets $s_{n+1} := 0$, and so $\mathcal{K}_{n+1} = \mathcal{K}_n$. Because

$$\left(\sum_{j=1}^{n+1} x_j\right)^{+,2} - \left(\sum_{j=1}^{n} x_j\right)^{+,2} \leq x_{n+1}^2 \leq c^2,$$

we again obtain (8) from (7).

□

Proof of Lemma 5. Sceptic's strategy in Lemma 6 is prudent (it is obvious that $\mathcal{K}_n \geq 0$ for all n), and (7) implies

$$\mathcal{K}_N \geq \frac{1}{c^2 N}\left(\sum_{n=1}^{N} x_n\right)^{+,2}.$$

Combining this inequality with the assumption that $N \geq c^2/\delta_1\delta_2^2$, we see that when the event $\frac{1}{N}\sum_{n=1}^{N} x_n \geq \delta_2$ happens, $\mathcal{K}_N \geq 1/\delta_1$. □

Appendix B: Proof of Theorem 2

This proof is modelled on the very simple proof of Dvoretzky, Erdős, and Kakutani's result given by Burdzy [2]. We will only prove that, almost surely, ω has no points of semi-strict increase in $(0,\infty)$ (the argument given in the direct proof of Corollary 2, with $a = 0$, shows that almost surely 0 cannot be a point of semi-strict increase).

It suffices to prove that, for any given positive constants C and D, the following event, denoted $E_{C,D}$, is null: the price process ω starts from 0, before hitting the level C reaches a point of semi-strict increase t such that $\omega(t) = \max_{s\in[0,t]}\omega(s)$, then reaches the level $\omega(t)+D$ before hitting $\omega(t)$ again. Indeed, suppose ω in the original game has a point of semi-strict increase, say $t > 0$. There are positive rational numbers $a \in [0,t)$, C, and D such that $\omega(s) \leq \omega(t) \leq \omega(a) + C$, for all $s \in [a,t)$, and ω hits $\omega(t) + D$ before hitting $\omega(t)$ strictly after moment t. The latter event, denoted by $E_{a,C,D}$ is null since it is a translation of the null event $E_{C,D}$ (namely, $E_{a,C,D} = \{\omega \in \Omega \mid \omega|_{[a,\infty)} - \omega(a) \in E_{C,D}\}$). By Lemma 2, the union of all $E_{a,C,D}$ is also null, which completes the proof.

Fix positive C and D; our goal is to prove that $E_{C,D}$ is null. For each $\epsilon \in (0,1)$ (intuitively, a small constant), define sequences of stopping times U_n and T_n and a sequence of functions M_n on Ω, $n = 0, 1, \ldots$, as

$$M_0 := 0, \quad U_0 := 0,$$

$$
\begin{aligned}
T_n &:= \inf\{t > U_n \mid \omega(t) \in \{M_n - \epsilon, M_n + D\}\}, \quad n = 0, 1, \dots, \\
M_{n+1} &:= \sup\{\omega(t) \mid t \in [0, T_n)\}, \quad n = 0, 1, \dots, \\
U_{n+1} &:= \inf\{t > T_n \mid \omega(t) = M_{n+1}\}, \quad n = 0, 1, \dots;
\end{aligned}
$$

as usual, $\inf \emptyset$ is interpreted as ∞. We also set

$$
X_n := M_n - M_{n-1}, \;\; n = 1, 2, \dots, \quad N := \left\lfloor \frac{1}{\epsilon\sqrt{\ln \frac{1}{\epsilon}}} \right\rfloor .
$$

It suffices to establish, for an arbitrarily large constant $K > 0$ and a small enough ϵ, the existence of two positive elementary capital processes having strictly positive initial values and satisfying the following conditions when $\omega \in E_{C,D}$:

(a) The first process increases K-fold if $T_{N-1} < \infty$ and the price level C is not attained before time T_{N-1}.

(b) The second process increases K-fold if $T_{N-1} = \infty$ or the price level C is attained before time T_{N-1}.

An elementary trading strategy leading to (b) chooses portfolio 1 at time U_0, portfolio 0 at time T_0, portfolio 1 at time U_1, portfolio 0 at time T_1, etc.; finally, portfolio 1 at time U_{N-1} and portfolio 0 at time T_{N-1}. The strategy is started with initial capital ϵN to ensure that its capital process is positive. If $T_{N-1} = \infty$ or the price level C is attained before T_{N-1}, it will be true that $\omega(T_n) = M_n + D$ for some $n \in \{0, \dots, N-1\}$, and so the final capital will be at least D. By the definition of N, we can ensure $\epsilon N \le D/K$ by choosing a small ϵ.

It remains to prove the existence of an elementary trading strategy leading to (a). Intuitively, this strategy will implement a law of large numbers; in this paragraph we will discuss the situation informally considering the case of Brownian motion. Why can we expect that the price level C will be attained? For $x \ge 0$,

$$
\mathbb{P}\{X_n \ge x\} = \begin{cases} \epsilon/(x+\epsilon) & \text{if } x \in [0, D] \\ 0 & \text{otherwise,} \end{cases}
$$

and so we can compute the expectation of the truncated version $\tilde{X}_n := X_n \wedge (\sqrt{\epsilon} - \epsilon)$ of X_n as

$$
\mathbb{E}\,\tilde{X}_n = \int_0^{\sqrt{\epsilon}-\epsilon} \frac{\epsilon \mathrm{d}x}{x + \epsilon} = \frac{\epsilon}{2} \ln \frac{1}{\epsilon}
$$

(ϵ is assumed small throughout; in particular, $\sqrt{\epsilon} - \epsilon < D$); it is clear that the variance of $\tilde{X}_n$ does not exceed ϵ. The expectation of the sum $\tilde{X}_1 + \cdots + \tilde{X}_N \le X_1 + \cdots + X_N$ will exceed or be approximately equal to $N \frac{\epsilon}{2} \ln \frac{1}{\epsilon} \approx \frac{1}{2}\sqrt{\ln \frac{1}{\epsilon}} \gg 1$ and its variance will be at most $N\epsilon \approx 1/\sqrt{\ln \frac{1}{\epsilon}} \ll 1$. Therefore, the sum of X_n can be expected to exceed C. The purpose of this paragraph has been to get a sense of direction in which we are moving, and now we resume the actual proof.

Choose a positive constant $\delta > 0$ (intuitively, small even as compared with ϵ) such that the ratio $M := (\sqrt{\epsilon} - \epsilon)/\delta$ is integer. The Darboux sums for the Riemann integral used earlier for computing $\mathbb{E}\,\tilde{X}_n$ are

$$L := \sum_{m=1}^{M} \frac{\epsilon\delta}{m\delta + \epsilon} \le \int_0^{\sqrt{\epsilon}-\epsilon} \frac{\epsilon \mathrm{d}x}{x + \epsilon} \le \sum_{m=0}^{M-1} \frac{\epsilon\delta}{m\delta + \epsilon};$$

we will be interested in the lower Darboux sum L. Fix temporarily an $n \in \{1, \dots, N\}$. For each $m \in \{1, \dots, M\}$, there is a positive elementary capital process starting at time U_{n-1} from $\epsilon\delta/(m\delta + \epsilon)$ and ending at:

- δ if and when ω hits $M_{n-1} + m\delta$ (provided this happens before T_{n-1});
- 0 if and when ω hits $M_{n-1} - \epsilon$ (at time T_{n-1}) before hitting $M_{n-1} + m\delta$.

Indeed, such a process can be obtained by choosing portfolio $\delta/(m\delta+\epsilon)$ at time U_{n-1} and then choosing portfolio 0 when $M_{n-1} + m\delta$ or $M_{n-1} - \epsilon$ is hit. The sum S_n of such elementary capital processes over $m = 1, \dots, M$ will also be a positive elementary capital process.

The initial capital $S_n(U_{n-1})$ of S_n is L, and it is easy to see that $S_n(T_{n-1}) = \delta\lfloor \tilde{X}_n/\delta \rfloor \le \tilde{X}_n$. The elementary capital process $L - S_n$ starts from 0 and ends up with at least $x_n := L - \tilde{X}_n$ at time T_{n-1}.

Let us take δ so small that $L \ge \frac{\epsilon}{3} \ln \frac{1}{\epsilon}$. Lemma 6 gives an explicit elementary capital process $\mathcal{K}$ that starts from 1 and ends with at least

$$\frac{1}{\sqrt{\epsilon}^2 \left\lfloor \frac{1}{\epsilon\sqrt{\ln \frac{1}{\epsilon}}} \right\rfloor} \left(\frac{\epsilon}{3} \ln \frac{1}{\epsilon} \left\lfloor \frac{1}{\epsilon\sqrt{\ln \frac{1}{\epsilon}}} \right\rfloor - \sum_{n=1}^{N} \tilde{X}_n \right)^{+,2}$$
$$\ge \sqrt{\ln \frac{1}{\epsilon}} \left(\frac{1}{4} \sqrt{\ln \frac{1}{\epsilon}} - \sum_{n=1}^{N} \tilde{X}_n \right)^{+,2} \qquad (10)$$

at time T_{N-1}. On the event $\sum_{n=1}^{N} \tilde{X}_n \le C$, the final capital (10) can be made arbitrarily large by choosing a small ϵ.

We still need to make sure that the elementary capital process $\mathcal{K}$ constructed in the last paragraph is positive: we did not show that it does not become strictly negative strictly between U_{n-1} and T_{n-1}. According to (9), Sceptic's move s_n never exceeds

$$\frac{2}{\sqrt{\epsilon}^2 N} N \frac{\epsilon}{2} \ln \frac{1}{\epsilon} = \ln \frac{1}{\epsilon},$$

and

$$S_n \le \sum_{m=1}^{M} \frac{\delta}{m\delta + \epsilon} m\delta \le M\delta \le \sqrt{\epsilon}$$

implies $L - S_n \ge -\sqrt{\epsilon}$. Therefore, our elementary capital process $\mathcal{K}$ is always at least $-\sqrt{\epsilon} \ln \frac{1}{\epsilon}$; $\sqrt{\epsilon} \ln \frac{1}{\epsilon}$ is a small amount that can be added to the initial capital to make $\mathcal{K}$ positive. This completes the proof.

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