A probability-free and continuous-time explanation of the equity premium and CAPM

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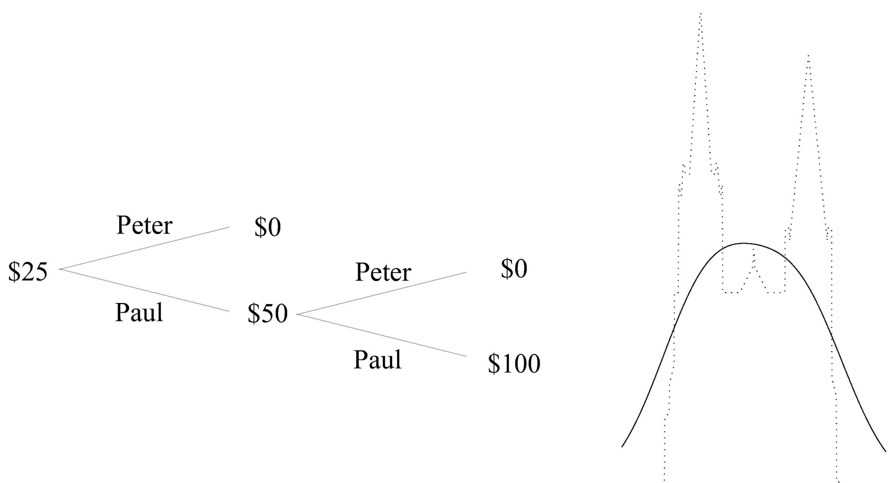

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Abstract

This paper gives yet another definition of game-theoretic probability in the context of continuous-time idealized financial markets. Without making any probabilistic assumptions (but assuming positive and continuous price paths), we obtain a simple expression for the equity premium and derive a version of the Capital Asset Pricing Model. Finally, we derive a probability-free version of Girsanov's theorem and explain how it implies the previous results.

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1 Introduction

This paper reviews and extends previous work in which we derived the existence of an equity premium and the validity of a Capital Asset Pricing Model (CAPM) from a purely game-theoretic hypothesis of market efficiency, without assuming the existence of probabilities for security prices.

For simplicity, we consider only two securities, a stock S and a traded market index I. We also make the following simplifying assumptions:

- Trading in S and I continues indefinitely. (The time horizon is infinite.)
- The prices of S and I are always positive and continuous.
- The interest rate is zero.

All these assumptions can be relaxed.

Our mathematical results have a practical interpretation if one adopts the hypothesis that the index I is efficient, in the sense that a strategy for trading in S and I will not multiply the capital it risks by a factor many times larger than what would be achieved by buying and holding I. We call this the *Efficient Index Hypothesis (EIH) for I*.

A typical mathematical result in this paper asserts the existence of a trading strategy that will multiply the capital it risks by a factor many times larger than what would be achieved by buying and holding I unless the price trajectories of S and I have a particular property. Here are some properties we consider:

- I grows at a rate determined by its volatility. (This is the equity premium.)
- I has the properties of geometric Brownian motion when time is appropriately rescaled.
- S obeys a CAPM with respect to I.

In each case, we prove the existence of a trading strategy that beats I by a large factor if the property does not hold. If you subscribe to the EIH for I, then you expect the property to hold.

The EIH is explained more fully in Section 2, where we specify the trading strategies we consider, define an extended class of approximate capital processes (supermartingales), and state the associated definition of upper probability. An upper probability measures how little initial capital must be risked to obtain unit capital if an event happens and thus how unlikely that event is.

We study the index I in Sections 3–5. In Section 3 we define I's cumulative growth rate and relative quadratic variation; these exist in a strong sense under the EIH: the trader can become infinitely rich as soon as they cease to exist. In Sections 4 and 5 we consider strategies for trading in I and show that under the EIH it grows at a rate determined by its relative quadratic variation. This growth is the equity premium.

Section 6 continues Section 3 by defining quantities involving both S and I. Section 7 then derives a CAPM that relates these quantities to each other and includes most of our results about the equity premium as special cases. Even more general results are discussed briefly in Section 8, and Section 9 concludes by discussing connections with the standard CAPM.

One purpose of this paper is to clarify the relation between two different methods that we used in previous work. We first established a probability-free CAPM fifteen years ago; essentially, our version was the conjunction of a probability-free version of the standard CAPM and a probability-free expression for the equity premium. We did this first in discrete time [20], and then we extended the argument to continuous time using nonstandard analysis [19]. The method used in those papers involved mixing, in a certain sense, the price paths of S and I (in the case of CAPM) or mixing I and cash (in the case of the equity premium). Ten years later, without using nonstandard analysis, one of us derived a probability-free version of the Dubins–Schwarz theorem [16], effectively reducing the probability-free setting to the Bachelier model, which for positive prices becomes the Black–Scholes model after a time change. The Black–Scholes model allows us to use standard probabilistic tools, including Girsanov's theorem (its standard measure-theoretic version), to obtain a stronger form of our version of the CAPM [13, 14, 15].

In this paper we apply and compare the two methods, mixing [20, 19] and probabilistic [13, 14, 15], implementing them both without using nonstandard analysis. In Section 5, we study the equity premium using the mixing method, and in Section 4, we study it using the probabilistic method in combination with the probability-free Dubins–Schwarz theorem. The results from the probabilistic method are stronger than those from the mixing method, in the sense that they assert higher lower probabilities for the approximations formalizing the equity premium phenomenon, but the difference is not great. Since the probability-free Dubins–Schwarz theorem is only applicable to one security, we cannot apply the probabilistic method to the CAPM, which involves both S and I. Therefore, we use the mixing method to obtain our version of the CAPM in Section 7.

2 The Efficient Index Hypothesis

The sample space of this paper is the set Ω of all pairs $\omega = (I, S)$ of positive continuous functions $I : [0, \infty) \to (0, \infty)$ and $S : [0, \infty) \to (0, \infty)$ such that $I(0) = 1$. Each $\omega = (I, S) \in \Omega$ will be identified with the function $\omega : [0, \infty) \to (0, \infty)^2$ defined by $\omega(t) := (I(t), S(t))$, $t \in [0, \infty)$. Intuitively, I is the price path of an index and S is that of a stock or another financial security. The assumption $I(0) = 1$ is made for simplicity and without loss of generality.

We equip Ω with the σ-algebra $\mathcal{F}$ generated by the functions $\omega \in \Omega \mapsto \omega(t)$, $t \in [0, \infty)$ (i.e., the smallest σ-algebra making them measurable). We often consider subsets of Ω and functions on Ω that are measurable with respect to $\mathcal{F}$. As shown in [18], the requirement of measurability is essential: without measurability, it is too easy to become infinitely rich infinitely quickly.

An *event* is an arbitrary subset of Ω (we will add the qualifier "$\mathcal{F}$-measurable" when needed), a *random vector* is an $\mathcal{F}$-measurable function of the type $\Omega \to \mathbb{R}^d$ for some $d \in \{1, 2, \ldots\}$, and an *extended random variable* is an $\mathcal{F}$-measurable function of the type $\Omega \to [-\infty, \infty]$. A *stopping time* is an extended random variable $\tau : \Omega \to [0, \infty]$ such that, for all ω and ω' in Ω,

$$\left(\omega|_{[0,\tau(\omega)]} = \omega'|_{[0,\tau(\omega)]}\right) \Longrightarrow \tau(\omega) = \tau(\omega'),$$

where $f|_A$ stands for the restriction of f to the intersection of A and f's domain. A random vector X is said to be *τ-measurable*, where τ is a stopping time, if, for all ω and ω' in Ω,

$$\left(\omega|_{[0,\tau(\omega)]} = \omega'|_{[0,\tau(\omega)]}\right) \Longrightarrow X(\omega) = X(\omega').$$

As customary in probability theory, we will often omit explicit mention of $\omega \in \Omega$ when it is clear from the context.

A *simple trading strategy* G is a pair $((\tau_1, \tau_2, \ldots), (h_1, h_2, \ldots))$, where:

- $\tau_1 \le \tau_2 \le \cdots$ is a nondecreasing sequence of stopping times such that, for each $\omega \in \Omega$, $\lim_{n\to\infty} \tau_n(\omega) = \infty$;
- for each $n = 1, 2, \ldots$, h_n is a bounded τ_n-measurable $\mathbb{R}^2$-valued random vector.

A *process* is a function $X : [0, \infty) \times \Omega \to [-\infty, \infty]$. A process X is *continuous* if each of its *paths*, $t \in [0, \infty) \mapsto X_t(\omega)$, is a continuous function. The *simple capital process* $\mathcal{K}^{G,c}$ corresponding to a simple trading strategy G and *initial capital* $c \in \mathbb{R}$ is defined by

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega) \cdot \big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\big), \quad t \in [0, \infty), \ \omega \in \Omega,$$

where "$\cdot$" stands for dot product and the zero terms in the sum are ignored (which makes the sum finite for each t).

The vector $h_n(\omega)$ tells the trader how many units of I and S to hold between time $\tau_n(\omega)$ and $\tau_{n+1}(\omega)$, and thus $\mathcal{K}_t^{G,c}(\omega)$ is his *capital* at time t. Negative components for h_n indicate short selling. Because I and S are continuous, a strategy G can sell one or both of them short and yet produce a nonnegative simple capital process $\mathcal{K}^{G,c}$; the τ_n and h_n can be chosen so that the short selling always stops before $\mathcal{K}_t^{G,c}(\omega)$ gets below zero.

For $\omega = (I, S)$ and $t \in [0, \infty)$, we often let $I_t(\omega)$ stand for $I(t)$ and $S_t(\omega)$ for $S(t)$. When we omit ω, this makes I_t (resp. S_t) synonymous with $I(t)$ (resp. $S(t)$).

Let us say that a class $\mathcal{C}$ of processes (not necessarily continuous) is lim inf-*closed* if the process

$$X_t(\omega) := \liminf_{k\to\infty} X_t^k(\omega) \tag{1}$$

is in $\mathcal{C}$ whenever each process X^k is in $\mathcal{C}$. A nonnegative process X is a *test supermartingale* if it belongs to the smallest lim inf-closed class of processes containing all nonnegative simple capital processes. Intuitively, test supermartingales are nonnegative capital processes (as they can be approximated by nonnegative simple capital processes; in fact, they can lose capital as the approximation is in the sense of lim inf).

We call processes of the type $X_t(\omega)/I_t(\omega)$, where X is a test supermartingale, *test I-supermartingales*; they are like test supermartingales but use I as the numéraire. Notice that test supermartingales and test I-supermartingales are not required to satisfy any continuity properties (such as being càdlàg).

The initial value X_0 of a test supermartingale X is always a constant. Given a subset E of Ω, we set

$$\overline{\mathbb{P}}(E) := \inf\bigl\{X_0 \bigm| \forall\omega\in\Omega : \liminf_{t\to\infty} X_t(\omega) \ge \mathbf{1}_E(\omega)\bigr\} \tag{2}$$

and

$$\overline{\mathbb{P}}^I(E) := \inf\bigl\{X_0 \bigm| \forall\omega\in\Omega : \liminf_{t\to\infty} X_t(\omega)/I_t(\omega) \ge \mathbf{1}_E(\omega)\bigr\}, \tag{3}$$

X ranging in each case over the test supermartingales.[1] We call $\overline{\mathbb{P}}(E)$ E's *upper probability*, and we call $\overline{\mathbb{P}}^I(E)$ its *I-upper probability*. The definition (3) can be rewritten as

$$\overline{\mathbb{P}}^I(E) = \inf\bigl\{X_0 \bigm| \forall\omega\in\Omega : \liminf_{t\to\infty} X_t(\omega) \ge \mathbf{1}_E(\omega)\bigr\}, \tag{4}$$

X ranging over the test I-supermartingales.

Recalling that $I_0(\omega) = 1$ for all $\omega\in\Omega$, we see from (3) that a value of $\overline{\mathbb{P}}^I(E)$ close to zero indicates the existence of a trading strategy that beats the index I by a large factor if E happens. The EIH for I says that we should not expect to beat I by a large factor, and so we should not expect E to happen. The EIH for E has a lot of empirical support when I is an index, such as the S&P500, which can be approximately traded with low transaction costs; see, e.g., [7, 8].

We do not interpret small values of $\overline{\mathbb{P}}(E)$ in the same way. Saying that E will not happen when $\overline{\mathbb{P}}(E)$ is small would amount to adopting an efficiency hypothesis for cash or for a bank account (recall that the interest rate is zero)—i.e., to asserting that no trading strategy will beat holding cash by a large factor. We do not assert this. In fact, the EIH for nontrivial I implies the opposite. It implies that we can expect holding I to beat holding cash by an infinite factor as time goes to infinity; this is a consequence of our results for the equity premium in Sections 4 and 5. (The efficiency hypothesis for cash, in contrast, implies that the price of I, or any other traded security, will tend to a constant. See Theorem 3.1 in [16].)

Remark 2.1. The EIH can be considered to be a special case of Cournot's principle [11].

Remark 2.2. An equivalent definition of the class $\mathcal{C}$ of test supermartingales can be given using transfinite induction over the countable ordinals α (see, e.g., [2], 0.8). Namely, define $\mathcal{C}^\alpha$ as follows:

[1] Here, as always in game-theoretic probability, upper probability is a special case of upper expected value. Upper expected values $\overline{\mathbb{E}}(F)$ and $\overline{\mathbb{E}}^I(F)$, where $F:\Omega\to[0,\infty]$, are defined by substituting the function F for $\mathbf{1}_E$ in (2) and (3), respectively. We do not use $\overline{\mathbb{E}}^I$ in this paper but do use $\overline{\mathbb{E}}$ on one occasion.

- $\mathcal{C}^0$ is the class of all nonnegative simple capital processes;
- for $\alpha > 0$, $X \in \mathcal{C}^\alpha$ if and only if there exists a sequence $X^1, X^2, \dots$ of processes in $\mathcal{C}^{<\alpha} := \cup_{\beta<\alpha}\mathcal{C}^\beta$ such that (1) holds.

It is easy to check that the class of all test supermartingales is the union of the nested family $\mathcal{C}^\alpha$ over all countable ordinals α. The *class* of a test supermartingale X is defined to be the smallest α such that $X \in \mathcal{C}^\alpha$; in this case we will also say that X is *of class* α.

Remark 2.3. The hierarchy $(\mathcal{C}^\alpha)$ described in Remark 2.2 is somewhat analogous to the Baire hierarchy of Borel functions on a Polish space: see, e.g., [6], Section 24.

Remark 2.4. We can split the requirement that a class of processes be lim inf-closed into two: that it be min-closed (if a finite number of processes are in the class, there minimum is also required to be in the class) and that it be lim-closed (if processes $X^1, X^2, \dots$ are in the class and the limit $X := \lim_{k\to\infty} X^k$ exists, X is also required to be in the class).

Remark 2.5. Our definition of upper probability is similar to the one given by Perkowski and Prömel [9] (who modified the definition given in [16]). The main differences are that Perkowski and Prömel define the upper probability (2) using the test supermartingales in the class $\mathcal{C}^1$ rather than $\mathcal{C}$ (in the notation of Remark 2.2) and that they consider a finite horizon (our time interval is $[0,\infty)$ instead of their $[0,T]$). The proofs of our results given below work for any $\mathcal{C}^n$, $n \ge 2$, in place of $\mathcal{C}$.

Remark 2.6. The motivation for our terminology is the analogy with measure-theoretic probability. Namely, let us suppose that I and S are local martingales on a measure-theoretic probability space. Each simple capital process is a local martingale. Since each nonnegative local martingale is a supermartingale ([10], p. 123), nonnegative simple capital processes are supermartingales. By Fatou's lemma, $\liminf_k X^k$ is a supermartingale whenever X^k are nonnegative supermartingales:

$$\mathbb{E}\left(\liminf_k X_t^k \mid \mathcal{F}_s\right) \le \liminf_k \mathbb{E}(X_t^k \mid \mathcal{F}_s) \le \liminf_k X_s^k,$$

where $0 \le s < t$. Therefore, our definition gives a subset of the set of all nonnegative measure-theoretic supermartingales. (We are using the definition of a measure-theoretic supermartingale that does not impose any continuity conditions, as in [10], Definition II.1.1.)

Remark 2.7. Let us check that, in the measure-theoretic setting of Remark 2.6 (where I and S are local martingales), $\overline{\mathbb{P}}(E) \ge \mathbb{P}(E)$ for each $\mathcal{F}$-measurable E. (In this sense our definition (2) of $\overline{\mathbb{P}}$ is not too permissive, unlike the definition ignoring measurability in [18].) It suffices to establish the "maximal inequality" for nonnegative measure-theoretic supermartingales X with X_0 a constant in the form

$$\mathbb{P}\left(\liminf_{t\to\infty} X_t \ge 1\right) \le X_0.$$

To check this, notice that, for each $\epsilon \in (0,1)$,

$$\begin{aligned}\mathbb{P}\left(\liminf_{t\to\infty} X_t \ge 1\right) &\le \mathbb{P}(X_t \ge 1-\epsilon \text{ from some } t \text{ on})\\ &= \liminf_{T\to\infty} \mathbb{P}(X_t \ge 1-\epsilon \text{ for all } t \ge T)\end{aligned}$$

$$\leq \liminf_{T\to\infty} \mathbb{P}(X_T \geq 1-\epsilon) \leq \frac{X_0}{1-\epsilon}$$

(the equality, where the lim inf is in fact lim, follows from the axiom of continuity of probability measures).

Remark 2.8. Let us say that X is a class α test I-supermartingale, where α is a countable ordinal, if XI is a class α test supermartingale, as defined in Remark 2.2. For each countable ordinal α and each class α test I-supermartingale we fix a sequence X^k of test I-supermartingales of smaller classes such that $X = \liminf_{k\to\infty} X^k$ (as usual, we are using the axiom of choice freely).

The following lemma says that the definition (4) is robust in that the lim inf in it can be replaced by lim sup or even sup. (The analogous statement is, of course, true for (2) as well.)

Lemma 2.9. *For any $E \subseteq \Omega$,*

$$\overline{\mathbb{P}}^I(E) = \inf\left\{X_0 \;\middle|\; \forall\omega\in\Omega : \sup_{t\in[0,\infty)} X_t(\omega) \geq \mathbf{1}_E(\omega)\right\}, \tag{5}$$

X ranging over the test I-supermartingales.

Proof. The only nontrivial part of the equality in (5) is the inequality "$\leq$", and it is clear that we can replace "$\geq \mathbf{1}_E(\omega)$" by "$> \mathbf{1}_E(\omega)$". This is what we will be proving.

For each test I-supermartingale X with $X_0 < 1$ we will define another test I-supermartingale X^*, satisfying $X_0^* = X_0$, as follows. If X is a simple capital process, set

$$X_t^* := \begin{cases} X_t & \text{if } \sup_{s\in[0,t]} X_s < 1 \\ 1 & \text{otherwise} \end{cases}$$

(intuitively, the trader spends all his capital to buy and hold the index as soon as X reaches 1). If X is a class α test I-supermartingale, we set $X^* = \liminf_{k\to\infty}(X^k)^*$, where X^k is the fixed sequence of I-supermartingales of classes smaller than that of X (see Remark 2.8).

It suffices to check that $\liminf_{t\to\infty} X_t^* = 1$ whenever $\sup_t X_t > 1$. We will prove that $X_t^* = 1$ whenever $\sup_{s\leq t} X_s > 1$. Fix a t. The proof is by transfinite induction. For nonnegative simple capital processes this is true by definition. Now let X be a class α test I-supermartingale such that $\sup_{s\leq t} X_s > 1$. Fix $s \leq t$ such that $X_s > 1$. Then $X_s = \liminf_{k\to\infty} X_s^k$ for the fixed X^k of smaller classes, and we have $X_s^k > 1$ from some k on. By the inductive assumption, $(X^k)_t^* = 1$ from some k on, which implies $X_t^* = 1$. □

We call a subset of $[0,\infty)\times\Omega$ a *time-dependent property* (or simply a property of t and ω). We say that a time-dependent property E holds *quasi-always* (q.a.) if there exists a test supermartingale (or equivalently, a test I-supermartingale) X such that $X_0 = 1$ and, for all $t \in [0,\infty)$ and $\omega \in \Omega$,

$$(\exists s < t : (s,\omega) \notin E) \Longrightarrow X_t(\omega) = \infty. \tag{6}$$

To put it differently, the trader can become infinitely rich as soon as such E is violated. Many of the results in this paper involve showing that some time-dependent property (such as the existence of relative quadratic variation or growth rate up to time t) holds quasi-always and therefore can be expected to hold always under the EIH.

Remark 2.10. In previous work on the topics of this paper (see for example [16]), we used only the very weak form of the EIH that states an event will not happen if it allows a trader to become infinitely rich infinitely quick. This weak hypothesis follows from the efficiency hypothesis for cash (which implies that an event with $\overline{\mathbb{P}}$-probability zero will not happen) just as easily as from the EIH for I (which implies that an event with $\overline{\mathbb{P}}^I$-probability zero will not happen). Indeed, if we set

$$\mathrm{Fail}_E := \{\omega \in \Omega \mid (s,\omega) \notin E \text{ for some } s \in [0,\infty)\}$$

when E is a time-dependent property, then E holding quasi-always implies $\overline{\mathbb{P}}(\mathrm{Fail}_E) = \overline{\mathbb{P}}^I(\mathrm{Fail}_E) = 0$; see (6). For this reason, we did not introduce $\overline{\mathbb{P}}^I$ in this previous work. Instead we discussed our results in terms of $\overline{\mathbb{P}}$, which is easier to define.

3 Existence of some basic quantities (1)

In this section we do the preparatory work needed to state our results about the equity premium; namely, we show the existence of all the quantities required in their statements.

All quantities will be defined in terms of the sequences of stopping times $T_0^n := 0$ and

$$T_k^n(\omega) := \inf\Big\{t > T_{k-1}^n(\omega) \mid \big|I(t) - I(T_{k-1}^n)\big| = 2^{-n}\Big\} \qquad (7)$$

for $k = 1, 2, \ldots$; here n is a positive integer, $n \in \{1, 2, \ldots\}$. The quadratic variation of I on the log scale (or *relative quadratic variation* of I) can be measured by the sums of squares of the relative increments of $I(t)$,

$$\Sigma_t^{I,n}(\omega) := \sum_{k=1}^{\infty} \left(\frac{I(T_k^n \wedge t) - I(T_{k-1}^n \wedge t)}{I(T_{k-1}^n \wedge t)} \right)^2, \quad n = 1, 2, \ldots . \qquad (8)$$

It follows from Theorem 3.1 in [16] and the properties of measure-theoretic Brownian motion that the limit of $\Sigma^{I,n}$ as $n \to \infty$ exists quasi-always; however, we will also check this independently in Section 6. The limit will be denoted $\Sigma_t^I(\omega)$. Moreover, the convergence is uniform over any compact time interval, so the limit is continuous quasi-always. (Formally, the property "$\Sigma_s^{I,n} \to \Sigma_s^I$ as $n \to \infty$ uniformly over $s \in [0,t]$" of t and ω holds quasi-always.)

Remark 3.1. Quasi-always, the limit

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\ln I(T_k^n \wedge t) - \ln I(T_{k-1}^n \wedge t)\right)^2$$

exists and is equal to Σ_t^I.

The quantity Σ^I_t measures the accumulated volatility of the index by time t. It can be interpreted as the intrinsic time that elapsed by the moment t of physical time; unlike physical time, intrinsic time flows faster during intensive trading.

We will simplify our exposition by requiring that Reality ensure that the function Σ^I exists and $\Sigma^I_\infty = \infty$. (Essentially, that the market exists forever and trading in it never dies out.)

The cumulative relative growth of the index I by time t is

$$\mathrm{M}^{I,n}_t(\omega) := \sum_{k=1}^{\infty} \frac{I(T^n_k \wedge t) - I(T^n_{k-1} \wedge t)}{I(T^n_{k-1} \wedge t)}, \quad n = 1, 2, \dots . \tag{9}$$

The existence of the limit of $\mathrm{M}^{I,n}$ as $n \to \infty$ and a simple expression for it are provided by the following lemma.

Lemma 3.2. *The limit* $\mathrm{M}^I := \lim_{n\to\infty} \mathrm{M}^{I,n}$ *exists and satisfies, quasi-always,*

$$\mathrm{M}^I_t = \ln I(t) + \frac{1}{2}\Sigma^I_t.$$

Proof. Let us show that the limit exists and is uniform over any compact time interval quasi-always. Applying Taylor's expansion

$$\ln(1 + m_k) = m_k - \frac{1}{2} m_k^2 + O\left(|m_k|^3\right)$$

to

$$m_k := \frac{I(T^n_k \wedge t) - I(T^n_{k-1} \wedge t)}{I(T^n_{k-1} \wedge t)}, \tag{10}$$

we obtain

$$\ln I(t) = \sum_{k=1}^{\infty} \ln \frac{I(T^n_k \wedge t)}{I(T^n_{k-1} \wedge t)}$$
$$= \mathrm{M}^{I,n}_t - \frac{1}{2}\Sigma^{I,n}_t + O\left(\sum_{k=1}^{\infty} \left| \frac{I(T^n_k \wedge t) - I(T^n_{k-1} \wedge t)}{I(T^n_{k-1} \wedge t)} \right|^3\right),$$

and it remains to notice that the last added, $O(\cdots)$, is $o(1)$ since the denominator in it can be ignored (remember that I is positive) and the variation index of I is at most 2 [16], quasi-always. □

4 Equity premium (1): reduction to the Black–Scholes model

In this section we will state two forms of our equity premium result: as a central limit theorem (which is trivial in the context of Brownian motion) and as a law of the iterated logarithm (LIL). Remember that we assume that $I(0) = 1$.

Lemma 4.1. *Set* $\tau_t := \inf\{s \mid \Sigma^I_s \geq t\}$ *for each* $t \in [0,\infty)$*. As function of* t*,* $\ln I(\tau_t) + t/2$ *is Brownian motion with respect to* $\overline{\mathbb{P}}$*.*

Formally, Lemma 4.1 says that the pushforward of the upper probability $\overline{\mathbb{P}}$ is the standard Wiener measure $\mathcal{W}$ on $C[0,\infty)$ under the following mapping $\phi : \Omega \to C[0,\infty)$: $\omega = (I,S) \in \Omega$ is mapped to the path $t \in [0,\infty) \mapsto \ln I(\tau_t) + t/2$. The domain of ϕ is the set of (I,S) such that $\Sigma^I_\infty = \infty$, which was our requirement for Reality in Section 3.

Proof of Lemma 4.1. Let E be a Borel set in $C[0,\infty)$. The set $\phi^{-1}(E)$ is time-superinvariant (as defined in [16], Section 3). According to Theorem 3.1 in [16], $\overline{\mathbb{P}}(\phi^{-1}(E))$ coincides with the standard Wiener measure $\mathcal{W}(\phi^{-1}(E))$ of $\phi^{-1}(E)$. This measure $\mathcal{W}$ is concentrated on the positive functions f whose quadratic variation is the identity. Applying the time transformation $f'(t') := f(t)$, where $t' := \int_0^t f^{-2}(s)\,\mathrm{d}s$ to those functions, we obtain a probability measure P (the pushforward of $\mathcal{W}$ under $f \mapsto f'$) concentrated on the functions whose relative quadratic variation Σ is the identity; we know that $\mathcal{W}(\phi^{-1}(E)) = P(\phi^{-1}(E))$. By the standard measure-theoretic Dubins–Schwarz theorem, P will coincide with the distribution of the measure-theoretic martingale

$$G_t := e^{W_t - t/2}, \tag{11}$$

W being the standard Brownian motion (started at 0). (Notice that (11) is a special case of geometric Brownian motion, i.e., the Black–Scholes model.) Therefore,

$$\overline{\mathbb{P}}(\phi^{-1}(E)) = \mathcal{W}(\phi^{-1}(E)) = P(\phi^{-1}(E)) = \mathcal{W}(E). \qquad \square$$

Remark 4.2. Lemma 4.1 can be strengthened to say that, for any nonnegative time-superinvariant Borel functional $F : \Omega \to [0,\infty)$, $\overline{\mathbb{E}}(F \circ \phi) = \int F\,\mathrm{d}\mathcal{W}$. Moreover, the same argument as in the proof of Lemma 4.1 (but using Theorem 6.3 instead of Theorem 3.1 in [16]) shows that the last line of the proof can be replaced by

$$\overline{\mathbb{E}}(F \circ \phi) = \int (F \circ \phi)\,\mathrm{d}\mathcal{W} = \int (F \circ \phi)\,\mathrm{d}P = \int F\,\mathrm{d}\mathcal{W}.$$

Corollary 4.3. *As function of* t*,* $\ln I(\tau_t) - t/2$ *is Brownian motion with respect to* $\overline{\mathbb{P}}^I$*.*

Proof. According to Lemma 4.1, $W_t := \ln I(\tau_t) + t/2$ is standard Brownian motion w.r. to $\overline{\mathbb{P}}$. To change the numéraire we apply Girsanov's theorem (see, e.g., [5], Corollary 3.5.2; this version, unlike Theorem 3.5.1 in [5], does not require the usual conditions). It suffices to show, for each $T > 0$, that $\ln I(\tau_t) - t/2$, $t \in [0,T]$, is Brownian motion over $[0,T]$ with respect to $\overline{\mathbb{P}}^I$. Fix such a T. By Girsanov's theorem, $\tilde{W}_t := W_t - t = \ln I(\tau_t) - t/2$ is standard Brownian motion w.r. to the measure $\tilde{P}$ on $C[0,T]$ whose density with respect to P (as defined in the proof of Lemma 4.1 but restricted to $C[0,T]$) is

$$Z_T := e^{W_T - T/2} = I(\tau_T).$$

Let $\tilde{\phi} : \Omega \to C[0,T]$ map each $\omega = (I,S) \in \Omega$ to the path $t \in [0,T] \mapsto \ln I(\tau_t) - t/2$; we will continue to use the notation ϕ for the function that maps each $\omega = (I,S) \in \Omega$ to the path $t \in [0,T] \mapsto \ln I(\tau_t) + t/2$. Let us check that $\tilde{P}$ is the pushforward of $\overline{\mathbb{P}}^I$ under $\tilde{\phi}$: for each Borel $E \subseteq C[0,T]$, by the definition (3) and Remark 4.2,

$$\begin{aligned}\overline{\mathbb{P}}^I(\tilde{\phi}^{-1}(E)) &= \inf\left\{X_0 \mid \forall\omega \in \Omega : X_{\tau_T}(\omega)/I_{\tau_T}(\omega) \geq \mathbf{1}_{\tilde{\phi}^{-1}(E)}(\omega)\right\} \\ &= \inf\left\{X_0 \mid \forall\omega \in \Omega : X_{\tau_T}(\omega) \geq I_{\tau_T}(\omega)\,\mathbf{1}_{\tilde{\phi}^{-1}(E)}(\omega)\right\} \\ &= \overline{\mathbb{E}}\left(I_{\tau_T}\,\mathbf{1}_{\tilde{\phi}^{-1}(E)}\right) = \overline{\mathbb{E}}\left(I_{\tau_T}\,\mathbf{1}_E \circ \tilde{\phi}\right) \\ &= \mathbb{E}_P\left(Z_T\,\mathbf{1}_E \circ \tilde{W}\right) = \mathbb{E}_{\tilde{P}}\left(\mathbf{1}_E \circ \tilde{W}\right) = \mathcal{W}(E). \qquad \square\end{aligned}$$

Let us first derive a central limit theorem for the index from Corollary 4.3. Let z_p be the upper p-quantile of the standard Gaussian distribution $N_{0,1}$; i.e., z_p is defined by the requirement that $\mathbb{P}(\xi \geq z_p) = p$, where $\xi \sim N_{0,1}$.

Corollary 4.4. *If $\delta > 0$ and $T > 0$ are positive constants,*

$$\underline{\mathbb{P}}^I\left\{|\ln I(\tau_T) - T/2| < z_{\delta/2}\sqrt{T}\right\} = 1 - \delta. \tag{12}$$

We can interpret (12) by saying that there is a prudent trading strategy that beats the index by a factor of nearly $1/\delta$ unless $I(\tau_T)$ is close to $e^{T/2}$ in the sense

$$I(\tau_T) \in \left(e^{T/2 - z_{\delta/2}\sqrt{T}}, e^{T/2 + z_{\delta/2}\sqrt{T}}\right).$$

In other words, the efficient index can be expected to outperform cash $e^{T/2}$-fold. The case $\delta \geq 1$ in Corollary 4.4 is trivial, but we do not exclude it to simplify its statement; the upper quantile z_p is understood to be $-\infty$ when $p \geq 1$.

If we are only interested in a lower or upper bound on I, we can use the following corollary.

Corollary 4.5. *Let δ and T be as before. Then*

$$\underline{\mathbb{P}}^I\left\{\ln I(\tau_T) - T/2 > -z_\delta\sqrt{T}\right\} = 1 - \delta$$

and

$$\underline{\mathbb{P}}^I\left\{\ln I(\tau_T) - T/2 < z_\delta\sqrt{T}\right\} = 1 - \delta.$$

Corollaries 4.4 and 4.5 follow immediately from Corollary 4.3.

Corollary 4.3 also immediately implies the following law of the iterated logarithm for the equity premium:

Corollary 4.6. *It is $\overline{\mathbb{P}}^I$-almost certain that*

$$\limsup_{t\to\infty} \frac{\ln I(t) - \Sigma_t^I/2}{\sqrt{2\Sigma_t^I \ln\ln \Sigma_t^I}} = 1$$

and

$$\liminf_{t\to\infty} \frac{\ln I(t) - \Sigma_t^I/2}{\sqrt{2\Sigma_t^I \ln\ln \Sigma_t^I}} = -1.$$

5 Equity-premium (2): mixing method

In this section we will discuss an alternative approach to the equity premium phenomenon, which will also be used to derive a probability-free CAPM in Section 7. The test I-supermartingale whose existence is implicitly asserted in (12) is, in a sense, reckless: it beats the index $1/\delta$-fold if the event in the curly braces fails to happen but can (and does) lose everything if it happens. In this section we will discuss safer (more conservative) trading strategies instead of "all-or-nothing" trading strategies fine-tuned to the event of interest (such as the one in (12)).

Lemma 5.1. *For each $\epsilon \in \mathbb{R}$, the process*

$$\exp\left(\epsilon(\mathrm{M}_t^I - \Sigma_t^I) - \frac{\epsilon^2}{2}\Sigma_t^I\right) \tag{13}$$

is a test I-supermartingale q.a.

Remark 5.2. In other words, Lemma 5.1 says that the process (13) coincides with a test I-supermartingale quasi-always. This notion of a *test supermartingale q.a.* can be regarded as a generalization of the notion of a test supermartingale, and the former can be used in place of the latter when defining the notion of "quasi-always". However, it is easy to check that in fact this procedure does not extend our original notion of "quasi-always".

Proof of Lemma 5.1. Let us show that (13) is a class 1 test I-supermartingale (see Remark 2.8 for the definition).

The value of the index I at time T_K^n is $\prod_{k=1}^K(1+m_k)$ (where m_k is defined by (10)). Let us consider the simple capital process whose value at time T_K^n is $\prod_{k=1}^K(1+(1+\epsilon)m_k)$ (which should be stopped as soon as the capital hits 0). Intuitively, we are mixing the returns m_k of I and the returns 0 of cash (and this is a convex mixture when $\epsilon \in [-1,0]$); when $|\epsilon|$ is small (which case is important for limit theorems such as the law of the iterated logarithm in Corollary 5.6), the new simple capital process can be regarded as a perturbation of I.

We can see that

$$\ln\prod_{k=1}^K(1+(1+\epsilon)m_k) - \ln\prod_{k=1}^K(1+m_k)$$

is the value at time T_K^n of the log of a test I-supermartingale (of class 0). In combination with Taylor's expansion, this implies that

$$\epsilon\sum_{k=1}^K m_k - \epsilon\sum_{k=1}^K m_k^2 - \frac{\epsilon^2}{2}\sum_{k=1}^K m_k^2 + O\left(\sum_{k=1}^K |m_k|^3\right)$$

is the value at time T_K^n of the log of a test I-supermartingale. Passing to the limit as $n\to\infty$ (and remembering that the variation index of I over a compact time interval does not exceed 2 quasi-always [16]), we obtain that

$$\epsilon(\mathrm{M}_t^I - \Sigma_t^I) - \frac{\epsilon^2}{2}\Sigma_t^I$$

is the log of a test I-supermartingale (of class 1) q.a. $\square$

Corollary 5.3. *For any* $\epsilon > 0$ *and* $\delta > 0$,

$$\underline{\mathbb{P}}^I \left\{ \forall t \in [0,\infty) : \left| \mathrm{M}_t^I - \Sigma_t^I \right| < \frac{1}{\epsilon} \ln \frac{2}{\delta} + \frac{\epsilon}{2} \Sigma_t^I \right\} \geq 1 - \delta.$$

Proof. Fix $\epsilon > 0$ and $\delta > 0$. By Lemmas 2.9 and 5.1, with lower I-probability at least $1 - \delta/2$ we will have

$$\forall t \in [0,\infty) : \epsilon(\mathrm{M}_t^I - \Sigma_t^I) - \frac{\epsilon^2}{2} \Sigma_t^I < \ln \frac{2}{\delta}.$$

Dividing both sides by ϵ and considering the same test I-supermartingale but with $-\epsilon$ in place of ϵ, we obtain

$$\forall t \in [0,\infty) : \left| \mathrm{M}_t^I - \Sigma_t^I \right| < \frac{1}{\epsilon} \ln \frac{2}{\delta} + \frac{\epsilon}{2} \Sigma_t^I,$$

with lower probability at least $1 - \delta$. $\square$

The following corollary is in the spirit of Corollary 4.4.

Corollary 5.4. *If* $\delta > 0$, $\epsilon > 0$, *and* $\tau_T := \inf\{t \mid \Sigma_t^I \geq T\}$ *for some constant* $T > 0$,

$$\underline{\mathbb{P}}^I \left\{ \left| \mathrm{M}_{\tau_T}^I - T \right| < \frac{1}{\epsilon} \ln \frac{2}{\delta} + \frac{\epsilon}{2} T \right\} \geq 1 - \delta. \tag{14}$$

It is natural to optimize the ϵ in (14) given δ and T,

$$\min_{\epsilon > 0} \left(\frac{1}{\epsilon} \ln \frac{2}{\delta} + \frac{\epsilon}{2} T \right) = \sqrt{2T \ln \frac{2}{\delta}}, \tag{15}$$

which gives our final corollary in this direction.

Corollary 5.5. *If* $\delta > 0$ *and* $\tau_T := \inf\{t \mid \Sigma_t^I \geq T\}$ *for some constant* $T > 0$,

$$\underline{\mathbb{P}}^I \left\{ \left| \mathrm{M}_{\tau_T}^I - T \right| < \sqrt{2T \ln \frac{2}{\delta}} \right\} \geq 1 - \delta,$$

or, equivalently (by Lemma 3.2),

$$\underline{\mathbb{P}}^I \left\{ \left| \ln I_{\tau_T} - T/2 \right| < \sqrt{2T \ln \frac{2}{\delta}} \right\} \geq 1 - \delta. \tag{16}$$

It is instructive to compare (16) and (12), which are obtained using two very different methods (especially that for the more general CAPM-type results of the following sections we will be able to use only the second, more conservative, method). The difference between the two inequalities for $I(\tau_T)$ asserted with

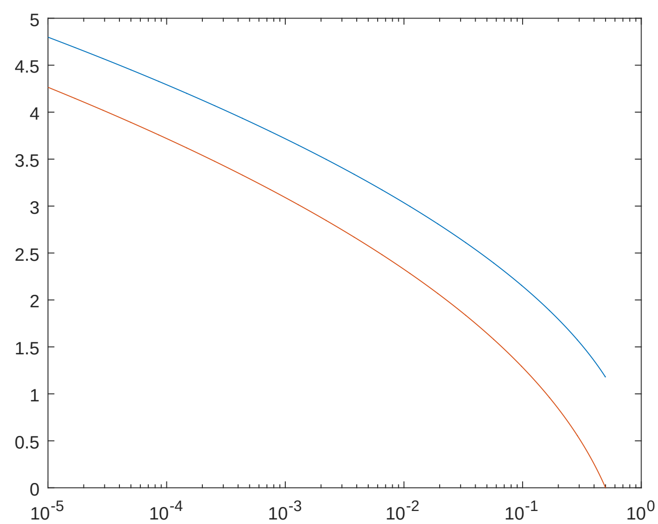

Figure 1: The functions X (in red, lower) and η (in blue, higher) over $[10^{-5}, 0.5]$

lower probability of (at least) $1-\delta$ boils down to the difference between the functions

$$X(q) := z_q \text{ and } \eta(q) := \sqrt{2 \ln \frac{1}{q}},$$

in the notation of Hastings [3], who considers $q \in (0, 0.5]$. Hastings gives two approximations to $X(q)$ (pp. 191–192, reproduced in [1], 26.2.22 and 26.2.23) as the optimal, in a minimax sense, product of $\eta(q)$ and a rational function of $\eta(q)$ (the two approximations correspond to different degrees of the polynomials in the numerator and denominator of the rational function).

It is easy to check that $X(q) \sim \eta(q)$ as $q \to 0$. Figure 1 compares the two functions over the range $q \in [10^{-5}, 0.5]$ ($q = 0.5$ corresponding to the trivial value $\delta = 1$).

It is easy to prove the validity part of a LIL for the equity premium using Ville's [12] method.

Corollary 5.6. *Almost surely w.r.t. to* $\overline{\mathbb{P}}^I$,

$$\limsup_{t \to \infty} \frac{\left| \mathrm{M}_t^I - \Sigma_t^I \right|}{\sqrt{2 \Sigma_t^I \ln \ln \Sigma_t^I}} \le 1.$$

Proof. This is part of Corollary 4.6 (combined with Lemma 3.2), so there is nothing to prove. But alternatively, we could mix the processes (which quasi-always coincide with test I-supermartingales: cf. Lemma 5.1)

$$\exp\left(-\epsilon \mathrm{M}_t^I + \epsilon \Sigma_t^I - \frac{\epsilon^2}{2} \Sigma_t^I\right)$$

and

$$\exp\left(\epsilon \mathrm{M}_t^I - \epsilon \Sigma_t^I - \frac{\epsilon^2}{2}\Sigma_t^I\right)$$

over $\epsilon > 0$ of the form $(1+\kappa)^{-k}$, $k = 1, 2, \ldots$, with weights $w_k = k^{-1-\delta}$ (so that $w_k \to 0$ slowly while still $\sum_k w_k < \infty$) for small $\kappa > 0$ and $\delta > 0$. For details, see Section 7, where we will prove a more general statement. □

Intuitively, our new mixing method still allows us to derive the upper LIL since $X(\delta/2) \sim \eta(\delta/2)$ as $\delta \to 0$, and the LIL is about almost sure convergence and so corresponds to small δ.

6 Existence of some basic quantities (2)

This section continues the series of definitions that we started in Section 3. Now we consider another traded security S. Since it is a traded security, we can define Σ^S and M^S analogously to Σ^I and M^I; however, this would involve stopping times defined in terms of S rather than I. It is more convenient to have just one family of stopping times. Therefore, we now define the sequence of stopping times T_k^n, $k = 0, 1, 2, \ldots$, inductively by $T_0^n(\omega) := 0$ and

$$T_k^n(\omega) := \inf\left\{t > T_{k-1}^n(\omega) \mid \left|I(t) - I(T_{k-1}^n)\right| \vee \left|S(t) - S(T_{k-1}^n)\right| = 2^{-n}\right\} \quad (17)$$

for $k = 1, 2, \ldots$. We let $T^n(\omega)$ stand for the *nth partition*, i.e., the set

$$T^n(\omega) := \{T_k^n(\omega) \mid k = 0, 1, \ldots\};$$

under our new definition, the partitions are not necessarily nested, $T^1 \subseteq T^2 \subseteq \cdots$ (as was the case for our old definition (7)).

The following lemma says that we can redefine $\Sigma_t^{I,n}$ as (8) using the new partitions.

Lemma 6.1. *The limit in* (8) *exists and is uniform over any compact time interval quasi-always for both partitions* (7) *and* (17)*; the limits coincide quasi-always.*

Proof. As shown in [17], the limit of (8) as $n \to \infty$ exists and is uniform over any compact time interval quasi-always if we ignore the denominator; namely, the sequence

$$\mathrm{A}_t^{I,n}(\omega) := \sum_{k=1}^{\infty} \left(I(T_k^n \wedge t) - I(T_{k-1}^n \wedge t)\right)^2, \quad n = 1, 2, \ldots,$$

converges to a function A^I uniformly over any compact time interval quasi-always. The function A^I is the same (quasi-always) for the sequences of partitions (7) and (17): to check this, apply the argument given in Section 5 of [17] to the nth partitions in sequences (7) and (17) rather than to the $(n-1)$th and nth partitions in the same sequence of partitions. It is clear that Σ^I is the Riemann–Stiltjes integral

$$\Sigma_t^I = \int_0^t \frac{\mathrm{dA}_s^I}{I^2(s)}$$

and that the statement of uniform convergence carries over to $\Sigma^{I,n}$. □

Next we state the analogue of Lemma 6.1 for M^I.

Lemma 6.2. *The limit in* (9) *exists and is uniform over any compact time interval quasi-always for both partitions* (7) *and* (17)*; the limits coincide quasi-always.*

Proof. The existence of the limit quasi-always is shown in [17], Section 4 (and the earlier work [9] by Perkowski and Prömel); the limit is nothing else than the Itô integral

$$\mathrm{M}_t^I = \int_0^t \frac{\mathrm{d}I(s)}{I(s)}.$$

The coincidence of the functions M^I quasi-always for the sequences of partitions (7) and (17) follows from the argument given in Section 4 of [17] applied to the nth partitions in (7) and (17) rather than to the $(n-1)$th and nth partitions in the same sequence. □

Using S in place of I, we obtain the definitions of Σ^S and M^S. The analogues of Lemmas 6.1 and 6.2 still hold.

As we are also interested in the covariance between (the returns of) S and I, we define

$$\Sigma_t^{S,I,n}(\omega) := \sum_{k=1}^{\infty} \frac{S(T_k^n \wedge t) - S(T_{k-1}^n \wedge t)}{S(T_{k-1}^n \wedge t)} \frac{I(T_k^n \wedge t) - I(T_{k-1}^n \wedge t)}{I(T_{k-1}^n \wedge t)}, \qquad n = 1, 2, \dots, \tag{18}$$

and then set $\Sigma^{S,I}$ to the limit as $n \to \infty$. The existence of the limit quasi-always is asserted in our next lemma.

Lemma 6.3. *The limit of* (18) *as* $n \to \infty$ *exists quasi-always uniformly over any compact time interval.*

Proof. First we notice that, if the denominators in (18) are ignored, the limit $\mathrm{A}_t^{S,I}(\omega)$ of

$$\mathrm{A}_t^{S,I,n}(\omega) := \sum_{k=1}^{\infty} \left(S(T_k^n \wedge t) - S(T_{k-1}^n \wedge t)\right) \left(I(T_k^n \wedge t) - I(T_{k-1}^n \wedge t)\right) = \frac{1}{2} \left(\mathrm{A}_t^{S+I,n} - \mathrm{A}_t^{S,n} - \mathrm{A}_t^{I,n}\right)$$

will exist quasi-always uniformly over any compact time interval; moreover, it will have bounded variation over any compact time interval q.a. It remains to notice that (18) are approximating sums for the Riemann–Stiltjes integral

$$\Sigma_t^{S,I} = \int_0^t \frac{\mathrm{d}\mathrm{A}_s^{S,I}}{S(s)I(s)}. \qquad \square$$

The last quantity that we will need quantifies the difference between the returns of S and I:

$$\Delta_t^{S,I,n}(\omega) := \sum_{k=1}^{\infty} \left(\frac{S(T_k^n \wedge t) - S(T_{k-1}^n \wedge t)}{S(T_{k-1}^n \wedge t)} - \frac{I(T_k^n \wedge t) - I(T_{k-1}^n \wedge t)}{I(T_{k-1}^n \wedge t)} \right)^2, \quad n = 1, 2, \ldots; \tag{19}$$

we then set $\Delta_t^{S,I}$ to the limit as $n \to \infty$. The final lemma of this section shows that the limit as $n \to \infty$ exists quasi-always and is closely related to the other quantities introduced in this section.

Lemma 6.4. *The limit $\Delta_t^{S,I}$ of (19) as $n \to \infty$ exists uniformly over any compact time interval quasi-always and satisfies*

$$\Delta_t^{S,I} = \Sigma_t^S + \Sigma_t^I - 2\Sigma_t^{S,I} \quad q.a.$$

Proof. It suffices to notice that, using the notation m_k and s_k for the returns of S and I (m_k is defined by (10) and s_k is defined in the same way using S in place of I),

$$\Delta_t^{S,I,n} = \sum_{k=1}^{\infty} (s_k - m_k)^2 = \sum_{k=1}^{\infty} s_k^2 + \sum_{k=1}^{\infty} m_k^2 - 2 \sum_{k=1}^{\infty} s_k m_k,$$

and pass to the limit as $n \to \infty$. $\square$

7 Capital Asset Pricing Model

Let us use the same mixing method as in Section 5, except that now we will apply it to I and S rather than to I and cash.

Lemma 7.1. *For each $\epsilon \in \mathbb{R}$, the process*

$$\exp\left(\epsilon(\mathrm{M}_t^S - \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I}) - \frac{\epsilon^2}{2} \Delta_t^{S,I} \right) \tag{20}$$

is a test I-supermartingale q.a.

Proof. The value of the index I at time T_K^n is $\prod_{k=1}^K (1 + m_k)$, and the value of the security S at time T_K^n is $\prod_{k=1}^K (1 + s_k)$, where as before we use s_k for the analogue of m_k for S. Let us consider the simple capital process whose value at time T^n_K is $\prod_{k=1}^K (1+(1-\epsilon)m_k+\epsilon s_k)$ (except that it is stopped if and when it hits 0); it can be considered to be a mixture (convex mixture when $\epsilon \in [0,1]$) of I and S. We can see that

$$\ln \prod_{k=1}^K (1+(1-\epsilon)m_k+\epsilon s_k) - \ln \prod_{k=1}^K (1+m_k)$$

is the log of a test I-supermartingale at time T^n_K for all K, which implies the analogous statement for

$$\epsilon \sum_{k=1}^K s_k - \epsilon \sum_{k=1}^K m_k + \epsilon \sum_{k=1}^K m_k^2 - \epsilon \sum_{k=1}^K s_k m_k \\ - \frac{\epsilon^2}{2} \sum_{k=1}^K m_k^2 + \epsilon^2 \sum_{k=1}^K s_k m_k - \frac{\epsilon^2}{2} \sum_{k=1}^K s_k^2 \\ + O\left(\sum_{k=1}^K |s_k|^3\right) + O\left(\sum_{k=1}^K |m_k|^3\right).$$

Passing to the limit as $n \to \infty$, we obtain that

$$\epsilon \mathrm{M}^S_t - \epsilon \mathrm{M}^I_t + \epsilon \Sigma^I_t - \epsilon \Sigma^{S,I}_t - \frac{\epsilon^2}{2} \Delta^{S,I}_t$$

is the log of a test I-supermartingale q.a. $\square$

Corollary 7.2. *For any $\epsilon > 0$ and $\delta > 0$,*

$$\underline{\mathbb{P}}^I \left\{ \forall t \in [0,\infty) : \left| \mathrm{M}^S_t - \mathrm{M}^I_t + \Sigma^I_t - \Sigma^{S,I}_t \right| < \frac{1}{\epsilon} \ln \frac{2}{\delta} + \frac{\epsilon}{2} \Delta^{S,I}_t \right\} \ge 1 - \delta.$$

Proof. For $\epsilon > 0$, the fact that (20) is a test I-supermartingale implies that

$$\forall t \in [0,\infty) : \epsilon \left(\mathrm{M}^S_t - \mathrm{M}^I_t + \Sigma^I_t - \Sigma^{S,I}_t \right) < \ln \frac{2}{\delta} + \frac{\epsilon^2}{2} \Delta^{S,I}_t \qquad (21)$$

with lower I-probability at least $1 - \delta/2$. It remains to divide both sides of the inequality in (21) by ϵ and consider the same test I-supermartingale but with $-\epsilon$ in place of ϵ (which should be stopped as soon as the capital hits 0). $\square$

The following corollary is in the spirit of Corollaries 4.4 and 5.4; however, now we wait until I and S become sufficiently different.

Corollary 7.3. *If $\delta > 0$, $\epsilon > 0$, and $\tau_T := \inf\{t \mid \Delta^{S,I}_t \ge T\}$ for some constant $T > 0$,*

$$\underline{\mathbb{P}}^I \left\{ \left| \mathrm{M}^S_{\tau_T} - \mathrm{M}^I_{\tau_T} + \Sigma^I_{\tau_T} - \Sigma^{S,I}_{\tau_T} \right| < \frac{1}{\epsilon} \ln \frac{2}{\delta} + \frac{\epsilon}{2} T \right\} \ge 1 - \delta. \qquad (22)$$

Our convention is that the event in the curly braces in (22) happens when $\tau_T = \infty$ (i.e., when the security essentially coincides with the index).

It is natural to optimize the ϵ in (22) given δ and T, as we did in (15), which gives us the following corollary.

Corollary 7.4. *If $\delta > 0$ and $\tau_T := \inf\{t \mid \Delta_t^{S,I} \geq T\}$ for some constant $T > 0$,*

$$\underline{\mathbb{P}}^I \left\{ \left| \mathrm{M}_{\tau_T}^S - \mathrm{M}_{\tau_T}^I + \Sigma_{\tau_T}^I - \Sigma_{\tau_T}^{S,I} \right| < \sqrt{2T \ln \frac{2}{\delta}} \right\} \geq 1 - \delta.$$

We will now give a complete proof of the validity part of a LIL for the CAPM; the following proposition generalizes Corollary 5.6 (we obtain the latter by taking cash as S).

Proposition 7.5. *Almost surely w.r.t. $\overline{\mathbb{P}}^I$,*

$$\Delta_t^{S,I} \to \infty \Longrightarrow \limsup_{t\to\infty} \frac{\left| \mathrm{M}_t^S - \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I} \right|}{\sqrt{2\Delta_t^{S,I} \ln\ln \Delta_t^{S,I}}} \leq 1. \tag{23}$$

Proof. We will implement in detail the idea mentioned in the proof of Corollary 5.6, namely, we will mix the processes

$$\exp\left(\epsilon \mathrm{M}_t^S - \epsilon \mathrm{M}_t^I + \epsilon \Sigma_t^I - \epsilon \Sigma_t^{S,I} - \frac{\epsilon^2}{2} \Delta_t^{S,I}\right) \tag{24}$$

and

$$\exp\left(-\epsilon \mathrm{M}_t^S + \epsilon \mathrm{M}_t^I - \epsilon \Sigma_t^I + \epsilon \Sigma_t^{S,I} - \frac{\epsilon^2}{2} \Delta_t^{S,I}\right),$$

which quasi-always are test I-supermartingales (see Lemma 7.1), over $\epsilon > 0$ of the form $(1+\kappa)^{-k}$, $k = 1, 2, \ldots$, with weights $w_k = k^{-1-\delta}$. We will only prove (23) with the operation of taking the absolute value omitted (the proof for the case where it is replaced by negation is analogous), and so we will only consider (24).

Since $\sum_k w_k$ converges,

$$\sum_k w_k \exp\left(\epsilon_k \mathrm{M}_t^S - \epsilon_k \mathrm{M}_t^I + \epsilon_k \Sigma_t^I - \epsilon_k \Sigma_t^{S,I} - \frac{\epsilon_k^2}{2} \Delta_t^{S,I}\right)$$

is also a test I-supermartingale with finite initial capital; therefore, it is bounded $\overline{\mathbb{P}}^I$-a.s. (cf. Lemma 2.9), and so we have

$$\mathrm{M}_t^S - \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I} \leq \frac{-\ln w_k + O(1)}{\epsilon_k} + \frac{\epsilon_k}{2} \Delta_t^{S,I} \quad \overline{\mathbb{P}}^I\text{-a.s.},$$

i.e.,

$$\mathrm{M}_t^S - \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I} \leq \frac{(1+\delta)\ln k + O(1)}{\epsilon_k} + \frac{\epsilon_k}{2} \Delta_t^{S,I} \quad \overline{\mathbb{P}}^I\text{-a.s.}$$

The value of $\epsilon = \epsilon_k$ that minimizes the right-hand side (let's forget for a minute that ϵ is a function of k and ignore the $O(1)$) is

$$\epsilon = \sqrt{2\frac{(1+\delta)\ln k}{\Delta_t^{S,I}}}.$$

Let us choose k such that this is approximately true, namely,

$$\epsilon_{k+2} = (1+\kappa)^{-k-2} \le \sqrt{2\frac{(1+\delta)\ln k}{\Delta_t^{S,I}}} \le (1+\kappa)^{-k} = \epsilon_k; \tag{25}$$

it is easy to check that such k exist when $\Delta_t^{S,I}$ is sufficiently large (for $k = 1$, the right-hand inequality in (25) always holds provided $\Delta_t^{S,I} > 0$; if the left-hand side inequality does not hold, we can then increment k by 1 until both inequalities in (25) hold, which will eventually happen for $\Delta_t^{S,I}$ sufficiently large). This gives us

$$\mathrm{M}_t^S - \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I} \le 2(1+\kappa)^2 \sqrt{(1+\delta)(\ln k + O(1))\frac{1}{2}\Delta_t^{S,I}} \quad \overline{\mathbb{P}}^I\text{-a.s.} \tag{26}$$

The right-hand inequality in (25) can be rewritten as

$$k\ln(1+\kappa) \le \frac{1}{2}\left(\ln\Delta_t^{S,I} - \ln 2 - \ln(1+\delta) - \ln\ln k\right) \le \frac{1}{2}\ln\Delta_t^{S,I}$$

(for large k), and plugging this into (26) gives

$$\begin{aligned}\mathrm{M}_t^S &- \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I} \\ &\le 2(1+\kappa)^2\sqrt{(1+\delta)\left(\ln\frac{1}{2} + \ln\ln\Delta_t^{S,I} - \ln\ln(1+\kappa) + O(1)\right)\frac{1}{2}\Delta_t^{S,I}} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\mathbb{P}}^I\text{-a.s.}\end{aligned}$$

It remains to mix over sequences of $\kappa \to 0$ and $\delta \to 0$. □

Theoretical performance deficit

By Lemma 3.2, the key component $\mathrm{M}_t^S - \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I}$ of Lemma 7.1, Corollaries 7.2–7.4, and Proposition 7.5 can be rewritten as follows:

$$\begin{aligned}\mathrm{M}_t^S - \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I} &= \ln S_{\tau_T} - \ln I_{\tau_T} + \frac{1}{2}\Sigma_{\tau_T}^S + \frac{1}{2}\Sigma_{\tau_T}^I - \Sigma_{\tau_T}^{S,I} \\ &= \ln S_{\tau_T} - \ln I_{\tau_T} + \frac{1}{2}\Delta_{\tau_T}^{S,I}.\end{aligned}$$

The subtrahend $\frac{1}{2}\Delta_{\tau_T}^{S,I}$ can be interpreted as measuring the lack of diversification in S as compared with I; we call it the *theoretical performance deficit*. Let us now rewrite Lemma 7.1, Corollaries 7.2–7.4, and Proposition 7.5 (in this order) in terms of the theoretical performance deficit.

Corollary 7.6. *For each $\epsilon \in \mathbb{R}$, the process*

$$\exp\left(\epsilon\left(\ln S_t - \ln I_t + \frac{1}{2}\Delta_t^{S,I}\right) - \frac{\epsilon^2}{2}\Delta_t^{S,I}\right)$$

is a test I-supermartingale q.a.

Corollary 7.7. *For any $\epsilon > 0$ and $\delta > 0$,*

$$\underline{\mathbb{P}}^I\left\{\forall t \in [0,\infty) : \left|\ln S_t - \ln I_t + \frac{1}{2}\Delta_t^{S,I}\right| < \frac{1}{\epsilon}\ln\frac{2}{\delta} + \frac{\epsilon}{2}\Delta_t^{S,I}\right\} \geq 1 - \delta.$$

Corollary 7.8. *If $\delta > 0$, $\epsilon > 0$, and $\tau_T := \inf\{t \mid \Delta_t^{S,I} \geq T\}$ for some constant $T > 0$,*

$$\underline{\mathbb{P}}^I\left\{\left|\ln S_{\tau_T} - \ln I_{\tau_T} + \frac{1}{2}\Delta_{\tau_T}^{S,I}\right| < \frac{1}{\epsilon}\ln\frac{2}{\delta} + \frac{\epsilon}{2}T\right\} \geq 1 - \delta.$$

Corollary 7.9. *If $\delta > 0$ and $\tau_T := \inf\{t \mid \Delta_t^{S,I} \geq T\}$ for some constant $T > 0$,*

$$\underline{\mathbb{P}}^I\left\{\left|\ln S_{\tau_T} - \ln I_{\tau_T} + \frac{1}{2}\Delta_{\tau_T}^{S,I}\right| < \sqrt{2T\ln\frac{2}{\delta}}\right\} \geq 1 - \delta.$$

Corollary 7.10. *Almost surely w.r. to $\overline{\mathbb{P}}^I$,*

$$\Delta_t^{S,I} \to \infty \Longrightarrow \limsup_{t\to\infty} \frac{\left|\ln S(t) - \ln I(t) + \frac{1}{2}\Delta_t^{S,I}\right|}{\sqrt{2\Delta_t^{S,I}\ln\ln\Delta_t^{S,I}}} \leq 1.$$

Informally, according to Corollaries 7.6–7.10, for large t the EIH for I implies

$$\ln S(t) \approx \ln I(t) - \frac{1}{2}\Delta_t^{S,I}.$$

Substituting cash for S, we obtain various statements of Sections 4–5 from those corollaries; e.g., Lemma 5.1 is a special case of Corollary 7.6.

8 A probability-free version of Girsanov's theorem

In conclusion of this paper we discuss mathematical (this section) and practical (the next one) implications of our results. In this section we introduce a probability-free notion of a continuous martingale and show how it can be used to simplify the results in the previous sections.

A continuous process X is a *continuous I-martingale* if it takes values in $\mathbb{R}$ and, for each $c \in \mathbb{R}$, both $c + X_{t\wedge\tau_+}$ and $c - X_{t\wedge\tau_-}$ are test I-supermartingales q.a., where $\tau_+ := \inf\{t \mid c + X_t \leq 0\}$ and $\tau_- := \inf\{t \mid c - X_t \leq 0\}$ (using the inequalities $c \pm X_t \leq 0$ rather than the equalities $c \pm X_t = 0$ takes care of the case $c \ll 0$).

Remark 8.1. Our use of the term "martingale" does not mean that our notion is particularly close to the measure-theoretic notion of a continuous martingale: e.g., measure-theoretic continuous local martingales fit our definition just as well. All our terminology in this paper is provisional.

Corollary 8.2. *The process* $\mathrm{M}_t^I - \Sigma_t^I$ *is a continuous* I*-martingale.*

Proof. Let X be the process (13). For any c and any $\epsilon > 0$, the process $c + (X-1)/\epsilon$ is a test I-supermartingale q.a. if stopped upon reaching 0. Passing to the limit as $\epsilon \to 0$ and similarly considering negative values of ϵ, we obtain the statement of the corollary. □

The following result, which is essentially a probability-free version of Girsanov's theorem (see, e.g., [4], (II.3.12)), can be regarded as the key mathematical result of this paper implying most of the other results.

Theorem 8.3. *The process* $\mathrm{M}_t^S - \Sigma_t^{S,I}$ *is a continuous* I*-martingale.*

Proof. Applying the argument in the proof of Corollary 8.2 to Lemma 7.1, we obtain that the process

$$\mathrm{M}_t^S - \mathrm{M}_t^I + \Sigma_t^I - \Sigma_t^{S,I} \tag{27}$$

is a continuous I-martingale. Combining this with the statement of Corollary 8.2 we obtain the statement of the theorem. □

Corollary 8.2 is a special case of Theorem 8.3: it can be obtained by setting $S := I$. To derive Lemma 7.1 (of which Lemma 5.1 is an easy corollary corresponding to $S = 0$), we can use the following probability-free version of the Doléans exponential.

Lemma 8.4. *If* X *is a continuous* I*-martingale with quadratic variation* $[X] := [X, X]$, $\exp(X - [X]/2)$ *is a test* I*-supermartingale q.a.*

A proof of Lemma 8.4 can be found in [21]. In combination with Corollary 8.2, this lemma implies Lemma 5.1, and in combination with (27) being a continuous I-martingale it implies Lemma 7.1.

Finally, we state the analogues of Lemma 7.1, Corollary 7.4, and Proposition 7.5 for the I-martingale of Theorem 8.3; they can be considered as simplified versions of the CAPM.

Lemma 8.5. *For each* $\epsilon \in \mathbb{R}$*, the process*

$$\exp\left(\epsilon(\mathrm{M}_t^S - \Sigma_t^{S,I}) - \frac{\epsilon^2}{2}\Sigma_t^S\right)$$

is a test I*-supermartingale q.a.*

Corollary 8.6. *If* $\delta > 0$ *and* $\tau_T := \inf\{t \mid \Sigma_t^S \geq T\}$ *for some constant* $T > 0$*,*

$$\underline{\mathbb{P}}^I \left\{ \left|\mathrm{M}_{\tau_T}^S - \Sigma_{\tau_T}^{S,I}\right| < \sqrt{2T \ln \frac{2}{\delta}} \right\} \geq 1 - \delta.$$

Corollary 8.7. *Almost surely w.r.t. to* $\overline{\mathbb{P}}^I$,

$$\Sigma_t^S \to \infty \Longrightarrow \limsup_{t\to\infty} \frac{\left|\mathrm{M}_t^S - \Sigma_t^{S,I}\right|}{\sqrt{2\Sigma_t^S \ln\ln \Sigma_t^S}} \le 1.$$

9 Comparisons to the standard CAPM

Assuming zero interested rates ($R_{\mathrm{f}} = 0$), the standard CAPM says, in the standard framework of measure-theoretic probability, that

$$\mathbb{E}(R_i) = \frac{\mathrm{Cov}(R_i, R_{\mathrm{m}})}{\mathrm{Var}(R_{\mathrm{m}})} \mathbb{E}(R_{\mathrm{m}})$$

in the notation of [22], where $\mathbb{E}(R_i)$ is the expected return of the ith security, $\mathbb{E}(R_{\mathrm{m}})$ is the expected return of the market, $\mathrm{Var}(R_{\mathrm{m}})$ is the variance of the return of the market, and $\mathrm{Cov}(R_i, R_{\mathrm{m}})$ is the covariance between the returns of the ith security and the market.

Replacing the theoretical expected values (including those implicit in $\mathrm{Var}(R_{\mathrm{m}})$ and $\mathrm{Cov}(R_i, R_{\mathrm{m}})$) by the empirical averages, we obtain an approximate equality

$$\mathrm{M}_t^S \approx \frac{\Sigma_t^{S,I}}{\Sigma_t^I} \mathrm{M}_t^I. \tag{28}$$

This approximate equality is still true in our probability-free framework (under the assumptions $\Sigma_t^I \gg 1$ and $\Sigma_t^S \gg 1$): indeed, our equity premium result implies $\mathrm{M}_t^I \approx \Sigma_t^I$ (see, e.g., Corollary 5.6), which makes (28) equivalent to $\mathrm{M}_t^S \approx \Sigma_t^{S,I}$, our game-theoretic CAPM (see, e.g., Corollary 8.7).

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