# Non-stochastic portfolio theory

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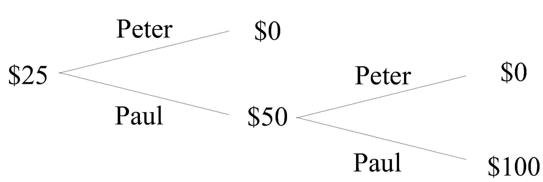


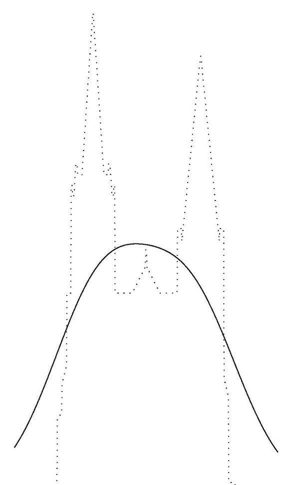

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## Abstract

This paper proposes a non-stochastic version of Fernholz's stochastic portfolio theory for a simple model of stock markets with continuous price paths. It establishes non-stochastic versions of the most basic results of stochastic portfolio theory, including the "master equation" (for a smooth portfolio generating function in this version of the paper) and its corollaries showing the possibility of beating the capital-weighted index in the case of a diverse and non-degenerate market.

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# 1 Introduction

Fernholz's stochastic portfolio theory [2, 3, 4], as its name suggests, depends on a stochastic model of stock prices. This paper proposes a non-stochastic version of this theory based on the framework of [12] (see the end of this section for a brief discussion of its relation to [10]).

A key finding (see, e.g., [2, Section 4], [3, Chapters 2 and 3], [4, Section 7]) of stochastic portfolio theory is that, under certain simplifying assumptions, there is a long-only portfolio that outperforms the capital-weighted index. The principal aim of this paper is to give a simple non-stochastic formalization of this phenomenon.

Section 2 defines our model of a stock market and introduces non-stochastic notions of a portfolio's value and its excess growth component. Section 3 is devoted to a non-stochastic version of the "master equation" of stochastic portfolio theory, and Section 4 to its applications. In particular, the latter covers the entropy-weighted portfolio (as in [2, Theorem 4.1] and [3, Theorem 2.3.4]) and diversity-weighted portfolios ([3, Example 3.4.4], [4, Section 7], going back to at least [1]). Section 5 is devoted to detailed interpretations and discussions of the results of the previous sections. Finally, Section 6 lists some directions of further research.

Another paper treating stochastic portfolio theory in a pathwise manner is [10], and it considers a wider class of portfolios. However, that paper relies on two assumptions that are not justified by economic considerations:

- it postulates a suitable "refining sequence of partitions";
- it postulates the existence of a continuous covariation between each pair of price paths w.r. to this refining sequence of partitions (in Föllmer's [6] sense).

# 2 Market and portfolios

This paper uses the definitions and notation of [12] and [3] (the latter, however, will always be repeated).

We consider a financial market in which $J$ idealized securities, referred to as stocks, are traded; their price paths $S_j : [0,\infty) \to (0,\infty)$, $j = 1,\dots,J$, are assumed to be continuous functions, and they never pay dividends. We let $\Omega$ stand for the set of all continuous real-valued functions on $[0,\infty)$. As in [12, Section 4], we fix a sufficiently rich language for defining sequences of partitions; all notions of non-stochastic Itô calculus used in this paper (such as Itô integral and Doléans exponent and logarithm) are relative to this language.

For convenience, we identify $S_j(t)$ with the total market capitalization of the $j$th stock at time $t \in [0,\infty)$. The *index* $I$ is defined as the process

$$I(t) := \sum_{j=1}^{J} S_j(t), \quad t \in [0,\infty),$$

and the *market share* of the $j$th stock is

$$\mu_j(t) := S_j(t)/I(t), \quad j = 1, \dots, J.$$

We take the index as our numéraire, which allows us to regard $\mu_1, \dots, \mu_J$ as the traded securities (cf. [12, Section 9]), just like our original securities $S_j$ but constrained by $\mu_1 + \cdots + \mu_J = 1$. (In fact, the original securities $S_j$ will never be used explicitly in the rest of this paper apart from an informal remark.)

Let $\Delta^J$ be the interior of the standard simplex in $\mathbb{R}^J$,

$$\Delta^J := \left\{ x = (x_1, \dots, x_J) \in (0,1)^J \mid x_1 + \cdots + x_J = 1 \right\}.$$

A *basic portfolio* is a continuous bounded function $\pi : \Delta^J \to \overline{\Delta^J}$ mapping $\Delta^J$ to its closure in $\mathbb{R}^J$; intuitively, it maps the current market shares $\mu = (\mu_1, \dots, \mu_J)$ to the fractions $\pi(\mu) = (\pi_1(\mu), \dots, \pi_J(\mu))$ of the current capital invested in the $J$ stocks. (In this paper we will only need these very primitive Markovian portfolios.)

The non-stochastic notions of Doléans exponent $\mathcal{E}$ and Doléans logarithm $\mathcal{L}$ used in this paper are defined in [12]. The most useful for us interpretation of Doléans logarithm is that $\mathcal{L}\, S$ is the cumulative return of a price path $S \in \Omega$, and Doléans exponent restores the price path from its cumulative return. The *value path* of $\pi$ is the Doléans exponent

$$Z_\pi := \mathcal{E}\Bigl(\pi(\mu) \cdot \mathcal{L}(\mu)\Bigr) := \mathcal{E}\left( \sum_{j=1}^J \pi_j(\mu) \cdot \mathcal{L}(\mu_j) \right) = \mathcal{E}\left( \sum_{j=1}^J \frac{\pi_j(\mu)}{\mu_j} \cdot \mu_j \right), \quad (1)$$

"$\cdot$" standing for Itô integration, $\mu : [0,\infty) \to \mathbb{R}^J$ defined by $\mu(t) := (\mu_1(t), \dots, \mu_J(t))$, $\pi_j(\mu) : [0,\infty) \to \mathbb{R}$ defined by $\pi_j(\mu)(t) := \pi_j(\mu(t))$, and $\pi(\mu) : [0,\infty) \to \mathbb{R}^J$ defined by $\pi(\mu)(t) := (\pi_1(\mu)(t), \dots, \pi_J(\mu)(t))$. The value path $Z_\pi \in \Omega$ exists quasi always.

The definition (1) involves Doléans logarithm, but stochastic portfolio theory emphasizes regular logarithm (cf. the logarithmic model in [3, Section 1.1]). On the log scale the definition (1) can be rewritten as

$$\ln Z_\pi = \ln \mathcal{E}\left( \sum_{j=1}^J \pi_j(\mu) \cdot \mathcal{L}(\mu_j) \right) \quad (2)$$

$$= \sum_{j=1}^J \pi_j(\mu) \cdot \mathcal{L}(\mu_j) - \frac{1}{2}\left[ \sum_{j=1}^J \pi_j(\mu) \cdot \mathcal{L}(\mu_j) \right] \quad (3)$$

$$= \sum_{j=1}^J \pi_j(\mu) \cdot \ln \mu_j + \frac{1}{2} \sum_{j=1}^J \pi_j(\mu) \cdot [\ln \mu_j] - \frac{1}{2}\left[ \sum_{j=1}^J \pi_j(\mu) \cdot \ln \mu_j \right] \quad \text{q.a.,} \quad (4)$$

where $\cdot$ now refers to Itô or Lebesgue–Stiltjes integration, as appropriate, and $[\cdots]$ stands for quadratic variation. The second equality in the chain (2)–(4) follows from the standard equality

$$\mathcal{E}(X) = \exp(X - [X]/2) \quad \text{q.a.} \tag{5}$$

and the third equality in (2)–(4) follows from

$$\mathcal{L}(Y) = \ln Y_t + \frac{1}{2}[\ln Y] \quad \text{q.a.} \tag{6}$$

and a slight generalization of

$$[\mathcal{L}(Y)] = [\ln Y] \quad \text{q.a.} \tag{7}$$

(see [12, Section 7] for (5)–(7)).

The part

$$\Gamma_\pi^* = \frac{1}{2}\sum_{j=1}^{J} \pi_j(\mu) \cdot [\ln \mu_j] - \frac{1}{2}\left[\sum_{j=1}^{J} \pi_j(\mu) \cdot \ln \mu_j\right] \tag{8}$$

of (2)–(4) consisting of the last two addends will be called the *excess growth term* (it corresponds to the cumulative excess growth rate in stochastic portfolio theory). We can use it to summarize (2)–(4) as

$$\ln Z_\pi = \sum_{j=1}^{J} \pi_j(\mu) \cdot \ln \mu_j + \Gamma_\pi^* \quad \text{q.a.} \tag{9}$$

The addend $\sum_{j=1}^{J} \pi_j(\mu) \cdot \ln \mu_j$ is the naive expression for the cumulative log growth in the value of $\pi$, and $\Gamma_\pi^*$ is the adjustment required to obtain the true cumulative log growth.

A particularly important special case is that of the market portfolio, $\pi = \mu$. To understand the intuition behind the excess growth term (8) in this case, we can rewrite $2\Gamma_\mu^*$ as

$$2\Gamma_\mu^*(t) = \int_0^t \sum_{j=1}^{J} \mu_j(s)\,\mathrm{d}[\ln \mu_j](s) - \left[\sum_{j=1}^{J} \mu_j \cdot \ln \mu_j\right](t) \tag{10}$$

$$= \int_0^t \sum_{j=1}^{J} \mu_j(s)\,\mathrm{d}[\ln \mu_j](s) = \int_0^t \sum_{j=1}^{J} \frac{\mathrm{d}[\mu_j](s)}{\mu_j(s)} \tag{11}$$

$$\geq \int_0^t \sum_{j=1}^{J} \mathrm{d}[\mu_j](s) = \sum_{j=1}^{J} [\mu_j](t).$$

where we have used the fact that the subtrahend in (10), being the quadratic variation of a monotonic function (remember that $\sum_j \mu_j = 1$), is zero. We can see that $2\Gamma_\mu^*(t)$ is bounded below by the total quadratic variation of the market shares.

# 3 Master equation

Let $\mathbf{S}$ be a $C^2$ positive function defined on an open neighbourhood $\operatorname{dom}\mathbf{S}$ of $\Delta^J$ in $\mathbb{R}^J$. For any $C^2$ function $F$ (such as $\ln \mathbf{S}$) defined on $\operatorname{dom}\mathbf{S}$ we let $D_j$ stand for its $j$th partial derivative,

$$D_j F(x) = \frac{\partial F}{\partial x_j}(x), \quad x = (x_1, \dots, x_J) \in \operatorname{dom}\mathbf{S},$$

and $D_{ij}$ stand for its second partial derivative in $x_i$ and $x_j$,

$$D_{ij} F(x) = \frac{\partial^2 F}{\partial x_i \partial x_j}(x).$$

The *portfolio generated by* $\mathbf{S}$ is defined by

$$\pi_j(x) := \left( D_j \ln \mathbf{S}(x) + 1 - \sum_{k=1}^{J} x_k D_k \ln \mathbf{S}(x) \right) x_j. \tag{12}$$

The main part of the expression in the parentheses is $D_j \ln \mathbf{S}(x)$; the rest is simply the normalizing constant $c = c(x)$ making $(D_j \ln \mathbf{S}(x) + c)x_j$ a portfolio (it is a constant in the sense of not depending on $j$).

Now we can state a non-stochastic version of the "master equation" of stochastic portfolio theory (see, e.g., [3, Theorem 3.1.5]).

**Theorem 3.1.** *The value process $Z_\pi$ of the portfolio $\pi$ generated by $\mathbf{S}$ satisfies*

$$\ln Z_\pi(t) = \ln \frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} + \Theta(t) \quad \textit{q.a.}, \tag{13}$$

*where*

$$\Theta(t) := \int_0^t \frac{-1}{2\mathbf{S}(\mu(s))} \sum_{i,j=1}^{J} D_{ij} \mathbf{S}(\mu(s)) \, \mathrm{d}[\mu_i, \mu_j](s). \tag{14}$$

*Proof.* The middle equality (3) in the chain (2)–(4) gives for the left-hand side of (13):

$$\begin{aligned}
\ln Z_\pi(t) &= \sum_j \int_0^t \frac{\pi_j(\mu(s))}{\mu_j(s)} \, \mathrm{d}\mu_j(s) - \frac{1}{2} \sum_{i,j} \int_0^t \frac{\pi_i(\mu(s))\pi_j(\mu(s))}{\mu_i(s)\mu_j(s)} \, \mathrm{d}[\mu_i, \mu_j](s) \\
&= \sum_j \int_0^t \left( D_j \ln \mathbf{S}(\mu(s)) + 1 - \sum_k \mu_k(s) D_k \ln \mathbf{S}(\mu(s)) \right) \mathrm{d}\mu_j(s) \\
&\quad - \frac{1}{2} \sum_{i,j} \int_0^t \left( D_i \ln \mathbf{S}(\mu(s)) + 1 - \sum_k \mu_k(s) D_k \ln \mathbf{S}(\mu(s)) \right) \\
&\quad \times \left( D_j \ln \mathbf{S}(\mu(s)) + 1 - \sum_k \mu_k(s) D_k \ln \mathbf{S}(\mu(s)) \right) \mathrm{d}[\mu_i, \mu_j](s)
\end{aligned}$$

$$= \sum_j \int_0^t \frac{D_j \mathbf{S}(\mu(s))}{\mathbf{S}(\mu(s))} \, \mathrm{d}\mu_j(s) \tag{15}$$

$$- \frac{1}{2} \sum_{i,j} \int_0^t \frac{D_i \mathbf{S}(\mu(s))}{\mathbf{S}(\mu(s))} \frac{D_j \mathbf{S}(\mu(s))}{\mathbf{S}(\mu(s))} \, \mathrm{d}[\mu_i, \mu_j](s) \quad \text{q.a.}, \tag{16}$$

where the last equality follows from $\sum_k \mu_k = 1$. Next we apply the Itô formula to the function $\ln \mathbf{S}$ on the right-hand side of (13); the Itô formula still holds in our non-stochastic setting: cf. [12, Section 6]. For the first addend on the right-hand side of (13) it gives us the expression

$$\ln \frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} = \sum_j \int_0^t \frac{D_j \mathbf{S}(\mu(s))}{\mathbf{S}(\mu(s))} \, \mathrm{d}\mu_j(s) + \frac{1}{2} \sum_{i,j} \int_0^t \frac{D_{ij} \mathbf{S}(\mu(s))}{\mathbf{S}(\mu(s))} \, \mathrm{d}[\mu_i, \mu_j](s)$$
$$- \frac{1}{2} \sum_{i,j} \int_0^t \frac{D_i \mathbf{S}(\mu(s)) D_j \mathbf{S}(\mu(s))}{\mathbf{S}(\mu(s))^2} \, \mathrm{d}[\mu_i, \mu_j](s)$$

equal, q.a., to (15)–(16) minus $\Theta$, as defined in (14). □

# 4 Special cases

A positive $C^2$ function $\mathbf{S}$ defined on an open neighbourhood of $\Delta^J$ is a *measure of diversity* if it is symmetric and concave. In this section we will discuss three examples of measures of diversity.

## 4.1 Fernholz's arbitrage opportunity

In [3, Section 3.3], Fernholz describes an arbitrage opportunity for his stochastic model of the market. In the non-stochastic setting of this paper his portfolio ceases to be an arbitrage opportunity but it is still interesting and suggests the possibility of beating the market (as discussed in the next section). Now we are interested in the measure of diversity

$$\mathbf{S}(x) := 1 - \frac{1}{2} \sum_{j=1}^J x_j^2.$$

The components (12) of the corresponding portfolio $\pi$ are

$$\pi_j(x) = \left( \frac{2 - x_j}{\mathbf{S}(x)} - 1 \right) x_j. \tag{17}$$

Now Theorem 3.1 gives the following non-stochastic version of [3, Example 3.3.3].

**Corollary 4.1.** *The value process $Z_\pi$ of the portfolio* (17) *satisfies*

$$\ln Z_\pi(t) = \ln \frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} + \sum_{j=1}^J \int_0^t \frac{\mathrm{d}[\mu_j](s)}{2\mathbf{S}(\mu(s))} \quad \textit{q.a.} \tag{18}$$

*Proof.* Plugging $D_{ij}\mathbf{S}(x) = -\mathbf{1}_{i=j}$ (where $\mathbf{1}_E$ stands for the indicator function of $E$) into (14), we indeed obtain

$$\Theta(t) = \int_0^t \frac{1}{2\mathbf{S}(\mu(s))} \sum_j \mathrm{d}[\mu_j](s). \qquad \square$$

A slightly cruder but simpler version of Corollary 4.1 is:

**Corollary 4.2.** *The value process $Z_\pi$ of the portfolio* (17) *satisfies*

$$\ln Z_\pi(t) \geq -\ln 2 + \frac{1}{2}\sum_{j=1}^{J} [\mu_j](t) \quad q.a. \tag{19}$$

*Proof.* It suffices to notice that $\mathbf{S} \in [1/2, 1]$. $\square$

## 4.2 Entropy-weighted portfolio

The archetypal measure of diversity [3, Examples 3.1.2 and 3.4.3] is the entropy function

$$\mathbf{S}(x) := -\sum_{j=1}^{J} x_j \ln x_j.$$

Using (12), the components of the corresponding *entropy-weighted portfolio* can be computed as

$$\pi_j(x) = -\frac{x_j \ln x_j}{\mathbf{S}(x)}. \tag{20}$$

Calculating the drift term $\Theta$ in Theorem 3.1, we obtain the following corollary (a non-stochastic version of [3, Theorem 2.3.4]).

**Corollary 4.3.** *The value process $Z_\pi$ of the entropy-weighted portfolio $\pi$ satisfies*

$$\ln Z_\pi(t) = \ln \frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} + \int_0^t \frac{\mathrm{d}\Gamma^*_\mu(s)}{\mathbf{S}(\mu(s))} \quad q.a. \tag{21}$$

*Proof.* Plugging $D_{ij}\mathbf{S}(x) = -\mathbf{1}_{i=j}/x_j$ into (14), we obtain

$$\Theta(t) = \int_0^t \frac{1}{2\mathbf{S}(\mu(s))} \sum_j \frac{\mathrm{d}[\mu_j](s)}{\mu_j(s)}.$$

It remains to compare this expression with (11). $\square$

## 4.3 Diversity-weighted portfolios with parameter $p$

Fix $p \in (0, 1)$. Define the *measure of diversity with parameter $p \in (0, 1)$* [3, Example 3.4.4] as

$$\mathbf{D}_p(x) := \left(\sum_{j=1}^{J} x_j^p\right)^{1/p}.$$

The *p-diversity-weighted portfolio* has components

$$\pi_j(t) := \frac{\mu_j(t)^p}{\sum_{i=1}^J \mu_i(t)^p}. \tag{22}$$

The following corollary is a non-stochastic version of [3, Example 3.4.4].

**Corollary 4.4.** *The value process $Z_\pi$ of the diversity-weighted portfolio $\pi$ with parameter $p \in (0,1)$ satisfies*

$$\ln Z_\pi(t) = \ln \frac{\mathbf{D}_p(\mu(t))}{\mathbf{D}_p(\mu(0))} + (1-p)\Gamma^*_\pi(t) \quad q.a. \tag{23}$$

*Proof.* Now (14) gives

$$\begin{aligned}
\Theta(t) = &\int_0^t \frac{-1}{2\mathbf{D}_p(\mu(s))} \\
&\times \left( \sum_{i,j} (1-p) \left(\mu_i(s)\mu_j(s)\right)^{p-1} \left( \sum_k \mu_k(s)^p \right)^{1/p-2} \mathrm{d}[\mu_i, \mu_j](s) \right. \\
&\left. + \sum_j (p-1) \left(\mu_j(s)\right)^{p-2} \left( \sum_k \mu_k(s)^p \right)^{1/p-1} \mathrm{d}[\mu_j](s) \right) \\
&= \frac{1-p}{2} \int_0^t \left( \sum_j \pi_j(\mu(s)) \frac{\mathrm{d}[\mu_j](s)}{\mu_j(s)^2} - \sum_{i,j} \pi_j(\mu(s)) \frac{\mathrm{d}[\mu_i, \mu_j](s)}{\mu_i(s)\mu_j(s)} \right) \\
&= (1-p)\Gamma^*_\pi(t).
\end{aligned}$$

□

Corollary 4.4 immediately implies:

**Corollary 4.5.** *The value process $Z_\pi$ of the diversity-weighted portfolio $\pi$ with parameter $p \in (0,1)$ satisfies*

$$\ln Z_\pi(t) \geq (1-p)\Gamma^*_\pi(t) - \frac{1-p}{p} \ln J \quad q.a. \tag{24}$$

*Proof.* Since $\mathbf{D}_p \in [1, J^{(1-p)/p}]$, we have

$$\ln \frac{\mathbf{D}_p(\mu(t))}{\mathbf{D}_p(\mu(0))} \geq -\frac{1-p}{p} \ln J$$

(cf. [4, (7.6)]); plugging this into (23) gives (24). □

# 5 Beating the market

The results of the previous section have striking implications for our idealized financial market. The easiest to discuss is Corollary 4.2. It can be interpreted, very informally, as the following Fisherian disjunction: either the variation of each stock in the market decays, in that the total quadratic variation $[\mu_j](\infty)$ of each of the $J$ market shares' returns over $[0,\infty)$ is finite, or we can beat the capital-weighted index in the sense that $\lim_{t\to\infty} Z_\pi(t) = \infty$ (cf. [5], p. 42). Notice that the second alternative of the disjunction also takes care of the "q.a." in (19).

The portfolio (17) is particularly tame (or admissible, in Fernholz's [3, Section 3.3] terminology): it is long-only, it never loses more than 50% of its value relative to the market portfolio (by (19)), and it never invests more than 3 times more than the market portfolio in any of the stocks.

A more specific possible interpretation of Corollary 4.2 is based on the efficient market hypothesis in the form that was so forcefully advocated in the bestseller [8] by Burton G. Malkiel; for him, "the strongest evidence suggesting that markets are generally quite efficient is that professional investors do not beat the market." Even if there are ways to beat the market, it is often believed that they should involve something unusual rather than merely simple portfolios such as (20), (22), or (17) (widely known since at least 2002). According to this interpretation, Corollary 4.2 implies that in efficient markets we expect market variation to die down eventually.

If we believe that the variation in our stock market will never die down, we are forced to admit that Corollary 4.2 "opens the door to superior long-term investment returns through disciplined active investment management" [7, Section 1.3]. This is the interpretation on which typical practical applications of stochastic portfolio theory are based (see, e.g., [1], which, however, is based on the stochastic versions of Corollaries 4.4 and 4.5 rather than Corollary 4.2).

Corollary 4.2 is a cruder version of Corollary 4.1 that replaces the first addend on the right-hand side of (18) by its lower bound and the denominator in the second addend by its upper bound. Corollary 4.1 is more precise in that it decomposes the growth in the portfolio's value into two components: one related to the growth in the diversity $\mathbf{S}(\mu)$ of the market shares and the other related to the accumulation of the variation of the market shares.

It is standard in stochastic portfolio theory to assume both that the market does not become concentrated, or almost concentrated, in a single stock and that there is a minimal level of stock volatility; precise versions of these assumptions are referred to as diversity and non-degeneracy, respectively. We will see that the results of the previous section can be interpreted as saying that we can beat the market unless it loses its diversity or degenerates. Corollary 4.2 says that, in fact, the condition of non-degeneracy alone is sufficient; this follows from the representation $\mathrm{d}[\ln \mu_j] = \mathrm{d}[\mu_j]/\mu_j^2$. (But remember that our exposition is in terms of market shares $\mu_j$ rather than prices $S_j$, which are usually used in stochastic portfolio theory.)

Corollary 4.3 relies on both assumptions, diversity and non-degeneracy. If the market maintains its diversity, we expect the first addend on the right-hand side of (21) to stay bounded below, and if, in addition, the market does not degenerate, we expect the second addend to increase steadily. As a result, the entropy-weighted portfolio outperforms the market.

To discuss Corollaries 4.4 and 4.5, it is convenient to extend our discussion of $\Gamma^*_\mu$ given in Section 2 to more general $\Gamma^*_\pi$. Let us now rewrite twice the excess growth term (8), $2\Gamma^*_\pi$, as

$$2\Gamma^*_\pi(t) = \int_0^t \sum_{j=1}^J \pi_j(\mu(s))\,\mathrm{d}[\ln \mu_j](s) - \left[\sum_{j=1}^J \pi_j(\mu)\cdot \ln \mu_j\right](t).$$

Define (using our fixed language) a sequence of partitions $T^1, T^2, \dots$ that is fine for all processes used in this paper and set, for a given partition $T^n = (T^n_k)_{k=0}^\infty$,

$$\begin{aligned} \mu_{j,k} &:= \mu_j(T^n_k \wedge t), && k = 0, 1, \dots, \\ \Delta \ln \mu_{j,k} &:= \ln \mu_{j,k} - \ln \mu_{j,k-1}, && k = 1, 2, \dots, \\ \pi_{j,k} &:= \pi(\mu_{j,k}), && k = 0, 1, \dots, \end{aligned}$$

with the dependence on $n$ suppressed. We can then regard

$$2\Gamma^{*,n}_\pi(t) := \sum_{k=1}^\infty \sum_{j=1}^J \pi_{j,k-1}\Delta\mu^2_{j,k} - \sum_{k=1}^\infty \left(\sum_{j=1}^J \pi_{j,k-1}\Delta \ln \mu_{j,k}\right)^2 \tag{25}$$

as the $n$th approximation to $2\Gamma^*_\pi(t)$; it can be shown that

$$2\Gamma^{*,n}_\pi(t) \to 2\Gamma^*_\pi(t) \quad \text{ucqa.}$$

Rewriting (25) as

$$2\Gamma^{*,n}_\pi(t) = \sum_{k=1}^\infty \sum_{j=1}^J \pi_{j,k-1}\left(\Delta\mu_{j,k} - \sum_{i=1}^J \pi_{i,k-1}\Delta \ln \mu_{i,k}\right)^2,$$

we can see that this expression is the cumulative variance of the logarithmic returns $\Delta \ln \mu_{j,k}$ over the time interval $[T^n_{k-1} \wedge t, T^n_k \wedge t]$ w.r. to the "portfolio probability measure" $Q(\{j\}) := \pi_{j,k-1}$. This makes the expression (9) very intuitive: the excess growth rate of the portfolio $\pi$ over the naive expression is determined by the volatility of the market shares w.r. to $\pi$.

As already mentioned, the stochastic versions of Corollaries 4.4 and 4.5 have been used for active portfolio management [1]. The remarks made above about the relation between Corollaries 4.1 and 4.2 are also applicable to Corollaries 4.4 and 4.5; the latter replaces the first addend on the right-hand side of (23) by its lower bound. Corollary 4.4 decomposes the growth in the value of the diversity-weighted portfolio into two components, one related to the growth in the diversity $\mathbf{D}_p(\mu)$ of the market shares and the other related to the accumulation of the diversity-weighted variance of the market shares. Corollary 4.5 ignores the first component, which does not make it vacuous since $\mathbf{D}_p$ is bounded, always being between 1 (corresponding to a market concentrated in one stock) and $J^{(1-p)/p}$ (corresponding to a market with equal capitalizations of all $J$ stocks).

Several explanations have been suggested for the somewhat counterintuitive disjunction stated at the beginning of this section:

- If we include all stocks traded in a real-world market in our model, perhaps making $J$ very large, the portfolio (12) and its special cases (17), (20), and (22) (particularly the last two) will not be efficient since they will be forced to invest into smaller and so less liquid stocks; it is known that portfolios generated by measures of diversity invest into smaller stocks more heavily than the market portfolio does [3, Proposition 3.4.2].
- If we include only $J$ largest stocks traded in a real-world market, for a moderately large $J$ (such as $J = 500$ for S&P 500), the performance of portfolios such as (17), (20), or (22) w.r. to the "market" (which is now, in fact, a market index) will be affected by the phenomenon of "leakage" [3, Example 4.3.5].
- Fernholz [2, 3] suggests that different dividend rates can serve as mechanism for maintaining market diversity.

# 6 Conclusion

These are some directions of further research:

- A natural direction is to try and strip other results of stochastic portfolio theory of their stochastic assumptions. First of all, it should not be difficult to extend Theorem 3.1 to functions $\mathbf{S}$ that are not smooth (as in [3, Theorem 4.2.1]); the existence of local time in a non-stochastic setting is shown in [9] and, in the case of continuous price paths, can be deduced from the main result of [11].
- Another direction is to extend this paper's results to general numéraires (this paper uses the value of the market portfolio as our numéraire).
- Finally, it would be very interesting to extend some of the results to càdlàg price paths.

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