Non-stochastic portfolio theory

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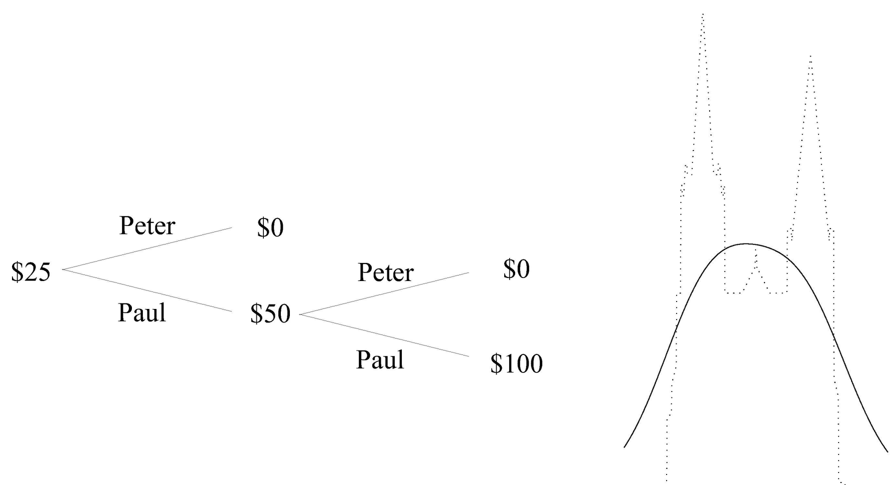

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Abstract

This paper studies a non-stochastic version of Fernholz's stochastic portfolio theory for a simple model of stock markets with continuous price paths. It establishes non-stochastic versions of the most basic results of stochastic portfolio theory and discusses connections with Stroock–Varadhan martingales.

Contents

1 Introduction

Fernholz's stochastic portfolio theory [3, 4, 5], as its name suggests, depends on a stochastic model of stock prices. This paper proposes a non-stochastic version of this theory based on the framework of [16] (see the end of this section for a brief discussion of its relation to [13]).

A key finding (see, e.g., [3, Section 4], [4, Chapters 2 and 3], [5, Section 7]) of stochastic portfolio theory is that, under certain simplifying assumptions, there is a long-only portfolio that outperforms the capital-weighted market portfolio. The principal aim of this paper is to give a simple non-stochastic formalization of this phenomenon.

Section 2 defines our model of a stock market and gives a very simple result that can be interpreted as the possibility of beating the market. The main technical tool of this section is non-stochastic Stroock–Varadhan martingales. We call the picture of the market painted in this section "additive", following the terminology of [8, 6] (but using it more widely and perhaps occasionally abusing it). Section 3 is devoted to Fernholz martingales, which are the non-stochastic counterpart of the "master equation" of stochastic portfolio theory, including such interesting special cases as the entropy-weighted portfolio (as in [3, Theorem 4.1] and [4, Theorem 2.3.4]) and diversity-weighted portfolios ([4, Example 3.4.4], [5, Section 7], going back to at least [2]). Fernholz martingales can be considered a boosted version of Stroock–Varadhan martingales, and the relation between Stroock–Varadhan martingales and Fernholz martingales is somewhat analogous to the relation between the additive Bachelier formula [14, Section 11.2] and the multiplicative Black–Scholes formula [14, Section 11.3] in option pricing. Both Section 2 and Section 3 give ways of "beating the market", and Section 4 complements them with a very different method, with some critical comments given in Section 5. Section 6 gives a finance-theoretic version of Jeffreys's law, introduced in a different context by Philip Dawid: two very successful stocks should have very similar price paths. Section 7 lists some directions of further research. Finally, Appendix A introduces a complementary "multiplicative" picture of financial markets, including non-stochastic notions of a portfolio's value and its excess growth component; they are not important for the main part of the paper but might be more familiar to some readers.

This version of the paper is not self-contained: it uses the definitions and notation of [16] (it also uses the definitions and notation of [4] and [5], but those, however, will always be repeated).

Another paper treating stochastic portfolio theory in a pathwise manner is [13]. However, that paper relies on some assumptions that are not justified by economic considerations:

- it postulates a suitable "refining sequence of partitions";
- it postulates the existence of a continuous covariation between each pair of price paths w.r. to this refining sequence of partitions (in Föllmer's [7] sense);

- a possible extension to non-smooth portfolio generating functions (as in [4, Chapter 4]) would require postulating the existence of local times (perhaps along the lines of [17]).

2 Stroock–Varadhan martingales

The notation $\int X \, \mathrm{d}Y$ is used for the process whose value at time t is $\int_0^t X(s) \, \mathrm{d}Y(s)$, both for Itô and Lebesgue–Stieltjes integration. The brackets $[\dots]$ always signify quadratic variation and are never used in the role of parentheses. The abbreviations "q.a." and "ucqa" stand for "quasi always" and "uniformly on compacts quasi always"; see [16] for definitions.

We consider a financial market in which J idealized securities, referred to as stocks, are traded; their price paths $S_j : [0, \infty) \to (0, \infty)$, $j = 1, \dots, J$, are assumed to be continuous functions, and they never pay dividends. As in [16, Section 4], we fix a sufficiently rich language for defining sequences of partitions; all notions of non-stochastic Itô calculus used in this paper (such as Itô integral and Doléans exponential and logarithm) are relative to this language.

For convenience, we identify $S_j(t)$ with the total market capitalization of the jth stock at time $t \in [0, \infty)$. The *total capitalization of the market* is defined as the process

$$S(t) := \sum_{j=1}^{J} S_j(t), \quad t \in [0, \infty),$$

and the *market weight* of the jth stock is

$$\mu_j(t) := S_j(t)/S(t), \quad j = 1, \dots, J.$$

We take the total capitalization of the market as our numéraire, which allows us to regard $\mu_1, \dots, \mu_J, 1$ as the traded securities (cf. [16, Section 9]), the first J of them being just like our original securities S_j but constrained by $\mu_1 + \dots + \mu_J = 1$. (In fact, the original securities S_j will never be used explicitly in the rest of this paper apart from an informal remark and Section 6.) To avoid any ambiguity, we will sometimes refer to a market weight μ_j as a *unit of* μ_j; the price of this security at time t is $\mu_j(t)$.

Let Δ^J be the interior of the standard simplex in $\mathbb{R}^J$,

$$\Delta^J := \left\{ x = (x_1, \dots, x_J) \in (0,1)^J \mid x_1 + \dots + x_J = 1 \right\}$$

(so that the market weights $(\mu_1, \dots, \mu_J)$ take values in Δ^J), and let f be a C^2 function defined on an open neighbourhood $\operatorname{dom} f$ of Δ^J in $\mathbb{R}^J$. For any such function we let D_j stand for its jth partial derivative,

$$D_j f(x) = \frac{\partial f}{\partial x_j}(x), \quad x = (x_1, \dots, x_J) \in \operatorname{dom} f,$$

and D_{ij} stand for its second partial derivative in x_i and x_j,

$$D_{ij} f(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

The non-stochastic Itô formula [16] implies that

$$f(\mu(t))-f(\mu(0))-\frac{1}{2}\sum_{i,j=1}^{J}\int_0^t D_{ij}f(\mu)\,\mathrm{d}[\mu_i,\mu_j]=\sum_{j=1}^{J}\int_0^t D_j f(\mu)\,\mathrm{d}\mu_j \quad \text{q.a.}, \quad (1)$$

and so the left-hand side of (1) is a continuous martingale, which we will refer to as the *Stroock–Varadhan martingale* generated by f [9, (5.4.2)]; it is a non-stochastic version of the classical martingales used by Stroock and Varadhan in their study of diffusion processes.

For a positive constant A, we define a stopping time τ_A by

$$\tau_A := \min\left\{t \mid \sum_{j=1}^{J}[\mu_j](t)=A\right\}. \quad (2)$$

We say that the market is *active* if $\sum_j[\mu_j](t)\to\infty$ as $t\to\infty$; equivalently, if $\tau_A<\infty$ for all A. In this section we will see a very simple instance of the central phenomenon of stochastic portfolio theory: one can beat an active market.

Setting

$$f(x) := -\frac{1}{2}\sum_{j=1}^{J}x_j^2, \quad (3)$$

we can rewrite the continuous martingale on the left-hand side of (1) as

$$Y(t) := \frac{1}{2}\sum_{j=1}^{J}\mu_j(0)^2-\frac{1}{2}\sum_{j=1}^{J}\mu_j(t)^2+\frac{1}{2}\sum_{j=1}^{J}[\mu_j](t)\ge-\frac{1}{2}+\frac{1}{2}\sum_{j=1}^{J}[\mu_j](t);$$

therefore, $X:=2Y+1$ is a nonnegative continuous martingale satisfying $X(0)=1$ and

$$X(\tau_A)\ge A, \quad (4)$$

with convention that $X(\infty):=\infty$.

Performance guarantees such as (4) are often referred to as arbitrages in stochastic portfolio theory. They cease to be arbitrages in non-stochastic theory, since the market activity is not part of our model.

The trading strategy that achieves (4) can be computed from the right-hand side of (1) using the definition (3): starting from initial capital of 1, we should hold $-\mu_j(t)$ units of $\mu_j(t)$ at time t (therefore, taking a short position in μ_j). The reader should remember that such representations of trading strategies are not unique: they ignore the amount invested in the market as a whole; e.g., increasing the number of units of μ_j by 1 for all j does not affect the capital process (since $\sum_j\mu_j=1$).

The papers [8] and [6] study the additive picture based on the notion of Stroock–Varadhan martingales in depth within the framework of stochastic portfolio theory. In particular, they give numerous interesting examples (including (3)).

3 Boosting Stroock–Varadhan martingales

Stroock–Varadhan martingales allow us to beat active markets but the growth rate of the right-hand side of (4) in A is linear, whereas an interesting growth rate would be exponential. This section discusses a way of boosting the growth rate of Stroock–Varadhan martingales using the non-stochastic notion of Doléans exponential $\mathcal{E}$ defined in [16]. We will need the standard equality [16, Section 7]

$$\mathcal{E}(X) = \exp(X - [X]/2) \quad \text{q.a.} \tag{5}$$

Let $\mathbf{S}$ be a C^2 positive function defined on an open neighbourhood of Δ^J in $\mathbb{R}^J$. Remember that we are using the terminology of [16]. The following is a non-stochastic version of a basic result of stochastic portfolio theory (see, e.g., [4, Theorem 3.1.5]).

Theorem 1. *The continuous process*

$$\frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} \exp\left(-\frac{1}{2}\sum_{i,j=1}^{J}\int_0^t \frac{D_{ij}\mathbf{S}(\mu)}{\mathbf{S}(\mu)}\,\mathrm{d}[\mu_i,\mu_j]\right) \tag{6}$$

is a continuous martingale.

We will refer to (6) as the *Fernholz martingale generated by* $\mathbf{S}$.

Proof. Let us check that the Fernholz martingale (6) is the Doléans exponential of the Stroock–Varadhan martingale on the left-hand side of (1) for $f := \ln \mathbf{S}$. Indeed, applying (5) gives the Doléans exponential

$$\frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} \exp\Bigg(-\frac{1}{2}\sum_{i,j=1}^{J}\int_0^t D_{ij}f(\mu)\,\mathrm{d}[\mu_i,\mu_j] \\ -\frac{1}{2}\sum_{i,j=1}^{J}\int_0^t D_i f(\mu) D_j f(\mu)\,\mathrm{d}[\mu_i,\mu_j]\Bigg)$$

of the left-hand side of (1), which is equal, by the identity

$$D_{ij}f = \frac{D_{ij}\mathbf{S}}{\mathbf{S}} - \frac{D_i\mathbf{S}}{\mathbf{S}}\frac{D_j\mathbf{S}}{\mathbf{S}} = \frac{D_{ij}\mathbf{S}}{\mathbf{S}} - D_i f D_j f,$$

to (6). □

Let us now find explicitly a trading strategy whose capital process is (6). The Doléans exponential $Y = \mathcal{E}(X)$ satisfies the equation

$$Y(t) = Y(0) + \int_0^t Y\,\mathrm{d}X$$

[16, Remark 7.2], and so the number of units of μ_j held by the Fernholz martingale is equal to the current capital times the number

$$D_j \ln \mathbf{S}(\mu) \tag{7}$$

of units of μ_j held by the corresponding Stroock–Varadhan martingale. Taking into account the fraction of the current capital that is not invested (i.e., held in the market as a whole), we obtain that the fraction of the current capital held in μ_j is

$$\pi_j := \left(D_j \ln \mathbf{S}(\mu) + 1 - \sum_{k=1}^{J} \mu_k D_k \ln \mathbf{S}(\mu) \right) \mu_j. \tag{8}$$

The main part of the expression in the parentheses is (7); the rest is simply the normalizing constant $c = c(\mu)$ making $(D_j \ln \mathbf{S}(\mu) + c)\mu_j$ sum to 1 over j (c is a constant in the sense of not depending on j). The factor μ_j in (8) turns the number of units of μ_j into the capital invested in μ_j.

Next we consider three standard special cases of the general formula (6) for Fernholz martingales. A positive C^2 function $\mathbf{S}$ defined on an open neighbourhood of Δ^J is a *measure of diversity* if it is symmetric and concave. All three special cases will be generated by measures of diversity.

3.1 Fernholz's arbitrage opportunity

In [4, Section 3.3], Fernholz describes an arbitrage opportunity for his stochastic model of the market. Now we are interested in the measure of diversity

$$\mathbf{S}(x) := 1 - \frac{1}{2} \sum_{j=1}^{J} x_j^2 \tag{9}$$

(cf. (3)). Theorem 1 gives the following non-stochastic version of [4, Example 3.3.3].

Corollary 2. *The Fernholz martingale generated by* (9) *is*

$$\frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} \exp \left(\frac{1}{2} \sum_{j=1}^{J} \int_0^t \frac{\mathrm{d}[\mu_j](s)}{\mathbf{S}(\mu(s))} \right). \tag{10}$$

Proof. It suffices to plug $D_{ij}\mathbf{S}(x) = -\mathbf{1}_{i=j}$ in (6). $\square$

A slightly cruder but simpler version of Corollary 2 is:

Corollary 3. *The Fernholz martingale Z generated by* (9) *satisfies*

$$Z(t) \geq \frac{1}{2} \exp \left(\frac{1}{2} \sum_{j=1}^{J} [\mu_j](t) \right). \tag{11}$$

Proof. It suffices to notice that $\mathbf{S} \in [1/2, 1]$. $\square$

By (8), in the case of the measure of diversity (9) the fraction of the current capital held in μ_j is

$$\pi_j = \left(\frac{2 - \mu_j}{\mathbf{S}(\mu)} - 1 \right) \mu_j. \tag{12}$$

This portfolio is particularly tame (or admissible, in Fernholz's [4, Section 3.3] terminology): it is long-only, it never loses more than 50% of its value relative to the market portfolio (by (11)), and it never invests more than 3 times more than the market portfolio in any of the stocks.

A possible interpretation of Corollary 3 is based on the efficient market hypothesis in the form that was so forcefully advocated in the bestseller [11] by Burton G. Malkiel; for him, "the strongest evidence suggesting that markets are generally quite efficient is that professional investors do not beat the market." Even if there are ways to beat the market, it is often believed that they should involve something unusual rather than merely simple portfolios such as (12), (15), or (18) (widely known since at least 2002). According to this interpretation, Corollary 3 implies that in efficient markets we expect market variation to die down eventually.

If we believe that the variation in our stock market will never die down, we are forced to admit that Corollary 3 "opens the door to superior long-term investment returns through disciplined active investment management" [10, Section 1.3]. This is the interpretation on which typical practical applications of stochastic portfolio theory are based (see, e.g., [2], which, however, is based on the stochastic versions of Corollaries 5 and 6 rather than Corollary 3).

Corollary 3 is a cruder version of Corollary 2 that replaces the first factor in (10) by its lower bound and the denominator in the second factor by its upper bound. Corollary 2 is more precise in that it decomposes the growth in the portfolio's value into two components: one related to the growth in the diversity $\mathbf{S}(\mu)$ of the market weights and the other related to the accumulation of the variation of the market weights.

Notice that (11) in Corollary 3 implies

$$Z(\tau_A) \geq \frac{1}{2} e^{A/2} \quad \text{q.a.}, \tag{13}$$

where A is a positive constant, τ_A is defined by (2), and $Z(\infty) := \infty$.

Now we have two methods for achieving the same qualitative goal, $X(\tau_A) \to \infty$ as $A \to \infty$. Quantitatively the additive result (4) appears weaker: it does not feature the exponential growth rate in A. However, there is a range of A (roughly between 0.7 and 4.3) where the Stroock–Varadhan martingale X performs better: see Figure 1 (the green slightly concave function should be ignored for now).

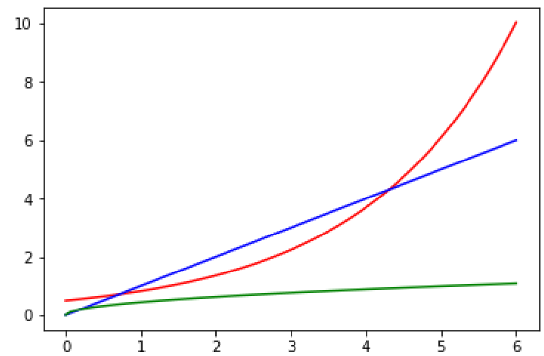

Figure 1: The values of the Fernholz martingale Z (red convex function), the Stroock–Varadhan martingale X (blue linear function), and the Dubins–Schwarz martingale X (green concave function) at time τ_A; the horizontal axis is labelled by the values of A

3.2 Entropy-weighted portfolio

The archetypal measure of diversity [4, Examples 3.1.2 and 3.4.3] is the entropy function

$$\mathbf{S}(x) := -\sum_{j=1}^{J} x_j \ln x_j. \tag{14}$$

Using (8), the components of the corresponding *entropy-weighted portfolio* can be computed as

$$\pi_j = -\frac{\mu_j \ln \mu_j}{\mathbf{S}(\mu)}. \tag{15}$$

This portfolio is also long-only. Theorem 1 now gives the following corollary (a non-stochastic version of [4, Theorem 2.3.4]).

Corollary 4. *The Fernholz martingale generated by* (14) *is*

$$\frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} \exp\left(\frac{1}{2}\sum_{j=1}^{J}\int_0^t \frac{\mathrm{d}[\mu_j]}{\mu_j \mathbf{S}(\mu)}\right) \geq \frac{\mathbf{S}(\mu(t))}{\ln J} \exp\left(\frac{1}{2\ln J}\sum_{j=1}^{J}[\mu_j](t)\right). \tag{16}$$

Proof. For the equality in (16), it suffices to plug $D_{ij}\mathbf{S}(x) = -\mathbf{1}_{i=j}/x_j$ into (6). For the inequality, use $\mathbf{S} \leq \ln J$ and $\mu_j \in (0,1)$. □

It is standard in stochastic portfolio theory to assume both that the market does not become concentrated, or almost concentrated, in a single stock and that there is a minimal level of stock volatility; precise versions of these assumptions are referred to as diversity and non-degeneracy, respectively. For the purpose of this paper it will be convenient to replace non-degeneracy by activity, $\sum_j [\mu_j](\infty) = \infty$. The inequality in Corollary 4 can be interpreted as saying that we can beat the market unless it loses its activity or degenerates. Corollary 3 says that, in fact, the condition of non-degeneracy alone is sufficient, but Corollary 4 relies on both assumptions, activity and non-degeneracy. If the market maintains its diversity, we expect the first factor on the right-hand side of (16) to stay bounded below, and if, in addition, the market stays active, we expect the second factor to grow exponentially fast. As a result, the entropy-weighted portfolio outperforms the market.

3.3 Diversity-weighted portfolios with parameter p

Fix $p \in (0,1)$. Define the *measure of diversity with parameter* $p \in (0,1)$ [4, Example 3.4.4] as

$$\mathbf{D}_p(x) := \left(\sum_{j=1}^{J} x_j^p \right)^{1/p}. \tag{17}$$

By (8), the corresponding *p-diversity-weighted portfolio* has positive components

$$\pi_j(t) := \frac{\mu_j(t)^p}{\sum_{i=1}^{J} \mu_i(t)^p}. \tag{18}$$

The following corollary is a non-stochastic version of [4, Example 3.4.4].

Corollary 5. *The Fernholz martingale generated by* (17) *is*

$$\frac{\mathbf{D}_p(\mu(t))}{\mathbf{D}_p(\mu(0))} \exp\bigl((1-p)\Gamma_\pi^*(t)\bigr), \tag{19}$$

where

$$\Gamma_\pi^* = \frac{1}{2} \sum_{j=1}^{J} \int \pi_j \, \mathrm{d}[\ln \mu_j] - \frac{1}{2} \left[\sum_{j=1}^{J} \int \pi_j \, \mathrm{d} \ln \mu_j \right]. \tag{20}$$

The term Γ_π^* defined by (20) will be referred to as the *excess growth term.*

Proof. Evaluating $D_{ij}\mathbf{D}_p$, we can rewrite (6) as

$$\begin{aligned} &\frac{\mathbf{D}_p(\mu(t))}{\mathbf{D}_p(\mu(0))} \exp\left(\frac{1-p}{2} \sum_j \int_0^t \mu_j^{p-2} \left(\sum_k \mu_k^p \right)^{-1} \mathrm{d}[\mu_j] \right. \\ &\qquad \left. + \frac{p-1}{2} \sum_{i,j} \int_0^t (\mu_i \mu_j)^{p-1} \left(\sum_k \mu_k^p \right)^{-2} \mathrm{d}[\mu_i, \mu_j] \right) \\ &= \frac{\mathbf{D}_p(\mu(t))}{\mathbf{D}_p(\mu(0))} \exp\left(\frac{1-p}{2} \left(\sum_j \int_0^t \pi_j \frac{\mathrm{d}[\mu_j]}{\mu_j^2} - \sum_{i,j} \int_0^t \pi_i \pi_j \frac{\mathrm{d}[\mu_i, \mu_j]}{\mu_i \mu_j} \right) \right), \end{aligned}$$

which is equal to (19). ☐

Corollary 5 immediately implies:

Corollary 6. *The Fernholz martingale Z generated by* (17) *satisfies*

$$Z(t) \geq J^{-(1-p)/p} \exp\bigl((1-p)\Gamma_\pi^*(t)\bigr). \tag{21}$$

Proof. Since $\mathbf{D}_p \in [1, J^{(1-p)/p}]$, we have

$$\frac{\mathbf{D}_p(\mu(t))}{\mathbf{D}_p(\mu(0))} \geq J^{-(1-p)/p}$$

(cf. [5, (7.6)]); plugging this into (19) gives (21). □

To see the intuition behind Corollaries 5 and 6, we will interpret twice the excess growth term (20),

$$2\Gamma_\pi^*(t) = \sum_{j=1}^J \int_0^t \pi_j \, \mathrm{d}[\ln \mu_j] - \left[\sum_{j=1}^J \int \pi_j \, \mathrm{d} \ln \mu_j\right](t)$$

as a kind of variance. Define (using our fixed language) a sequence of partitions $T^1, T^2, \ldots$ that is fine for all processes used in this paper and set, for a given partition $T^n = (T_k^n)_{k=0}^\infty$,

$$\begin{aligned} \mu_{j,k} &:= \mu_j(T_k^n \wedge t), & k &= 0, 1, \ldots, \\ \Delta \ln \mu_{j,k} &:= \ln \mu_{j,k} - \ln \mu_{j,k-1}, & k &= 1, 2, \ldots, \\ \pi_{j,k} &:= \pi(\mu_{j,k}), & k &= 0, 1, \ldots, \end{aligned}$$

with the dependence on n suppressed. We can then regard

$$2\Gamma_\pi^{*,n}(t) := \sum_{k=1}^\infty \sum_{j=1}^J \pi_{j,k-1} (\Delta \ln \mu_{j,k})^2 - \sum_{k=1}^\infty \left(\sum_{j=1}^J \pi_{j,k-1} \Delta \ln \mu_{j,k}\right)^2 \tag{22}$$

as the nth approximation to $2\Gamma_\pi^*(t)$; it can be shown that

$$2\Gamma_\pi^{*,n}(t) \to 2\Gamma_\pi^*(t) \quad \text{ucqa.}$$

Rewriting (22) as

$$2\Gamma_\pi^{*,n}(t) = \sum_{k=1}^\infty \sum_{j=1}^J \pi_{j,k-1} \left(\Delta \ln \mu_{j,k} - \sum_{i=1}^J \pi_{i,k-1} \Delta \ln \mu_{i,k}\right)^2, \tag{23}$$

we can see that this expression is the cumulative variance of the logarithmic returns $\Delta \ln \mu_{j,k}$ over the time interval $[T_{k-1}^n \wedge t, T_k^n \wedge t]$ w.r. to the "portfolio probability measure" $Q(\{j\}) := \pi_{j,k-1}$.

As already alluded to in Section 3, the stochastic versions of Corollaries 5 and 6 have been used for active portfolio management [2]. The remarks made above about the relation between Corollaries 2 and 3 are also applicable to Corollaries 5 and 6; the latter replaces the first factor in (19) by its lower bound. Corollary 5 decomposes the growth in the value of the diversity-weighted portfolio into two components, one related to the growth in the diversity $\mathbf{D}_p(\mu)$ of the market weights and the other related to the accumulation of the diversity-weighted variance of the market weights. Corollary 6 ignores the first component, which does not make it vacuous since $\mathbf{D}_p$ is bounded, always being between 1 (corresponding to a market concentrated in one stock) and $J^{(1-p)/p}$ (corresponding to a market with equal capitalizations of all J stocks).

4 Beating the market using the non-stochastic Dubins–Schwarz theorem

In this section we will see yet another method of achieving the qualitative goal of $\lim_{A\to\infty} X(\tau_A) = \infty$ for a nonnegative supermartingale X. The result will be weaker than those of the previous two sections, but it will shed light on a seemingly paradoxical feature of continuous-time game-theoretic probability.

The method uses the non-stochastic Dubins–Schwarz theorem presented in [15] and is based on the following apparent paradox, which we first discuss informally. As agreed in Section 2, we regard $\mu_1, \dots, \mu_J, 1$ as tradable securities. According to the non-stochastic Dubins–Schwarz theorem and a standard property of Brownian motion, with very high lower probability all J securities will eventually hit zero if their volatility is appreciable. When this happens, the normalized value of the market $\mu_1 + \cdots + \mu_J$ will be 0 rather than 1, which is impossible. Therefore, we expect an event of a low upper game-theoretic probability to happen, i.e., we expect to be able to outperform the market. This is formalized in the following statement:

Proposition 7. *For any constant $A > 0$, there is a nonnegative supermartingale X such that $X(0) = 1$ and*

$$X(\tau_A) \ge 1.25 J^{-3/2} A^{1/2} \quad \textit{q.a.}, \tag{24}$$

where τ_A is the stopping time (2) *and $X(\infty)$ is interpreted as ∞.*

Proof. For each $j \in \{1, \dots, J\}$, we will construct a nonnegative supermartingale X_j satisfying $X_j(0) = 1$ and

$$X_j(\tau_j) \ge 1.25 (A/J)^{1/2} \quad \text{q.a.}, \tag{25}$$

where

$$\tau_j := \min\{t \mid [\mu_j](t) = A/J\}.$$

(In this case we can set X to the average of all J of X_j stopped at time τ_j.) According to [9, (2.6.2)], the probability that a Brownian motion started from

1 (in fact μ_j is started from $\mu_j(0) < 1$) does not hit zero over the time period A/J is

$$1 - \sqrt{\frac{2}{\pi}} \int_{(J/A)^{1/2}}^{\infty} e^{-x^2/2}\, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{(J/A)^{1/2}} e^{-x^2/2}\, \mathrm{d}x \le \sqrt{\frac{2}{\pi}} (J/A)^{1/2}.$$

In combination with the non-stochastic Dubins–Schwarz result [15, Theorem 3.1] applied to μ_j, this gives (25) with

$$\sqrt{\frac{\pi}{2}} > 1.25$$

in place of 1.25. $\square$

The processes X in (4) and Z in (13) are nonnegative supermartingales in the sense of [16] (in fact, nonnegative continuous martingales). On the other hand, the process X in (24) is a nonnegative supermartingale in the sense of the more cautious definitions given in [15]. This can be regarded as advantage of (24) over (4) and (13). However, a disadvantage of (24) is that quantitatively it is much weaker than both (4) and (13); the right-hand side of (24) is always smaller than the right-hand side of (13), and it is greater than the right-hand side of (4) only for a small range of A (approximately $A \in (0, 0.2)$ for $J = 2$). See Figure 1.

5 Can we really beat the market?

Sceptics have come up with several explanations for the apparent possibility of beating the market (idealized or real) discussed in the previous sections; let me order some of those explanations from most theoretical to most practical.

- A common feature of the portfolios discussed in this paper that outperform the market is increased weights of smaller stocks as compared with the market. It is well known in stochastic portfolio theory that portfolios generated by (8) from measures of diversity $\mathbf{S}$ invest into smaller stocks more heavily than the market portfolio μ does [4, Proposition 3.4.2]. If we model the market as an infinitely countable set of securities, those market-beating portfolios (and perhaps even the market portfolio) will cease to exist.

- If we restrict our attention only to J largest stocks traded in an idealized or real-world market, for a moderately large J (such as $J = 500$ for S&P 500), the performance of portfolios such as (12), (15), or (18) w.r. to this smaller "market" (which is now, in fact, a large cap market index) will be affected by the phenomenon of "leakage" [4, Example 4.3.5 and Figure 7.5].

- If we make our model more realistic by including dividends, we will no longer be forced to conclude that portfolios such as (12), (15), and (18) will beat the market. If larger companies tend to pay higher dividends, those portfolios will be disadvantaged. As a practical matter, over the last decades, such portfolios have indeed been adversely affected by the tendency of larger companies to pay higher dividends (cf., e.g., [4, Figure 7.4], describing the performance of an index that has been used in investment practice). The role of differential dividend rates in maintaining market diversity is emphasized in [3].

- If we include all stocks traded in a real-world market in our model, perhaps making J very large (but of course still finite), portfolios (12), (15), and (18) (particularly the last two) will not be efficient since they will be forced to invest into smaller and so less liquid stocks.

- A standard objection from the practical point of view is that transaction costs are usually ignored in stochastic portfolio theory (and we have ignored them in our non-stochastic theory).

6 Jeffreys's law in finance

Consider two securities and choose one of them as the numéraire. We get a market with two securities, 1 and μ, where $\mu \in (0, \infty)$. Setting $\mathbf{S}(\mu) := \mu^p$ for $p \in (0, 1)$ in (6), we obtain that

$$\frac{\mu(t)^p}{\mu(0)^p} \exp\left(-\frac{p(p-1)}{2} \int_0^t \mu^{-2} \,\mathrm{d}[\mu]\right) = \left(\frac{\mu(t)}{\mu(0)}\right)^p \exp\left(\frac{p(1-p)}{2} [\ln \mu](t)\right)$$

is a continuous martingale in the market $(1, \mu)$. Setting $p := 1/2$ gives us the following corollary (where "continuous martingale" is understood in the sense of the original market).

Corollary 8. *For any two stocks S_1 and S_2, the process*

$$\sqrt{S_1(t) S_2(t)} \exp\left(\frac{1}{8} \left[\ln \frac{S_1}{S_2}\right](t)\right) \tag{26}$$

is a continuous martingale.

Corollary 8 can be interpreted as a version of the phenomenon sometimes referred to as Jeffreys's law in statistics: if two forecasting systems are both successful, they should be in agreement with each other; if they are not, we will be able to outperform greatly at least one of them (see, e.g., [1]). Indeed, according to (26), if the stocks S_1 and S_2 disagree in the sense of the ratio S_1/S_2 being very volatile, we will be able to outperform greatly their geometric mean and, therefore, at least one of them.

7 Conclusion

Figure 1 gives three functions g such that a final capital of $g(A)$ is achievable at time τ_A. It would be interesting to characterize the class of such functions g. A related question is: what is the best growth rate of $g(A)$ as $A \to \infty$? This question can be asked in both stochastic and non-stochastic settings. These are some directions of further research for non-stochastic theory:

- A natural direction is to try and strip other results of stochastic portfolio theory of their stochastic assumptions. First of all, it should be possible to extend Theorem 1 to functions $\mathbf{S}$ that are not smooth (as in [4, Theorem 4.2.1]); the existence of local time in a non-stochastic setting is shown in [12] and, in the case of continuous price paths, can be deduced from the main result of [15].
- Another direction is to extend this paper's results to general numéraires (this paper uses the value of the market portfolio as our numéraire).
- Finally, it would be very interesting to extend some of the results to càdlàg price paths.

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A Multiplicative picture of financial markets

This appendix will give more familiar alternative definitions of basic notions of (non-)stochastic portfolio theory. A *basic portfolio* is a continuous bounded function $\pi : \Delta^J \to \overline{\Delta^J}$ mapping Δ^J to its closure in $\mathbb{R}^J$; intuitively, it maps the current market weights $\mu = (\mu_1, \dots, \mu_J)$ to the fractions $\pi(\mu) = (\pi_1(\mu), \dots, \pi_J(\mu))$ of the current capital invested in the J stocks. For the purposes of this paper these very primitive Markovian portfolios would have been sufficient, since they cover all Fernholz martingales: cf. (8).

The non-stochastic notion of Doléans logarithm $\mathcal{L}$ is defined, alongside Doléans exponential, in [16]. The most useful for us interpretation of Doléans logarithm is that $\mathcal{L}(Y)$ is the cumulative return of a positive price path Y, and Doléans exponential restores the price path from its cumulative return. The *capital process* of π is the Doléans exponential

$$\begin{aligned} Z_\pi &:= \mathcal{E}\Big(\int \pi(\mu) \,\mathrm{d}\, \mathcal{L}(\mu)\Big) \\ &:= \mathcal{E}\left(\sum_{j=1}^J \int \pi_j(\mu) \,\mathrm{d}\, \mathcal{L}(\mu_j)\right) = \mathcal{E}\left(\sum_{j=1}^J \int \frac{\pi_j(\mu)}{\mu_j} \,\mathrm{d}\mu_j\right), \end{aligned} \tag{27}$$

where $\mu : [0, \infty) \to \mathbb{R}^J$ is defined by $\mu(t) := (\mu_1(t), \dots, \mu_J(t))$, $\pi_j(\mu) : [0, \infty) \to \mathbb{R}$ is defined by $\pi_j(\mu)(t) := \pi_j(\mu(t))$, and $\pi(\mu) : [0, \infty) \to \mathbb{R}^J$ is defined by $\pi(\mu)(t) := (\pi_1(\mu)(t), \dots, \pi_J(\mu)(t))$. The capital process Z_π is defined and continuous quasi always.

The definition (27) involves Doléans logarithm, but stochastic portfolio theory emphasizes regular logarithm (cf. the logarithmic model in [4, Section 1.1]). On the log scale the definition (27) can be rewritten as

$$\ln Z_\pi = \ln \mathcal{E}\left(\sum_{j=1}^J \int \pi_j(\mu) \,\mathrm{d}\, \mathcal{L}(\mu_j)\right) \tag{28}$$

$$= \sum_{j=1}^J \int \pi_j(\mu) \,\mathrm{d}\, \mathcal{L}(\mu_j) - \frac{1}{2}\left[\sum_{j=1}^J \int \pi_j(\mu) \,\mathrm{d}\, \mathcal{L}(\mu_j)\right] \tag{29}$$

$$= \sum_{j=1}^J \int \pi_j(\mu) \,\mathrm{d} \ln \mu_j + \frac{1}{2} \sum_{j=1}^J \int \pi_j(\mu) \,\mathrm{d}[\ln \mu_j] \tag{30}$$

$$- \frac{1}{2}\left[\sum_{j=1}^J \int \pi_j(\mu) \,\mathrm{d} \ln \mu_j\right] \quad \text{q.a.} \tag{31}$$

The second equality in the chain (28)–(31) follows from (5), and the third equality in (28)–(31) follows from

$$\mathcal{L}(Y) = \ln Y_t + \frac{1}{2}[\ln Y] \quad \text{q.a.} \tag{32}$$

(showing that the first term in (29) can be represented as (30)) and a slight generalization of

$$[\mathcal{L}(Y)] = [\ln Y] \quad \text{q.a.} \tag{33}$$

(showing that the second term in (29) can be rewritten as (31)). See [16, Section 7] for (32)–(33) (and (5), as already mentioned).

The part

$$\begin{aligned}
\Gamma^*_\pi &= \frac{1}{2}\sum_{j=1}^{J}\int \pi_j(\mu)\,\mathrm{d}[\ln\mu_j] - \frac{1}{2}\left[\sum_{j=1}^{J}\int \pi_j(\mu)\,\mathrm{d}\ln\mu_j\right] \\
&= \frac{1}{2}\sum_{j=1}^{J}\int \pi_j(\mu)\,\mathrm{d}[\ln\mu_j] - \frac{1}{2}\sum_{i,j=1}^{J}\int \pi_i(\mu)\pi_j(\mu)\,\mathrm{d}[\ln\mu_i,\ln\mu_j]
\end{aligned}$$

of (28)–(31) consisting of the last two addends was called the excess growth term in Section 3: cf. (20); it corresponds to the cumulative excess growth rate in stochastic portfolio theory. We can use it to summarize (28)–(31) as

$$\ln Z_\pi = \sum_{j=1}^{J}\int \pi_j(\mu)\,\mathrm{d}\ln\mu_j + \Gamma^*_\pi \quad \text{q.a.} \tag{34}$$

The addend $\sum_{j=1}^{J}\int \pi_j(\mu)\,\mathrm{d}\ln\mu_j$ is the naive expression for the cumulative log growth in the value of π, and Γ^*_π is the adjustment required to obtain the true cumulative log growth. Equation (23) makes the expression (34) very intuitive: the excess growth rate of the portfolio π over the naive expression is determined by the volatility of the market weights w.r. to π.

A particularly important special case is that of the market portfolio, $\pi = \mu$. To understand the intuition behind the excess growth term (20) in this case, we can rewrite $2\Gamma^*_\mu$ as

$$\begin{aligned}
2\Gamma^*_\mu(t) &= \sum_{j=1}^{J}\int_0^t \mu_j(s)\,\mathrm{d}[\ln\mu_j](s) - \left[\sum_{j=1}^{J}\int \mu_j\,\mathrm{d}\ln\mu_j\right](t) \qquad (35)\\
&= \sum_{j=1}^{J}\int_0^t \mu_j(s)\,\mathrm{d}[\ln\mu_j](s) = \sum_{j=1}^{J}\int_0^t \frac{\mathrm{d}[\mu_j](s)}{\mu_j(s)} \\
&\ge \sum_{j=1}^{J}\int_0^t \mathrm{d}[\mu_j](s) = \sum_{j=1}^{J}[\mu_j](t),
\end{aligned}$$

where we have used the fact that the subtrahend in (35), being the quadratic variation of a monotonic function (remember that $\sum_j \mu_j = 1$), is zero. We can see that $2\Gamma^*_\mu(t)$ is bounded below by the total quadratic variation of the market weights.