SUR
L'USAGE
DU
PRINCIPE DE LA RAISON SUFFISANTE DANS LE CALCUL DES PROBABILITÉS*

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I.

I have indicated in a preceding Memoir that the doctrine of the probabilities was uniquely based on the principle of sufficient reason; therefore it would not be surprising that mathematicians were not in accord among themselves in the solution of problems which have probability for object; their calculations are necessarily true, but the nature of the subject to which they apply them is not. The contingent truths can be demonstrated only by starting from an assumption; & however plausible that one assumption be, it does not exclude necessarily the others, which can serve as basis to some other calculations, & to give consequently some different results.

An illustrious author[1], geometer & philosopher all at once, has published lately on the Calculus of probabilities, some doubts & some questions very worthy of being investigated thoroughly; he insinuates however that a great Geometer has not judged them such; the fact is not known to me, & I do not know likewise that which could establish here the diversity of sentiment between two Geometers of this order. Besides it would be ill advisable for me to attempt the decision in this dispute; I must say with Palémon:

Non nostrum inter vos tantas componere lites.[2]

That which I myself propose here, is simply to test how far the metaphysical principles can aid in clarifying some doubts, & to resolve the questions proposed on the Calculus of Probabilities.

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*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 1, 2009

[1] *Translator's note*: He refers to Jean D'Alembert. His most recent memoir on probability (Memoir 23) had been published in 1767.

[2] *Translator's note*: Virgil, Eclogue III. Palaemon: "It is not for us to end such great disputes." or "Not mine betwixt such rivals to decide" (Greenough translation.)

II. In order to avoid all ambiguity on this subject, we distinguish first the possibility of an event from its probability. Any combination which does not imply contradiction is possible, & as one would not involve by halves, all the possible combinations are equally possible; it is only improperly that one would say of a possible event, that is more or less possible than another; there is no mean, nor degrees to imagine, between that which can exist, & that which refuses to exist.

But the simple possibility does not suffice to give existence to an event; it is necessary moreover that there be a sufficient reason which determines the event to be rather the one which it is, than one of the equally possible others & it is here that probability begins.

III. In any event whatever, there are yet three things to distinguish which are able to be confused. 1. its necessity; 2. its probability; 3. its actuality. Each of these three things has its sufficient reason; but it is only that of the second one which is properly speaking the foundation of the probabilities. The sufficient reason of the necessity of an event, is the impossibility of the contrary event: if there are only black tickets in a wheel of fortune, it is necessary that the ticket which will be extracted be black. The sufficient reason of the probability of an event, is the preponderance of the reasons to expect this event over those to expect the contrary event; if there are in a wheel ninety-nine white tickets, & a single black ticket; there are ninety-nine reasons against one to expect that the ticket which one will extract by chance will be white. Finally the sufficient reason of the actuality of the event, is the actual competition of physical & mechanical causes capable of bringing about that event. Now, as one assumes not only, that the existence of this competition of causes does not depend on our will, but moreover that we do not know how to disentangle it even when that it exists, it is evident that we do not know the sufficient reason for the existence of an event which one names fortune; & that it can consequently be contrary to the one which we had a sufficient reason to expect, that is to regard as the most probable. But it is evident also that the opposition between the event & the calculus which foretold it, not weaken the solidity of the principles of the calculus. These principles lead only to determine what is the most probable event, setting aside some imperceptible physical & mechanical causes which must combine to determine its existence.

IV. If the calculus of probabilities is not based on the physical causes which produce the event, it is no longer on the caprices of chance that one imagines to preside at the birth of the events of a certain specie. The strangeness of chance could not give the least hold to the calculus, & the most able analyst would never say, either that which will produce the lot, or even that which is probable that it will wish to produce; there is a perfect repugnance between the idea of chance & that of probability; the last assumes some fixed principle, the other excludes all principle. One can predict infallibly the effect of a mechanical agent submitting to some immutable laws; one can foresee probably the action of an intelligent being who follows the laws of its nature; but one will never guess with the least degree of possibility the operation of a being who would not be controlled by any kind of law, neither physical, nor moral, nor necessary, nor contingent.[3] The Calculus of probabilities takes therefore a mean

[3]. . .*Incerta haec si tu postules*
Ratione certa facere, nihilo plus agas,

between contingent arbitrariness, & physical necessity; it decides what will be the event, not so much as it is controlled by chance, not so much as it is determined by mechanical causes, but by assuming it prescribed by the laws of suitability, by the equity of an impartial judge. If among one hundred possible cases, & equally probable, there is only one of them which makes me win one hundred écus, & ninety-nine which make me to win nothing, there are odds of 99 against 1 that I will not win. Why does one say in this case, that the probability that I have of winning is worth precisely one écu while it is very possible that I win one hundred écus, & when it is absolutely impossible that I win one? It is that one does not calculate that which the chance will be, but that which it should make if it distributed its favors with an exact impartiality.

V. After these preliminary clarifications, the first question which presents itself to resolve, is to know *if symmetric & regular events, attributed to chance, are (all things equal besides) as probable as the events which have neither order nor regularity, & in the case which they have the same degree of probability, whence comes their regularity which strikes us, & which appears to us so singular?*

We choose first a proper example to clarify this subject:

I cast onto the table six dice, A, B, C, D, E, F. There are precisely 46656 possible combinations; thus, any combination that I produce, there are odds of 46655 against 1, that it would not come on the first cast. But as meager this probability which seems to exclude successively each particular combination, it is necessary that one of them come, if I produce for example the combination $A2$, $B5$, $C3$, $D4$, $E3$, $F1$, no person will mark the least surprise, if on the contrary I produce on the first cast a rafle[4] of six, or of five, etc. one will exclaim at the singularity of the case. There is more, it is that if one casts these six dice 46656 times in sequence, there is a sufficient reason to expect that each particular combination will appear one time in 46656 trials; & I swear that I see nothing which must exclude the rafles in their right to be contained in this probability. However it is certain that the trial, whatever it be, which will produce them, will appear always extraordinary, while their exclusion will never astonish.

There are two reasons, it seems to me, which must make a cast of rafle appear more singular than any other combination. The one is deduced from the perfect regularity which distinguishes this case. The dice are mixed at random, & cast likewise. The regularity supposes the contrary to chance, a choice, an arrangement, a sufficient reason. To find in a production at random a similar effect to the one which one would have been able to expect on a premeditated design, is an event for which one is not prepared, it must confuse by its singularity, & appear less probable consequently than others which without having a higher degree of probability have no singularity.

The second reason which must make finding the combination where all the dice present the same face more strange, than all other determined combinations, is that this latter being irregular has nothing which renders it remarkable, nothing which fixes the attention. It offers no marked character which happens to distinguish it from a great number of other equally irregular combinations; now the number of these latter being

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Quam si des operam, ut cum ratione insanias. Terence. *Eunuch.*

Translator's note: "If you tried to turn these uncertainties into certainties by a system of reasoning, you'd do no more good than if you set yourself to be mad on a system." Trans. John Sargeaunt, Loeb Classical Library translation, 1912.

[4]*Translator's note*: A *rafle* occurs when all dice show the same face.

without contradiction the greatest, the cast which will bring forth one of the irregular combinations, confounding with all the other similar casts, must appear a quite common event, to which one has everywhere to expect, whence it happens naturally that a contrary event will seem very singular. Six dice give 6^6 different combinations; of which there are of them only 6 exactly regular, all the others in number of $6^6 - 6$ deviate more or less from regularity; it is therefore not astonishing that the cast which would produce a rafle of six appears more singular that one which would produce the combination $A2$, $B5$, $C3$, $D4$, $E3$, $F1$;since this combination resembles 46649 others, whereas the cast of a rafle of six, admits only 5 other similar casts. Consequently, if one is not aware from the individual determination, one will judge the cast which produces 2; 5; 3; 4; 3; 1. more probable, seven thousand seven hundred sixty-five times[5], & as many times less singular, than the cast which would produce 6; 6; 6; 6; 6; 6.

We suppose nonetheless that one really pays attention to the determined numbers 2; 5; 3; 4; 3; 1; the case which will produce them must however seem much less singular than the one which will give all *six*, because provided that one sees these numbers of different points on the dice, one scarcely notices to examine scrupulously to precisely which die each number belongs. Now six dice can give this combination 2; 5; 3; 4; 3; 1; in seven hundred twenty different ways, it must therefore appear even by paying attention to the specie determined by irregularity, that the case which produces it is seven hundred twenty times less singular than the unique case of the rafle of six.

VI. The same reasons which make the regular combinations attributed to chance be the cause of surprise, the combinations where one perceives neither order nor regularity are likewise found strange by us, when these combinations are attributed to the will of an intelligent Being. Among all the possible combinations, a wise Being will not choose one of those which composes the most numerous specie, by the sole consideration that this class is the most numerous. He will choose the combination which corresponds most exactly to his plan, were it unique in its specie, as it is necessarily unique by its individual determination. The orbits of the planets of our sun, for example, could without doubt have among themselves some inclinations very different from those that the author of the universe has assigned them. But these latter being the result of his free choice, one can be assured without temerity that this was the most appropriate combination for the most perfect plan; & that there is in this plan a sufficient reason for the actual arrangement of these orbits. It would be perhaps bold to attempt to determine precisely this reason. But it is quite permitted to imagine if one may be the most satisfactory final cause, provided that one does not claim to decide peremptorily, that it is due to being the unique motif of the actual arrangement. One has effectively accomplished by finding it from very plausible reasons, & in order to speak only of the newest Work that I know on this subject, the wise author of the Cosmological Letters[6] has indicated with much sagacity, that the preferred arrangement was the most proper of all to collect & to make rotate without uncertainty around the same Sun, the greatest possible number of planets in circular & elongated orbits.

VII. From that which we just spoke in the two preceding articles, I believe that

[5] *Translator's note*: This is 6^5.

[6] J. H. Lambert, *Cosmologische Briefe über die Einrichtung des Weltbaues* (1761).

one can conclude legitimately; 1. that the regularity or irregularity of an individual determined combination, neither adds, & nor subtracts anything to its real probability. 2. that the most symmetric combination will appear nonetheless the most singular, the most unexpected, & the least probable of all, when it will be produced by the fortuitous concurrence of purely mechanical causes. 3. That on the other hand this same combination will seem the most natural, & the most probable of all, if one regards it as the effect of free choice by an intelligent Being. 4. that in this last assumption one is authorized to seek the sufficient reason of the actuality of the event, because it must have a final cause; & finally 5. that one must not seek the reason for the existence of an event which one attributes to chance, since there is no final cause here to discover, & since the physical causes are too complicated, & too hidden in order that one can disentangle them.

VIII. Until here there has been question only of the case of a unique combination, of which the existence excludes that of any other equally possible combination. But one proposes another more difficult question to discuss: it is that *when one same event has already happened one or more times in sequence, one asks if this event maintains as much probability for its future existence, as the contrary event which with an equal original probability does not happen again*. It is not necessary to caution that the question concerns fortuitous events, or at least those that one regards such lack of knowing the causes which produce them. For as soon as the question would concern events produced by a constant mechanical cause, or events directed by the will of an intelligent Being, it is evident that these events must succeed themselves without variation as long as their final & mechanical causes will not change; & that if these causes are known one could predict at one sure stroke the return of the same effect. The fortuitous events have equally their determined cause, but in the impossibility where we are to perceive it, all that we can do is to examine if it is probable that the same concurrence of circumstances which have produced one time or two an event, will subsist long enough invariable in order to reproduce this same event a third & a fourth time. Now if one agrees, as it seems that one could not doubt it, that each event depends on a great number of separate causes which work together to determine it, & that these causes have among themselves no necessary connection; if one considers moreover that all nature, as its proper activity, passes continually from one state to another state, one will recognize without difficulty that it is not probable that the same concurrence of circumstances of which the accidental reunion has produced an event, returns many times in sequence without the least alteration; & since any alteration in the assembly of causes, can produce a diversity in the effect which results from it, it seems probable that an event produced by the accidental concurrence of diverse partial causes will not be the same many times in sequence. It seems therefore that when there is question of a repeated event, the probability of its return must be by reason composed of the absolute probability of this event, & of the probability of the combined causes which can restore it.

IX. In order to clarify further the question, we suppose a lottery of only two tickets, one white, the other black; let one extract only one ticket in each drawing, & let the extracted ticket be replaced each time into the wheel for the following drawing. If the ticket which will be extracted is white, I lose my wager; if it is black, the entrepreneur of the lottery pays me the double of it. It is evident that in the first drawing the probability is equal on both sides & that we play evenly.

But after the first drawing, one asks, if this trial must have an influence on the following as to the calculus of probabilities, or if one must consider each new drawing as an isolated act which has connection neither with those which have preceded it nor with those which will follow it. There are some plausible reasons for one & for the other opinion. In fact one can say on the one hand that by the reentry of the withdrawn ticket all is returned to the original state; that one is no more justified to combine the drawing which has immediately preceded with the one which is going to follow, than any drawing whatever which would have preceded this one by many centuries; or than a drawing which would have been made in one hundred places hence on a similar plan; that before proceeding to the first drawing one would have quite well by right to suppose, & to imagine whatever number of drawings anterior to this first, which would change nonetheless nothing in relation to this latter, neither in the event, nor in its probability; in a word that each drawing is evidently a unique act, independent, without any relation whatever to all others, & of which consequently the probability remains invariably determined by the ratio of the number of winning tickets, to the losing tickets.

In adopting this principle, the lots would be here constantly the double of the wager; & as I am obliged, in order for me to recoup some losses made in the preceding drawings, to double the wager in each new drawing, if the first has been a half-écu, one will have the following table.

Drawing	Wager	Lot
1st	$\frac{1}{2}$	1
2nd	1	2
3rd	2	4
4th	4	8
$\vdots$	$\vdots$	$\vdots$
tth	$\frac{2^t}{4}$	$\frac{2^t}{2}$

As plausible as this opinion be, there results from it nonetheless a consequence which tends to refute it invincibly, it is that early or late the entrepreneur of the lottery will be the dupe of the calculus. The player is made whole of all the unfortunate drawings by a single drawing which will be favorable to him, while on the contrary one hundred & one thousand favorable drawings to the entrepreneur never shelter him from the loss in a single trial the advantage of all these fortunate drawings. Where would the equality be here which must be between the condition of the interested parties? There is only a single case which can compensate the disadvantage of the entrepreneur, but this case is strange to the calculus of probabilities; it is that the player doubling the wager at each drawing, can find himself at the end of a certain number of unfortunate drawings out of state of sustaining the stake; that it can be from the impossibility to continue the game for lack of silver for a further wager, instead of which the entrepreneur never risks of his other than the value of the first wager: but this consideration, very important for the interested parties, would not influence the correctness of the abstract calculus of the probabilities.

X. The partisans of the opinion that I just examined will say perhaps that the disadvantage of the entrepreneur is real only under the assumption that the drawings continue to infinity, or at least at the wish of the player; but that if the number of drawings has been fixed in advance, the probability will be equal on both sides, since if the entrepreneur risks more loss by the repetition of the drawings, he risks also to win a sum proportioned to this repetition. But will he not there admit tacitly a connection between the sequence of drawings, & to acknowledge in some fashion that the ones are an influence on the others, & that it is not presumable that a great number of successive drawings can give constantly the advantage on a same side? Because if in each drawing there were equal odds *one* against *one*, that the ticket which will exit will be a white ticket, it must be quite indifferent to the entrepreneur if the number of drawings be limited in advance, or that it not be.

XI. We see therefore also that one can say in favor of the sentiment which establishes a connection among the successive drawings, & what would be the new probability which would result from it. When one reflects on the principle of this calculus one finds nothing which prevents applying it as well to the successive events, as to the simultaneous events. In fact this principle is only the relationship of the sufficient reasons for or against an event. We assume that a wheel where one will have mixed 500 white tickets, with as many black tickets, one extracts from it at once two, at random. There is precisely as much reason to expect that both will be white, as to expect that both will be black. But the two tickets would not know how to be at once white & black; there is therefore a sufficient reason to think that one of the two will be a white ticket, and the other a black ticket. The same reasoning will hold if one extracts from it at once 4; 6; 8; or such even number as one will wish; the probability will remain always that half of the withdrawn tickets will be of one specie, & the other half of the other specie. Now I ask if there is any difference capable of altering this probability, either that one extracts for example twelve tickets at once by one simultaneous act, or, that one extract these twelve tickets two by two, in six successive acts? I say in six times, because if one extracts them in twelve trials, the probability for the alteration of the two species would be again strengthened, by consideration that there would remain in the wheel, after the first drawing, one ticket more, of the color opposed to that which would come from extraction.

But if one agrees that the probability is the same, either that one draws at once these twelve tickets, or that one make twelve successive drawings from it, one agrees also that the probability is that for 12 drawings, there will be six of them which will give some white tickets, & six others which will give of them some blacks; & by the same reason the probability will signify that out of two drawings, the species alternate; & that, either the withdrawn tickets will reenter, or that they no longer reenter But in this last case the probability for the alternation increases by reason of the numerical ratio of the remaining tickets of each specie.

XII. In order to determine the law of this probability, one can make different assumptions; the most natural is to consider each specie as uniting the rights of all the individuals which compose it, & each individual as having an equal right to exit; they support its claim for its exiting, & subsist as long a time as it will not exit; if the species contain an equal number of individuals, they are in the case of this latter, for the equality of claims. If, for example, one makes two drawings, a specie has the same right as the other, of exiting on the first & on the second trial. But that which exits on the first drawing, has no longer so to speak a reason of claim to exit, instead of which the other keeps its two degrees of claim; there is therefore here odds of 2 against 1, that the specie which does not exit in the first drawing, will exit in the second.

For the same reason, if the first three drawings have produced the same specie, there would be odds of 4 against 1, that the other specie will exit in the fourth drawing, & in general as much as there will be in the wheel so many tickets of one specie as the other; if one extracts a number of them whatever t, either at one time, or in t successive drawings, & if they be all of the same specie, there are odds of $t + 1$ against 1, that the following ticket will be the other specie.

XIII. If the withdrawn tickets no longer reenter for competing in the following drawing, there results from this condition a new probability for the alternation of the species.

Let the number of tickets of each of the two species A, & B, have been first $= n$, let one have already made a certain number of drawings $= t$, & let all the withdrawn tickets have been of species A, their remaining number will be $= n - t$, & the one of B will be yet $= n$. One asks the probability that there is that in the following drawing $t + 1$, one will see exit a ticket of specie B?

If one pays no attention to the preceding drawings, & if one considers only the number of remaining tickets, the probability for specie B will be $= \frac{n}{2n-t}$, & for specie A it will be $= \frac{n-t}{2n-t}$; thus it would have odds n against $n - t$, that the specie B will exit in the next drawing.

But, if one regards in this that each specie has first $t + 1$ claims to exit in $t + 1$ drawings, that specie B has conserved all its rights, & that specie A has no more than one degree of claim on the $t + 1$st drawing, there would be yet odds $t + 1$ against 1, that specie B will exit on this drawing here: & consequently, if one combines these two probabilities, there will be odds for specie B, $nt + n$ against $n - t$.

It is remarkable that this which is only the probability when it concerns species, becomes necessity when the species are reduced to the individual; if $n = 1$ & $t = 1$, there are odds 1 against 0 that B will exit in the second drawing.

It seems that, in the case that I suppose here, namely that the withdrawn tickets are not reentered, the combination of the probabilities what I propose must be admitted. Because the successive events must alternate by the same probability which presumes that a simultaneous multiple event will be composed of simple alternate events. But, when the withdrawn tickets are not reentered, the reason for the alternation increases in measure as the cases which could have prevented it diminish: this probability increases even to the point, that when the diminution of the contrary cases comes to annihilation, the alternation, or rather the passage from one specie to the other, becomes necessary, instead of being simply probable. I say the *passage*, rather than the *alternation*, because that latter contains the idea of a later return to the first specie; a return which is no longer possible when all the cases which would have produced this former specie are depleted, this which happens when one has $t = n$, or $n - t = 0$.

XIV. From that which has been said in article XII under the assumption that the withdrawn ticket reenters in each following drawing, there results that, if the number of drawings were counted infinite, there would be also odds of infinity against unity that one will not have always extracted the white ticket; consequently, in the problem that Mr. *Nicolas Bernoulli* has proposed to Mr. de *Montmort*, Paul wagering that Pierre will produce *tails*, could according to all the rules, I say not of prudence, but of probability, wager an infinite sum if the thing were false, against a single écu, if sooner or later *tails* will be produced.[7]

XV. This known problem has embarrassed the Mathematicians by the difficulty that its solution creates. Here is to what it is reduced if one brings it back to our perpetual lottery, composed only of two tickets, one white & one black. Pierre permits Paul to extract a ticket; & is bound to give to him as many half-écus, as the number 1 expresses of it doubled as many times drawings will be necessary in order to produce the black ticket. If, for example, this ticket exits only on the twelfth drawing, Paul will receive $\frac{2^{12}}{2} = 2^{11} = 2048$ écus, & in general if it exits only on the drawing t, Pierre promises the sum of 2^{t-1} écus, a sum which will be enormous, if t is a slightly considerable number.

It seems first by the statement of this Problem, & this is that which has seemed singular to some celebrated Geometers, that the fortune of Paul is assured, & that he has expectation to receive an immense sum: in fact, by the ordinary calculus of probabilities, one finds that the sum of the expectations of Paul is expressed in écus by this series

$$\frac{1}{2}t \times \frac{2}{2} + \frac{1}{4}t \times \frac{4}{2} + \frac{1}{8}t \times \frac{8}{2} + \cdots + \frac{1}{t}t \times \frac{t}{2} = \frac{1}{2}t + \frac{1}{2}t + \frac{1}{2}t \text{ etc. } = \frac{1}{2}tt \text{ écus.}$$

But, as among all the drawings, of which one assumes here that the number is $= t$, there is only one case which makes Paul win, his mean expectation is reduced to the sum of $\frac{tt}{2t} = \frac{1}{2}t$ écus. He could therefore expect an infinite sum, if the number t of the drawings went to infinity before the black ticket exits. However one falls rather generally to an agreement that it would be folly to give him for this expectation beyond about twenty écus.

We see therefore, in assuming our principles, what will be the sum that one can reasonably offer to Paul for his expectation.

At each drawing t prepared to be made, the sum promised by Pierre is $= 2^{t-1}$ écus, & the probability for *white* & *black* being supposed equal, the expectation of Paul in this sum is $= \frac{2^{t-1}}{2} = 2^{t-2}$. But we have seen (article XII) that, if the black ticket had not exited in the $t-1$ preceding drawings, there would be odds t against 1, that it would exit on the drawing t. Therefore, by the same reason, before every drawing there would be odds $t-1$ against 1, that the black ticket would exit before the drawing t; $t-2$ against 1, that it would exit before the drawing $t-1$; $t-3$ against 1 that it would exit before the drawing $t-2$, and so forth. So that by combining these wagers there would be odds of $1 \times 2 \times 3(\cdots)(t-1)$ against 1, that Paul will not realize the expectation 2^{t-2} écus; his expectation for this sum would be worth to him therefore only:

$$2^{t-2} \times \frac{1}{1.2.3(\cdots)(t-1)-1} \text{ écus.}$$

[7] *Translator's note*: This is the Petersburg problem.

One can, by developing this formula, find the value of the expectation $= e$ of Paul for each prize 2^{t-2}; for the number of drawings in which one will be agreed to advance

$$\begin{array}{llll}
t=1 & \text{gives} & e=\dfrac{1}{2}\times\dfrac{1}{0+1} & \cdots \quad =\dfrac{1}{2}\text{ écu}\\
t=2 & \cdots & e=1\times\dfrac{1}{1+1} & \cdots \quad =\dfrac{1}{2}\text{ écu}\\
t=3 & \cdots & e=2\times\dfrac{1}{1.2+1} & \cdots \quad =\dfrac{2}{3}\text{ écu}\\
t=4 & \cdots & e=2.2\times\dfrac{1}{1.2.3+1} & \cdots \quad =\dfrac{4}{7}\text{ écu}\\
t=5 & \cdots & e=2.2.2\times\dfrac{1}{1.2.3.4+1} & \cdots \quad =\dfrac{8}{25}\text{ écu}\\
t=6 & \cdots & e=\dfrac{2.2.2.2}{1.2.3.4.5+1} & \cdots \quad =\dfrac{16}{121}\text{ écu}\\
t=7 & \cdots & e=\dfrac{2.2.2.2.2}{1.2.3.4.5.6+1} & \cdots \quad =\dfrac{32}{721}\text{ écu}\\
\text{etc.} & & \text{etc.} &
\end{array}$$

Whence one sees that the series which would express the total value of the expectation of Paul, by supposing that the drawings be unlimited, or pushed to infinity, would be

$$\frac{1}{2}+\frac{1}{2}+\frac{2}{2+1}+\frac{2.2}{2.3+1}+\frac{2.2.2}{2.3.4+1}+\frac{2.2.2.2}{2.3.4.5+1}+\text{etc. to infinity.}$$

Now it is evident that after the first three terms of this series the value of each of the following diminishes more and more; since each term which accedes has for new factor the fraction $\frac{2}{t-1}$, of which the numerator remains constant, while the denominator increases uniformly to infinity. The twentieth term, for example, which corresponds to the twentieth drawing, is worth no more than $\frac{2^{18}}{1.2.3\cdots19+1} = \frac{1}{462972000000}$ of an écu[8]. Thus, although the series which expresses the sum of the expectations of Paul, really goes to infinity, without any term ceasing to be real & positive, this sum is nonetheless so little considerable that the value of the 20th term is already no more than one infinitely small part of an écu, & that one can neglect without error all the terms beyond the first eight or ten. The sum of the expectations of Paul would reduce therefore to

$$\frac{1}{2}+\frac{1}{2}+\frac{2}{3}+\frac{4}{7}+\frac{8}{25}+\frac{16}{121}+\frac{32}{721}+\frac{64}{5040}+\frac{16}{5040}+\frac{3\frac{1}{2}}{5040}+\text{etc.}=\frac{1331}{504}=2.454\,\text{écus},$$

so that it could not go beyond $2\frac{1}{2}$ écus, & that this would be also all that which one could reasonably offer to him for a claim which would seem first to have no bounds.

XVI. One has again expressed this singular problem in one other fashion, which does not differ however in the first with regard to the calculation. One supposes that Pierre & Paul wish to play evenly, & under this assumption one asks what is the stake which Paul must set in commencing the game. The calculus of probabilities determines

[8] *Translator's note*: I obtain $\frac{262144}{121645100408832001} \approx \frac{1}{464039231906}$.

this stake at $\frac{1}{2}t$ écus; that is that Paul must set in the game as many half-écus as one will have fixed the number of trials for each game. Now, if instead of determining in advance the number of trials, one was agreed to end the game only when one will produce *tails*, or when the black ticket will exit, as it implies no contradiction that this ticket not exit, the game of Paul according to this calculation ought to be an infinite sum of half-écus, since absolutely speaking it is not impossible that the number of drawings t go beyond any limited number. One senses well that it would be absurd in any sense to impose on Paul such a game, & it is no less true that all the formulas that the Mathematicians have given until here require it, as soon as one does not limit them by some conditions entirely foreign to the problem, such as the consideration of the faculties of Pierre, that of the fortune of Paul, or that of the limits that his moderation can set on his cupidity. This is not all yet: we show that the reasonable expectation of Paul does not go to 3 écus; he would therefore be wrong to risk in this game a greater sum; on the other hand we have seen (art. XIV) that he could chance in a parallel game an infinite sum against a single écu, & finally one sees by article IX that the advantage would be again on the side of Paul when, in order to win the same lots that Pierre offers to him here, he would risk, whatever be the number of drawings t, I say no longer as many half-écus as t contains of units, but as many of them as the incomparably greater number $2^{t-1} - 1$ contains of them.

XVII. In order to clarify these paradoxes, I believe that it suffices to pay attention to the nature of the case. When one asks, in the problem in question, what must be the stake, or expectation of Paul, one does not ask what is the sum which will not imply contradiction that Paul win; this question is not the province of probability; but one asks simply what is the sum that Paul can reasonably expect to win. Now there is no reason to expect that the black ticket will come only at the end of an infinite number of drawings; there is therefore no reason to require of Paul a stake of an infinite value. There is very little reason to expect that this ticket will exit only at the 10^{th} drawing, & much less again that it will exit only on the twentieth. Paul would have very little reason to chance a stake of *five* écus, & much less again to risk on it one of *ten*. The mathematical calculation gives very correctly the proportion between the stake & the corresponding prize, for such number of drawings as one would wish from zero to infinity; this is all that one demands of this calculation, & more even than one would demand; but, if one wishes to know how many drawings there will be probably before the black ticket exits, it is a new problem, which demands another calculation, & it is this other calculation which I am going to attempt; if one admits it, it will result that Paul must not expect more than five drawings, & that he must risk consequently only a stake of 2 & a half écus.

But why ought he risk only a so small stake, since he can wager an infinite sum against one écu, that the black ticket will exit sooner or later (XIV)? It is precisely because he has odds of infinity against one that an infinite number of drawings will not be necessary to produce a black ticket, that Paul must expect to see it exit before the sixth drawing; & that thus his stake must be proportional to the number of drawings which he can reasonably anticipate; if he would chance a stake of 20 écus, & if the black ticket exits on the first, second, third or fourth drawing, he would have risked against all feasibility 19, 18, 16, 12 or 4 écus, on the simple possibility, that becomes more and more less probable, to win on the 6th drawing, 12 écus, on the 7th, 44 etc.

The same consideration raised the difficulty which seems to result from the comparison of the two cases of art. IX & XV. In the first of these cases, Paul, we have said, can risk for example with advantage $31\frac{1}{2}$ écus, on the expectation to receive 32, & in the second case, we find that he would have great wrong to risk these $31\frac{1}{2}$ écus, in order to win more than 4 trillions. But the diversity of the two cases is sensible: in the first, Paul has already lost *five* consecutive drawings, he chances actually only 16 écus against 32 in an equal game, & with the moral certitude that the black ticket could not come much later. In the second case on the contrary, no drawing has yet preceded, the same moral certitude must be presumed by Paul that the black ticket will exit in one of the first drawings, which does not repay him his stake.

XVIII. I am however quite remote to regard the calculation of the stake of Paul, such as the calculation of article XV gives it, as a demonstrated truth; I believe on the contrary that the problem in question suffers as many different solutions as there are diverse ways to envisage the probabilities. Any solution which admits nothing impossible or absurd, can be good, or at least is not susceptible to a demonstrative refutation. The two extreme solutions are, that which gives a stake increasing until becoming infinitely great, & that which would fix the stake constant at a half-écu; both must be excluded, the one because Paul would risk probably more than Pierre; the other because Pierre would risk presumably more than Paul. The most admissible solution will be that where the mean gain which must always be equal on all sides, will be thus accompanied by an equal possibility for both of the players; & it is of this genre, if I do not fool myself, that are the following solutions.

XIX. 1. Each particular prize proposed by Pierre, combined with its probability, is reduced to the value of the first prize: this is on what the Mathematicians agree. The first prize is one écu, & the expectation of Paul to this prize is worth precisely one half-écu; thus one can say that the expectation of each of the other particular prizes is not worth more; & since Paul could win only one single prize, it seems that his expectation, or that which comes to the same his stake, must not exceed this half-écu, either when the number of drawings has been fixed to one or to many, or when it has not been limited at the beginning of the game. But, on the other hand, the more there will be of drawings, the more probability there is that Paul will win this prize, of which the *absolute* or *reduced* value, is one écu. If one plays to a single drawing, there are odds *one* against *one* that he will win it; thus his stake is $e = 1 \times \frac{1}{2}$ écu: if one plays to one hundred thousand trials, the stake would therefore be by the same reason $e = 1 \times \frac{100000}{100001}$ écus, & if the number of trials is unlimited, there are odds of infinity against 1, that Paul will win the prize. Thus the stake in this former case must be $e = 1 \times \frac{\infty}{\infty+1} = 1$ écu; the value of the stake would vary therefore to infinity between the two extreme values $\frac{1}{2}$ & 1, & the formula which expresses these values in general will be $e = 1 \times \frac{t}{t+1} = \frac{t}{t+1}$.

But, to take it this way, Pierre could never win, when the number of trials is unlimited, while Paul is nearly sure to win $\frac{t}{t+1}$ écus. It is necessary therefore, in order to play evenly, that Pierre have an equal probability to win as much. Now, there is precisely as much probability for the first trial as for all the others together; thus, by doubling the stake of Paul, there will be as much probability that Pierre will win $\frac{2t}{t+1} - \frac{t}{t+1} = \frac{t}{t+1}$, as he has of it that he will not win it. Thus the formula of the Stake of Paul must be $e = \frac{2t}{t+1}$, when the number of trials is unlimited; or all the times that being limited, the value of $\frac{2t}{t+1}$ will be smaller than that of $\frac{1}{2}t$, which is the general formula of the geometers. In this calculation the value of the diverse stakes would vary from its *minimum* $\frac{1}{2}$ écu, to its *maximum* 2 écus, in the following order:

$$\frac{1}{2};\ 1;\ \frac{3}{2};\ \frac{8}{5};\ \frac{10}{6};\ \frac{12}{7};\ \frac{14}{8};\ \frac{16}{9};\ \frac{18}{10};\ \frac{20}{11};\ \cdots \frac{2\infty}{\infty+1}.$$

XX. 2. A simple metaphysical reasoning seems to lead to the same solution. Chance confines itself neither to the order, nor to the formulas: but the calculation of the probabilities supposes tacitly, as we have observed (art. IV) that chance distributes its favors with an impartial equity. Now, in the game of *heads* & *tails*, the most equitable referee would be quite embarrassed to decide for *heads* or *tails* on the first trial; setting aside some strange circumstances, he would be found exactly in the case of the freedom of indifference, & would choose the one of the two which would present itself first to the intellect; but in the following trial embarrassment would cease; *tails* would succeed undoubtedly to *heads*; & consequently the game would terminate itself infallibly on the second trial, if it had not ended on the first. The greatest value of the stake must be therefore that of the second prize$=$ 2 écus, & its least value the half of the first prize $= \frac{1}{2}$ écu; this which for a number of unlimited trials returns to the formula which we just found $e = \frac{2t}{t+1}$.

XXI. 3. One will arrive, if not to the same formula, at least to the same extreme values of the stake, by another consideration which seems rather plausible.

When the Geometers have calculated the stake of Paul, for whatever drawing t, they have found this stake equal to $\frac{1}{2}t$, by saying: the expectation of Paul for the greatest prize p is two times less probable than the expectation for the preceding prize $p-1$; this latter is in its turn two times less probable than the expectation for the prize $p-2$, & so forth, in descending to the first prize $= 1$ écu. Now the expectation for this prize 1 is $= \frac{1}{2}$, therefore that for the particular prize p is $= p \times \frac{1}{2} \times \frac{1}{2} \times (\cdots) = \frac{p}{2^t}$. But p is worth 2^{t-1} écus, therefore the expectation for this prize is worth $\frac{2^{t-1}}{2^t} = \frac{1}{2}$ écu, & consequently the expectation for all the prizes up to p inclusively is worth $\frac{1}{2}t$ écus. It is evident that in this calculation the Geometers are not satisfied to calculate the probability on the absolutely possible combinations, but on the most probable, & that this probability itself is evaluated not on the absolute number of possible cases, but on the better regular order that one supposes reigns in their respective existence. Because, as the celebrated author of the Doubts has quite well observed, there is for each prize $p = 2^{t-1}$ only $t+1$ possible cases, of which one alone wins this prize for Paul, one alone makes him him lose his stake, & the others in number $t-1$, wins some inferior prize for him: the expectation of Paul for prize p will be therefore expressed by the probability $\frac{1}{t+1}$; thus the sequence of expectations by retrograding would be

$$\frac{2^{t-1}}{t+1}, \frac{2^{t-2}}{t}, \frac{2^{t-3}}{t-1}, \ldots, \frac{2^0}{t-t+2},$$

now in the natural order, $\frac{1}{2}$; $\frac{2}{3}$; $\frac{4}{4}$; $\frac{8}{5}$; $\frac{16}{6}$ etc. & one would have the formula $e = \frac{2^{t-1}}{t+1}$, this which is quite different from $e = \frac{1}{2}t$.

Now the same reason which has made the stake fixed at $\frac{1}{2}t$, & not at $\frac{2^{t-1}}{t+1}$, proves, it seems to me, that this value $\frac{1}{2}t$ demands again a second reduction in order to be generally applicable, since it is more probable that *heads* will not be produced, twenty, thirty, fifty, one thousand, one hundred thousand times in sequence, than it is probable that one will produce it as many times. One knows that the successive or simultaneous arrangement of individuals of two different species can vary in as many ways, as there are units in the number 2 raised to the power which expresses the quantity of things that one wishes to arrange. If, for example, a wheel which contains many white & black tickets, one extracts from it *twelve*, at once or one after the other, it is certain that they can exit in 2^{12} different orders, & that of this great number of diverse arrangements, there is only one, out of 4096, which can produce twelve black tickets. There are therefore odds of $2^t - 1$ against 1, that this combination will not happen when one will extract a number t of tickets, either that one extracts them at once or by t extractions in sequence. Now the probability that the game will continue until the drawing t is equal to the probability of producing *heads* $t-1$ times in sequence, & this probability, as we just said, is only $\frac{1}{2^{t-1}}$. It seems therefore that the calculated expectation $\frac{1}{2}$ écu of winning the highest prize which corresponds to the number of trials t, must yet be weakened by the probability that there is to push the game to this number of trials t, this which would change the constant value of this expectation on each prize to a lesser and variable value $= \frac{1}{2^{t-1}}$ écus, for all the cases where $\frac{1}{2^{t-1}}$ is smaller than $\frac{1}{2}t$, & consequently the stake for an unlimited number of prizes will be:

$$e = \frac{1}{2^{t-1}} + \frac{1}{2^{t-2}} + \frac{1}{2^{t-3}} + \cdots + \frac{1}{2^{t-t}} = \frac{1+2+4+\cdots+2^{t-1}}{2^{t-1}} = \frac{2^t - 1}{2^{t-1}}.$$

So that the stake would approach always more to the value of 2 écus in measure as the number of trials increases, without nonetheless arriving at this value as when one will suppose the number of trials, either unlimited, or infinite. In fact, the least stake being $= \frac{1}{2}$ écu, the successive growth of this stake is expressed by the infinite series

$$\frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^\infty} = 1\frac{1}{2},$$

& the stakes themselves for a number of trials fixed in advance, will be

$$\frac{1}{2};\ 1;\ 1\frac{1}{2};\ \frac{15}{8};\ \frac{31}{16};\ \frac{63}{32};\ \text{etc.}$$

XXII. 4. The Geometers with the well-known formula $e = \frac{1}{2}t$ take into account truly already the small probability that the number of trials can go very far; this is that which reduces each particular prize, although increasing in geometric progression, to enter only as the value of a half-écu in the mean sum which constitutes the expectation & the stake of Paul. The question returns therefore to ask if this first reduction suffices, or if the nature of the subject requires yet a second. Now it seems that the expectation in each particular prize must be modified for two different reasons, the one is that the half of the possible cases is absorbed into the first trial, the fourth into the second trial, the eighth into the third trial; etc. the other reason is that the possibility of the possible cases remaining diminished also in measure as the number of trials is assumed greater. One can in fact represent the sums that Pierre offers by the ordinates of a logarithmic curve of which the axis would express the total number of trials. The calculation of the Mathematicians has reduced the mean value of these sums, or the expectation of Paul, to a parallelogram which has for height the half of the first ordinate, & for base the entire axis of the curve. But by this reduction the center of the figure, or, if I can express myself in this way, the center of the expectations changes continually in position, & advances in the end to correspond to the midle of the axis extended as far as infinity; it falls also very nearly on the last prize to which there is no likelihood to happen; than on the first prize which has the greatest probability for pouring itself out. It is very nearly as if, by extracting a white, one is satisfied to double the prize in measure as one would move the target away, & as one required of the shooter to proportion his wager to the distance. It is evident that beyond a certain range to which the shooter could expect to reach, the proportion would become more and more disadvantageous for him. It seems therefore that it is necessary to keep the logarithmic figure, bringing together the center of the expectations to the origin of the axis, & to see to it that whatever be the number of trials, its center remains always between the first two divisions of the axis; that is by leaving to the axis all its length, it is necessary to describe in the parallelogram a new logarithm, of which the greatest ordinate is towards the origin of the axis, while the others decrease to infinity; or, that which reverts to the same, it is only to take on the wrong side the first logarithm, & since the first prize is infinitely more assured than the last, to regard the greatest ordinate as that which represents the unit, & all the others as some fractions decreasing to infinity.

By envisioning the problem from this point of view, one will observe that the center of the expectations passes from one term of the series, or from one ordinate, to the other, by moving from its first point, all the time that the number of trials t is double. If $t = 4$, the stake $\frac{1}{2}t = 2$ écus correspond to the second ordinate, or to the third term of the sequence of prizes; if $t = 8$, the expectation is moved back to the 3rd prize; if $t = 16$, it falls on the fourth, & so forth. In order to fix it at the beginning of the second term, where the probability is the same on both sides of the series, it is necessary to take account of the difference that there is between the value of a deferred expectation, & that of a next expectation. Let therefore $t = 2^n$, the center of the expectation $\frac{1}{2}t$ will fall on the prize 2^{n-1} which is the n^{th} term of the series; the probability of reaching will consequently be $= \frac{1}{2^{n-1}}$, while the probability of reaching to prize 2, which corresponds to the fixed center of the expectations, is $= \frac{1}{2^{2-1}} = \frac{1}{2}$. One has therefore the inverse analogy:

$$\frac{1}{2} : \frac{1}{2^{n-1}} = \frac{1}{2}t : e, \quad \text{therefore } e = \frac{t}{2^{n-1}} = \frac{2t}{t} = 2 \text{ écus.}$$

A value which will hold, whatever be the number of drawings, provided that $\frac{1}{2^{2-1}}$ is not smaller than $\frac{1}{2^{n-1}}$, or, this which reverts to the same, that e is not found greater than $\frac{1}{2}t$; or in the end, as soon as t is a number over 3, & that consequently the ordinary formula would advance the center of the expectations beyond its fixed point.

In this account the diverse values of the stake would be

$$\begin{array}{lll}
\text{on } \textit{one} \text{ trial} & e = \frac{1}{2}t & = \frac{1}{2} \text{ écu} \\
\text{on } \textit{two} \text{ trials} & e = \frac{1}{2}t & = 1 \text{ écu} \\
\text{on } \textit{three} \text{ trials} & e = \frac{1}{2}t & = 1\frac{1}{2} \text{ écu} \\
\left.\begin{array}{l}\text{on } \textit{four} \text{ trials} \\ \text{\& beyond to infinity}\end{array}\right\} & e = \frac{1}{2}t & = 2 \text{ écus.}
\end{array}$$

But, as it suffices that the center of the expectations not pass from the second to the third term, there is no sufficient reason to suppose it entirely immobile, since in fact it could not be in the first trials, & that in measure as their number increases, the expectation acquires some small degree of further intensity; it seems therefore more natural to leave to it the latitude which results of the logarithmic space between the second & the third ordinate; & the stake represented by the logarithmic space will be

$$e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \text{etc. to infinity,}$$

or, that which reverts to the same $e = 1 + \frac{1}{2} + \frac{1}{4}$+etc. precisely as the solution of article XXI has given.

XXIII. One will find perhaps strange that the stake of Paul is so modified, while Pierre risks so enormous sums. I believe in fact that no person would wish to commit himself to pay these prizes for an unlimited number of trials, against a stake of two écus, or of two & a half écus: he would have however much less to risk without doubt to commit himself, if he would not have to pay in advance an exorbitant stake in order to seize probably in exchange only a prize of one écu or of two. But, as the probability is not decided by the event, it would be imprudent to chance large sums on the probability most exactly calculated; because besides it is easy to prove that in all the rules of the possible Pierre can risk all the prizes of the problem, against a stake of 2 écus. In fact, let the first prize be an écu, the second 2 écus, the third x''', the fourth x^{iv}, the fifth x^{v}, & thus in sequence to infinity. It is evident if *tails* coming on the first trial, Pierre wins *one* écu; if it is on the second, he neither wins nor loses; if it is on the third, he loses $x''' - 2$; on the fourth $x^{iv} - 2$, & thus in sequence to infinity.

Now, by admission of the Mathematicians, the probability of winning on the first trial is $= \frac{1}{2}$. That of loss on the 3rd is $= \frac{1}{8}$, on the 4th it is $= \frac{1}{16}$ etc. thus in proportion the probable loss to the probable gain, one will have:

$$\begin{array}{ll}
\frac{x''' - 2}{8} = 1 \times \frac{1}{2}, & \text{or } x''' - 2 = 4 \text{ écus} \\
\frac{x^{iv} - 2}{16} = 1 \times \frac{1}{2}, & \text{or } x^{iv} - 2 = 8 \\
\frac{x^{v} - 2}{32} = 1 \times \frac{1}{2}, & \text{or } x^{v} - 2 = 16 \\
\text{etc.} & \text{etc.}
\end{array}$$

whence one sees that the stake must not even go to 2 écus, if one supposes the number of trials, neither infinite, nor at least unlimited.

XXIV. 5. This consideration can lead to a sixth solution. Since the game must be fair, it is necessary that the expectation of Pierre compensate the risk to which he exposes himself. Now, as each trial considered separately can equally produce *heads* or *tails*, & as the first trial producing tails, excludes all the other trials, it is evident that the probability of terminating the game on the first trial, is also greater than that of terminating it on one of all the following trials, taken together. It is necessary therefore, (Note well as soon as one plays on more than one trial,) that the stake $= e$ is greater than the prize that Pierre promises if tails is produced on the first trial. By this means Pierre will have a probability $= \frac{1}{2}$ to win the surplus of the stake for the first prize, namely $e - 1$ écus; joined to the expectation more or less deferred as each trial will not produce tails. Equally therefore the gain that Pierre can make, multiplied by the probability of winning, to the sum that he can lose combined with the probability of losing this sum, one will have the following equation:

$$(e-1) \times \frac{1}{2} + e \times \frac{1}{2^n} = (2^{n-1} - e) \times \frac{1}{2^n}$$

an equation whence one must deduce the value of e.

Now there is here by the nature of the subject three cases to distinguish:

1°. when $n = 1$, that is when one has agreed to play only to a single trial, the first term $\frac{e-1}{2}$ has place no longer, because it represents here the case where Pierre would lose, where it would produce tails; a case which is expressed by the second member of the equation $\frac{2^{n-1}-e}{2^n}$. It is necessary therefore either to suppress $\frac{e-1}{2}$, & then the equation gives $e \times \frac{1}{2^n} = (2^{n-1} - e) \times \frac{1}{2^n}$, thus $e = \frac{1}{2}$ écu; or it is necessary to suppress the second member of the equation; & one will have the sum of the gain & of the loss, $(e-1)\frac{1}{2} + e \times \frac{1}{2^n} = 0$, this which gives equally $e = \frac{1}{2}$ écu.

2°. When $n = \infty$, that is to say when the number of trials is unlimited, the term $e \times \frac{1}{2^n}$ becomes infinitely small; consequently the equation gives for this case: $(e - 1)\frac{1}{2} = \frac{1}{2}$, therefore $e = 2$ écus.

3°. Between these two extreme values of the stake, the equation will give for such determined number of trials as one would wish above from (1) the general formula: $e = \frac{2^n}{2^{n-1}+2}$. Thus one will have the values of the stake as follows:

Number of trials	1	2	3	4	5	6	7	$\cdots$	∞
Value of the stake in écus	$\frac{1}{2}$	1	$\frac{4}{3}$	$\frac{8}{5}$	$\frac{16}{9}$	$\frac{32}{17}$	$\frac{64}{33}$	$\cdots$	2

XXV. One could object against this solution that, in the first member of the equation, one enters two expectations of Pierre, of which he can nonetheless ever realize but one; because if *tails* comes on the first trial, the expectation $e \times \frac{1}{2^n}$ no longer takes place; & if this latter takes place, the first trial has not produced tails. But it is necessary to consider that, in order to determine the greatest and the least stake, we have effectively made to vanish that of the two expectations which could not be realized, by reducing the equation, when $n = 1$ to $e \times \frac{1}{2^n} = (2^{n-1} - e) \times \frac{1}{2^n}$ & when $n = \infty$ to $(e-1)\frac{1}{2} = (2^{n-1} - e)\frac{1}{2^n}$. Now it would be absurd if between these two extreme cases, the stake be greater, than it is when the number of trials & of prizes is counted infinite; this is nonetheless that which would happen if one wished to take the mean value of the two expectations, & by forming the equation

$$\left((e-1)\frac{1}{2}+e\times\frac{1}{2^n}\right)\frac{1}{2}=(2^{n-1}-e)\frac{1}{2^n},$$

this which would give $e=(2^n\times 1\frac{1}{2})\left(\frac{1}{2^{n+1}+12}\right)$, a value which would exceed 2 écus, as soon as the number of trials would go beyond 3; although even here the greatest stake would be only three écus, for an infinite number of trials.

Excepting the two extreme cases, each of the two expectations compete to modify the value of the stake, one could exclude none; but these expectations themselves are in their turn determined by the value of the stake, & vary with the number of trials; as the following table indicates.

Number of trials	1	2	3	4	5	n	$\cdots$	∞
Value of the expectation $\frac{e-1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{3}{10}$	$\frac{7}{18}$	$\frac{2^{n-2}-1}{2^{n-1}+2}$	$\cdots$	$\frac{1}{2}$
Value of the expectation $e\times\frac{1}{2^n}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{18}$	$\frac{1}{2^{n-1}+2}$	$\cdots$	0

These two expectations have an equal value in the case which gives $2^{n-1}-1=1$, that is when one plays to three trials; but as the expectation $e\times\frac{1}{2^n}$ enters doubly in the determination of the stake, while the expectation $(e-1)\times\frac{1}{2}$ enters only one time, there is not any case where they contribute equally to fix this stake. If there were such a case, this would be when the value of e given by each of these expectations taken alone, would be the same: it would be necessary therefore to have $2^{n-2}=2^n\times\frac{1}{2^{n-1}+1}$, this which would suppose $2^{n-1}+1=4$, & consequently n equal to the fractional number $3\frac{1}{2}$.

XXVI. Moreover this problem having nothing of interest than the difficulty which resulted from its solution, it suffices, I believe, to have raised this difficulty without deciding among the different solutions that one could imagine in order to fix the precise value of the stake; whatever it is of those which I just proposed, there will result at least that the illustrious author of the Doubts has had some very good reasons to reject the stake increasing uniformly to infinity. I would not wish nonetheless to say with him that it is *physically impossible* that the same combination return constantly more than a certain number of times; because if at trial t, for example, the probability of producing again *heads* were null, it would be necessary not only that it had diminished successively until vanishing precisely at this latter trial, but it seems that it would be necessary again, that by taking a greater number of trials $t+n$, this probability became then negative, whence there would result that the stake would decrease in measure as the number of trials of which one would be agreeing would exceed the one where the probability of producing again *heads* would be null. Now it is evident that, if it is not absolutely necessary that the stake increase with the number of prizes, it is at least incontestable that it must not diminish in measure as those increase.

One can say in truth that the nature of the subject does not admit of negative probability; & that the physical impossibility results here from the vicissitudes attached to the ordinary course of nature. But it seems to me that, as soon as there is question of the simple probability & of its calculation, one would not need to infer from the order of nature that which I have inferred (art. VIII), a new probability, or all the more a moral certitude on the non-return of the physical & mechanical causes of an event always possible in itself, & which, by this likewise that it has already existed one or many times, can never become in rigor physically impossible. I swear that it is physically impossible that the state of things remain the same in a world where reigns a perpetual movement; but, as the same event can be produced in many different ways, it appears to me that all that one can legitimately conclude from the physical order, is that it is morally impossible, that is infinitely small probable, that a like fortuitous event always return.