Calcul des probabilités. Solution d'une problème du calcul des probabilités se rettachant avec élections.*

Jules Bienaymé

L'Institut 327, Vol. 8, (1840), pp. 119–120
Soc. Philomat. Paris Extraits, Ser. 5, 23–26.

Session of 21 March 1840

Mr. Jules Bienaymé states that he is arrived to resolve exactly a question of probabilities of which there existed no rigorous solution. The concern is to determine, among a great number of packs of cards taken at random in a great quantity of cards of two colors in given proportion, how many packs there are able to be found in which one of the two colors, designated in advance, carries away onto the other.

This problem merited that one research the true solution because it offers some special difficulties, and at the same time because it is susceptable of an interesting application. One knows effectively that it is the quite simple translation of an electoral question, and Mr. Bienaymé has care to point out that the first idea of this question does not belong to it, but that the solution of it which has been given is fundamentally wrong.

Here is the statement of this electoral problem. One supposes the very great number of the electors of a great country divided between two opinions in a known ratio; one supposes moreover the electors apportion at random into numerous colleges, and one demands what is, with a great probability, the number of the colleges in which the majority will belong to the opinion which possesses a known plurality in the electoral body.

The solution that this double problem had received was based on the erroneous application of a very true proposition, namely: that the probability to find the cards of a designated color, in plurality in any one of the packets, depends only on the number of cards of this packet, and not on the rank which it has been able to occupy in the repartition of the mass of given cards. In order that this theorem of probability conserves all exactitude, it is necessary that the packet considered is isolated from all others, and one have no regard to the composition of those, and that the ratio of the two colors in each is able to have all possible values. But one imagines that if one comes to envision

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. June 21, 2010

simultaneously two or many packets, the probability of the composition of an influence on the probability of the composition of the others; so that one is able, without paralogism, to conclude from the preceding theorem only the probability to find a color in plurality remains constant in a sequence of packets formed from an equal number of cards; or, if the packets are unequal, that this probability changes only by reason of the number of cards which they contain.

Despite the evident error of this conclusion, chance has wanted that it influenced little the numeric solution which had taken it for base. In order to explain this effect, Mr. Bienaymé recalls that the probabilities of the results of great numbers are expressed ordinarily by an integral, which is represented in the greater part by the problems of physics, etc., and of which the limits decide the same limits which he agrees to attribute to the most probable values. Moreover, these last limits are composed of two terms, of which the one is proportional to the number of the events considered, while the second term is proportional only to the square root of this number. Now, in the actual question, chance has wished that the term proportional to the number of the packets of cards, or of the electoral colleges, has received no alteration of the inexactitude of the reasoning. It has altered only the second term, which determines the magnitude of the limits of the probable values, and which is only proportional to the square root of the number of colleges. The rigorous solution will modify therefore only the extent of these limits. But it has given place to rectify in the expression of the first term an error of calculation. It is one of these errors which, in the astronomical calculations, have occasioned sometimes some rather lively contestations, because one attaches to the problems which they resolve a very great importance. It is the omission of a quantity of the order conserved in the approximation of which one makes this first term depend.

There results therefore from the researches of Mr. Bienaymé that the numerical values furnish by the solution of which it indicates the defect, will receive notable changes only within these limits, but the mean values will remain very nearly the same. Thus, for example, one had found that 208,000 electors, apportioned into 440 electoral colleges, ought to give to the opinion which the majority possessed of $\frac{1}{20}$ (which counts around 104,000 against 94,000), around 85 colleges out of 100. The rigorous solution will change with difficulty these numbers. The electors who account a plurality of $\frac{1}{20}$ should therefore carry away at least one voice in nearly 374 colleges. The electors in minority in the mass would obtain it however at least by one vote in 66 colleges. One has drawn from this disproportion between the ratio of the numbers of the colleges and the ratio of the majority to the minority, a consequence not very favorable to the system of elections. One has said that an opinion which possesses a very strong minority would have only a very small number of representatives; and that thus the representative system would be a deception, without the influences that create the reunion of the electors of the same locality and of other similar causes. Mr. Bienaymé believes that there is place to deduce from the results of the calculus a contrary consequence. In the older states where there existed only a single assembly, the strongest minority was necessarily suppressed. Now one sees that the distribution in colleges assures to a minority of $\frac{9}{20}$ around 66 deputies out of 440. It is thence a guarantee completely to the advantage of the modern system, which holds also more count of the individualities, and hence of the minorities, than the older system made it.

In finishing, Mr. Bienaymé further showed that his solution could be able, under the logical point of view, to present some interest, because it is by means of a pure artifice of analysis that he reused to extricate the problem from a set of very difficult reasonings to follow, if same difficulties that he has been able immediately to replace the ancient solution, although he had perceived the defect in it since the moment that it has been published. This defect exists indeed totally similar in the solution of the game of *trente et quarante*, in which Mr. Bienaymé had recognized previously. But the idea to employ the analytic artifice which leads to an exact solution has come to him only recently.