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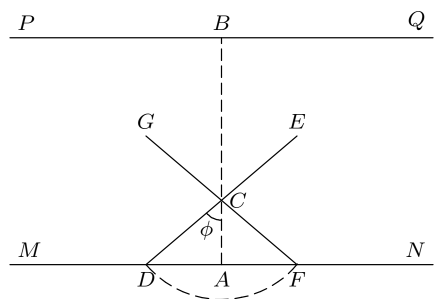
64. **Problem IX. (Buffon's Needle problem.)** *A plane is divided by parallel, equidistant straight lines into strips; a cylindrical, very thin needle, whose length equals at most the mutual distances of the parallels, is cast randomly onto the same. How great is the probability, that does the needle meet one of the lines of division?*

Solution. Let MN, PQ (Fig. 30) be any two adjacent lines of division; their interval $AB = a$. Assume, the midpoint of the needle falls at C, so that its distance from MN, that is $AC = x$; the related probability is $\frac{dx}{a}$. One describes from C with the half length r of the needle the arc of the circle DF, records the positions DCE, FCG of the needle, such that MN is met by this arc, if it falls within the angle $DCF = 2\phi$, the related probability is $\frac{2\phi}{\pi}$.



From this the entire compound probability follows, that MN is met by the needle, while its midpoint falls over MN,

$$\frac{2}{\pi a}\int_0^r \phi\, dx = \frac{2}{\pi a}\int_0^r \arccos\frac{x}{r} dx = \frac{2r}{\pi a};$$

the same value is produced on the condition, that the midpoint of the needle comes to lie under MN. The demanded probability is therefore

$$p = \frac{4r}{\pi a}.$$ [1]

65. **Historical Note.** The present problem was one of the first which has been put and solved out of the area of geometrical probability. It is noteworthy, that Buffon, whose area of research lay distant from mathematics, was he that succeeded to the solution to such a strange problem the correct way and thereby has laid the foundation for a new branch of the probability calculus.

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[1]A second solution, at the same time a generalization of this problem. See No. 82.

He opens Chapter XXIII of his "Essai d'Arithmétique Morale" with the words: "Analysis is the only instrument by which one is served until this day in the science of probabilities, to determine & to fix the ratios of risk; Geometry appeared ill-suited to a work so delicate; however if one considers it closely, it will be easy to recognize that this advantage of Analysis on Geometry, is completely accidental, & that risk according as it is modified & composed, is found as a result of geometry as well as that of analysis;. . . therefore to put Geometry in possession of its rights on the science of risk, the concern was only to invent some games which turn on size & on their ratios, or to reckon the small number of those of that nature which are already found." So then he turns himself to the solution of some problems concerning the game of "franc-carreau," which has in it, that one casts a coin onto a floor tiled with equal, regular tiles and wagers, that none or one, two, three. . .joints are covered. The principle used for the solution is correct, the separation of the favorable and unfavorable cases though is not always correctly accomplished.[2]

Buffon notices further, that the problem demands "somewhat more Geometry," if the cast piece possesses a form other than the circle, and then he passes to the needle problem, which he solves in a wholly correct manner with help of the integral calculus. The event, that the plane is covered instead of with one but with two sets of equidistant parallels, which they divide into congruent squares, he treats imperfectly. (See No. 68)

Apart from the simple examples concerning the "franc-carreau" game, concerning which Buffon had made already a short memorandum in 1733 to the academy (see Histoire de l'Academie de France pour 1733, page 43–45), was the composition of the "Essai d'Arithmétique Morale" placed, according to Gourand's "Histoire du Calcul des Probabilités depuis ses origines jusqu'à nos jours" (Paris 1848) in the time around 1760,[3] which was therefore described as the time of the foundation of the theory of geometrical probabilities.

From 1760 until 1812 nothing is recorded about the subject. In his "Théorie analytique des Probabilités" appearing in the last mentioned year Laplace treats Buffon's needle-problem again on pages 359–362, (3rd edition, 365–369) and a more difficult case of the same, where namely the plane has been divided into congruent rectangular areas, without mention however the origin of these problems.[4] They are constructed in Laplace's great work at the end of Chapter 5: "Application du Calcul des Probabilités, à la recherche des phénomènes et de leur causes." After he spoke of the application of the probability calculus to astronomy, physiology, medicine, political economics, to the influence of moral causes, to the investigation of games of luck, which complication does not permit a direct treatment and where therefore observations must be taken to help, there is conveyed the transition to that present problem in speech by the words: "Finally, one is able to make use of the calculus of probabilities in order to rectify curves or square their surfaces. Without doubt, geometers did not employ this means; but as it gives me place to speak of a particular kind of combinations of chance, I am going to expose it in a few words." The insertion of the problems into the scheme is therefore peculiar; it follows in a position where one should not suppose it.

For perhaps thirty years the new area of the probability calculus is cultivated namely by English and French mathematicians with preference; the vastly numerous problems bear witness to it, which especially appeared in English journals and that solutions repeatedly

[2]Moreover some of the results are distorted through misprints.

[3]See J. Todhunter "A History of the mathematical Theory of Probability from the time of Pascal to that of Laplace" (Cambridge and London, Macmillan, 1865), page 344.

[4]Subsequently Laplace frequently is described as the originator of the needle-problem.

have given occasion to an interesting exchange of opinion about the fundamentals of the new subject.

Two works on integral calculus, in fact J. Todhunter's "A Treatise on the Integral Calculus etc." (Cambridge and London, Macmillan, 2nd edition, 1862) and B. Williamson's "An elementary Treatise on the Integral Calculus etc." (London, Longmans, 3rd edition, 1880) have a special chapter devoted to the geometrical probability and the geometrical mean-value; in the last named work this chapter has the author Crofton.

66. **Comment.** Buffon's needle-problem offers further interest for this reason, that it is one of the few about geometrical probability, which was confirmed also by experimental means.

The essential difficulty with execution of such trials is due to the requirement that the experiments be so arranged so that the ideas of the random count be sustained. Random points are assumed on a somehow bounded planar surface, then, ever greater their number, their distribution over the surface should fall out all the more uniformly. But one finds, that the density of the points decreases towards the boundary of the figure, that is from this reason, since the requirement that the points should be assumed within the figure has to a certain extent kept back from the border and therefore has a lesser point density in their proximity. In order to encounter this influence, one must assume the points without regard to the boundary of the figure in the extended plane, and all points, which have fallen outside the figure, discard. Similar remarks apply over the assumption of points in lines, in space.

Likewise by assumption special care must be used with straight lines in a plane, that all directions, also possible positions, could appear with equal facility, that no tendency exists, to bring about this or that directions or situations more frequent than another.

Professor Dr. R. Wolf in Zürich, who has taken away for years numerous series of experiments toward the confirmation of the law of large numbers, also extended the same out of Buffon's needle-problem, that had become known to him from L. Lalanne's "Un million de faits" (Paris, 3rd edition, 1843), though without establishment of the outcome. On a table of 1 foot square a series of parallels are drawn at the mutual interval of 45 mm and from a knitting needle a piece of length 36 mm length broken out. Of the three series of experiments employed he has only here an interest in the third; there were executed 50 times each 100 casts, and with the "arbitrary direction" of the needle calculation to occur, the table maintained a constant rotation. Among each 100 casts was the number of events, where the parallel was met:

41	in	1	trial	53	in	2	trials,
42	"	3	"	54	"	1	"
43	"	2	"	55	"	1	"
45	"		"	56	"	2	"
46	"		"	57	"	1	"
47	"		"	58	"	1	"
48	"		"	59	"	1	"
49	"		"	60	"	2	"
50	"		"	61	"	1	"
51	"		"	62	"	2	
52	"		"	63	"	1	"

2532 in 50 trials$= 50 \times 100$ casts

From this the derived ratio of the number of favorable cases to the total number of experiments is

$$p' = \frac{2532}{5000} = 0,5064,$$

whereas the theoretical formula for $2r = 36$ and $a = 45$ produces

$$p = 0,5093.$$

The correspondence of the experiment with the theory could be described thus as a very satisfactory one, in that the difference of both outcomes amounts to only.$0,0029$.

Wolf has subjected the numbers of the first vertical series to an equalization according to the method of least squares, in that he regards them as the results of 50 equal exact observations; according to this count (see Wolf's Handbuch der Mathem., Phys. etc., Book 1, page 277) itself produces the most likely number of cases under 100 casts, where the parallels are met, equal to 50,64 with the average error $\pm 0,84$, after which the ratio of the favorable to the number of all trials itself varies between the average limits $0,5064 \pm 0,0084$, between which the theoretical value actually is contained.

Since Lalanne in the place cited makes the observation that one can arrive to an all the more exact determination of the number π through experiments in the described manner, the larger one makes the series of trials, so Wolf uses the outcomes of his experiments also to this account. From the theoretical formula follows

$$\pi = \frac{4r}{a} \cdot \frac{1}{p};$$

here one sets for p that value derived from the observation $p' = 0,5064$, and calculates with that average error $\mu = \pm 0,0084$ the average error of the foregoing function, so provides itself

$$\pi' = \frac{4r}{a} \cdot \frac{1}{p'} \pm \frac{4r}{a} \cdot \frac{1}{p'^2} \cdot \mu = 3,1596 \pm 0,0524;$$

here also falls the theoretical value of π between these limits delivered through the experiment.

Wolf's trials admit yet occasion to speak about it, how the experiment is to be arranged so a priori the greatest possible agreement with the theory may be expected with the given number of the attempts. Lalanne claims namely in the place cited: "The error will be the least possible for a given number of tests, when the length a of the needle will be equal to the fourth of the product of the interval d of the divisions by the ratio π," thus for $a = \frac{2\pi r}{4}$, and this corresponding piece of information has furnished Wolf with the dimensions in his attempts.

In a supplement to the previously cited work, which is thereby of special interest, when therein one Professor Rud. Merian of Basel has given the theoretical solution of the said needle-problem, Wolf says, Merian disputes the correctness of Lalanne's foregoing piece of information and makes the statement, that the greatest consensus between experiment and theory itself must result if $a = 2r$, that is the spacing of the parallels is made equal to the length of the needle. Wolf disputes this, defends the Lalanne's piece of information and takes a new series of 50×100 experiments, that according to Merian's supplied rule has delivered a less favorable outcome.

But Merian's piece of information is indeed correct. Because p signifies the a priori probability of an event, there is under s employed trials m of this favorable event, then one might expect according to Bernoulli's Theorem with the probability

$$\Pi = \frac{2}{\sqrt{\pi}} \int_0^\gamma \mathrm{e}^{-t^2} dt + \frac{\mathrm{e}^{-\gamma^2}}{\sqrt{2\pi s p(1-p)}}$$

the difference $p - \frac{m}{s}$ will be included between the limits $\pm\gamma\sqrt{\frac{2p(1-p)}{s}}$. But now with a given value of γ (or Π) and s these limits achieve the greatest extent for $p = \frac{1}{2}$, thus for $a = \frac{2\pi r}{4}$, Lalanne's rule is therefore completely wrong; on the other hand they actually become most narrow, if p assumes the largest value compatible with the conditions of the problem, that is for $a = 2r$ or $p = \frac{2}{\pi}$. That Wolf's second series of experiments produced less good agreement could be explained nevertheless from the only modest number of trials.