ESSAI D'ARITHMÉTIQUE MORALE

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I.

I do not undertake at all here to give some Essays on Morals in general; that would demand more enlightenment than I suppose myself of it, & more art of it than I recognize myself. The first & most wholesome part of morals, is rather an application of the maxims of our divine religion, than a human science; & I myself will take care well to dare to attempt some matters where the law of God makes our principles, & Faith our calculus. The respective recognition or rather the adoration that man owes to his creator; the fraternal charity, or rather the love which he owes to his neighbor, are natural sentiments & virtues written in a well made soul; all that which emanates from this pure source, carries the character of the truth; the wisdom of it is so lively that the illusion of the error cannot darken it, the evidence so grand that it admits neither reason, nor deliberation, nor doubt, & has no other measure than conviction.

The measure of the uncertain things make here my object, I am going to try to give some rules to estimate the ratios of truth, the degrees of probability, weights of testimony, the influence of chances, the inconvenience of risks; & to judge at the same time of the real value of our fears & of our expectations.

II.

There are some truths of different kinds, some certitudes of different orders, some probabilities of different degrees. The truths which are purely intellectual, as those of Geometry reduce themselves all to some truths of definition; the concern in order to resolve the most difficult problem is only to understand it well, & there is in the calculus & in the other purely speculative sciences, no other difficulties than those to disentangle that which we have set, & to loosen the knots that the human spirit has made to tie up & bind according to the definitions & the assumptions which serve of foundation & to progress in these sciences. All their propositions can always be demonstrated evidently, because one can always go up from each of these propositions to other antecedent propositions which are identical to them, & from those to others until to the definitions. It is by this reason that the evidence, properly said, belong to the mathematical sciences & belong only to them; because one must distinguish the evidence from the reasoning, from the evidence which comes to us by sense, that is to say, the intellectual evidence of corporal intuition; this is only a clear comprehension of objects & of images, the other is a comparison of similar or identical ideas; or rather it is the immediate perception of their identity.

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III.

In the physical sciences, the evidence is replaced by certitude; evidence is not susceptible to measure, because it has only a single absolute characteristic, which is the clear negation or affirmation of the thing which demonstrates it; but the certitude being never a positive absolute, has some relationships that one must compare & of which one can estimate the measure. Physical certitude, that is to say, the certitude most certain of all, is nevertheless only the near infinite probability as a result, an event which has never failed to happen, will arrive yet one time; for example, since the Sun is always risen, it is consequently physically certain that it will rise tomorrow; a reason in order to be, it is to have been, but a reason in order to cease to be, it is to have begun to be; & consequently one can not say that it is equally certain that the sun will rise always, at least to suppose to it an antecedent eternity, equal to the subsequent perpetuity, otherwise it will end because it has begun. Because we must judge for the future only by the view of the past; as soon as a thing has always been, or is always made in the same fashion, we must be assured that it will be or will be itself always of this same fashion: by *always,* I intend a very long time, & not an absolute eternity, the always of the future being never as equal to the always of the past. The absolute of any kind that it be, is neither of the activity of Nature nor of the one of the human spirit. Men have regarded as of the ordinary & natural effects, all the events which have this kind of physical certitude; an effect which always happens ceases to amaze us: to the contrary a phenomenon which would have never appeared, or which being always arrived in the same fashion, would cease to arrive or would arrive in a different fashion, would amaze us with reason, & would be an event which would appear so extraordinary, that we would regard it as supernatural.

IV.

Those natural effects which do not surprise us, have nonetheless all that which it is necessary in order to amaze us; what concurrence of causes, what assemblage of principles it is necessary to produce a single insect, a single plant! what prodigious combination of elements, of movements & of results in the animal machine! The smallest works of Nature are the subjects of the greatest admiration. That which makes that we are not at all astounded of all these marvels, is that we are born in this world of marvels, that we have always seen them, that our understanding & our eyes are equally accustomed; finally that all have been before & will make still after us. If we were born in another world with another form of body & other sense, we would have been in other relationships with the exterior objects, we would have seen other marvels & we would not have been more surprised by it; the ones & the others are based on the ignorance of causes, & on the impossibility to know the reality of things, of which it is permitted to us to perceive that the relations which they have with ourselves.

There are therefore two ways to consider the natural effects, the first is to see them such as they present themselves to us without paying attention to the causes, or rather without seeking causes in them; the second, is to examine the effects in the view of the relationship to some principles & to some causes; these two points of view are quite different & produce some different reasons of astonishment, the one causes the sensation of surprise, & the other gives birth to the sentiment of admiration.

V.

We will speak here only of this first manner to consider the effects of Nature; some incomprehensibles, however complicated that they appear to us, we judge them as most evident & most simple, & uniquely by their results; for example, we can not conceive nor even imagine why matter attracts itself, & we will content ourselves to be sure that it really attracts itself; we judge consequently that it is always attracted & that it will always continue to be attracted: it is likewise of other phenomena of each kind, however unbelievable that they can appear to us, we will believe them if we are sure that they are arrived very often, we will doubt if they have lacked as often as they are arrived, finally we will deny them if we believe to be sure that they have never arrived; in a word, according as we have seen & recognized them, or as we have seen & recognized the contrary.

But if experience is the base of our physical & moral knowledge, analogy is the first instrument of it, when we see that a thing arrives constantly in a certain fashion, we are assured by our experience that it will arrive still in the same fashion; & when one reports to us that a thing is arrived in such or such manner, if these facts have analogy with the other facts which we know by ourselves, consequently we believe them; to the contrary, if the fact has no analogy with the ordinary effects, that is to say, with the things which are known to us, we ought to doubt it; & if it is directly opposed to that which we know, we do not hesitate to deny it.

VI.

Experience & analogy can give us the different certitudes nearly equal & sometimes of the same kind; for example, I am nearly as certain of the existence of the city of Constantinople which I have never seen, as of the existence of the Moon which I have seen so often, & that because the testimonies of a great number can produce a certitude nearly equal to physical certitude, when they carry on some things which have a full analogy with those which we know. Physical certitude must be measured by an immense number of probabilities, since this certitude is produced by a constant series of observations, which are those which one calls the *experience of constancy*. Moral certitude must be measured by a smaller number of probabilities, since it supposes only a certain number of analogies with that which is known to us.

In supposing a man who had never seen anything, heard anything, we seek how the belief & the doubt would be produced in his mind; suppose him struck for the first time by the aspect of the sun; he sees it shine in the height of the Heavens, next to decline & finally disappear; what can he conclude? nothing, except that he has seen the sun, that he has seen it follow a certain route, & that he no longer sees it; but this star reappears & disappears again the following day; this second vision is a first experience, which must produce in him the expectation to see the sun again, & he begins to believe that he would be able to see again, however he doubts it much; the sun reappears anew; this third vision makes a second experience which diminishes the doubt as much as it increases the probability of a third return; a third experience increases it to the point that he scarcely doubts more that the sun returns a fourth time; & finally when he will have seen this star of light to appear & to disappear regularly ten, twenty, one hundred times in sequence, he will believe to be certain that he sees it always to appear, disappear & to move itself in the same fashion; the more he will have similar observations, the more the certitude to see the sun rise the following day will be great; each observation, that is to say, each day, produces a probability, & the sum of these reunited probabilities, as soon as it is very great, gives physical certitude; one can always express this certitude by numbers, by dating from the origin of time to our experience, & it will be likewise of all the other effects of Nature; for example, if one wishes to reduce here the age of the world & of our experience to six thousand years, the sun is risen for us[1] only 2 million 190 thousand times, & as to date from the second day which it is risen, the probabilities to rise the following day increases, as the sequence 1, 2,

[1]I say for us, or rather for our climate, because this would not be exactly true for the climate at the poles.

4, 8, 16, 32, 64 . . . or 2^{n-1}. One will have (when in the natural sequence of numbers, n is equal to 2,190000), one will have, I say, $2^{n-1} = 2^{2,189999}$, this which is already a number so prodigious that we ourselves can form no idea of it, & it is for this reason that one must regard the physical certitude as composed of an immensity of probabilities; since by deferring the date of creation only by two thousand years, this immensity of probabilities becomes 2^{2000} times more than $2.^{2,189999}$

VII.

But it is not so easy to make the estimation of the value by analogy, nor by consequently to find the measure of moral certitude; it is in truth the degree of probability which makes the force of the analogous reasoning; & in itself analogy is only the sum of the ratios with the known things; nevertheless according as this sum or this ratio in general will be more or less great, the consequence of the analogous reasoning will be more or less sure, without however ever being absolutely certain; for example, if a witness which I suppose of good sense, says to me that there comes to be born an infant in this city, I will believe him without hesitation, the fact of the birth of an infant having nothing but of ordinary strength, but having to the contrary an infinity of relationships with the known things, that is to say with the birth of all the other infants, I will believe therefore this fact without however being absolutely certain; if the same man said to me that this infant is born with two heads, I would believe it again, but more weakly, an infant with two heads having less relationship with known things; if he would add that this new born has not only two heads, but that it has further six arms & eight legs, I would have with good reason difficulty to believe it, & yet however weak that my belief was, I would not be able to refuse it of him entirely; this monster, although quite extraordinary, being nevertheless composed only of parts which have each some relationship with the known things, & having only their assemblage & their number quite extraordinary. The force of analogous reasoning will be always therefore proportional to the analogy itself, that is to say, to the number of the relationships with the known things, & it will not be of concern to make a good analogous relationship, but to set itself well to the fact of all the circumstances, to compare them with the analogous circumstances, to sum the number of those, to take next a model of comparison to which one will return this found value, & one will have the probability to the just, that is to say, the degree of force of the analogous reasoning.

VIII.

There is therefore a prodigious distance between physical certitude & the kind of certitude which one can deduce from the greater part of the analogies; the first is an immense sum of probabilities which force us to believe; the other is only a probability more or less great, & often so small that it leaves us in perplexity. Doubt is always in inverse ratio to the probability, that is to say, that it is so much greater as the probability is smaller. In the order of certitudes produced by analogy, one must place the moral certitude; it seems even to hold the middle between doubt & physical certitude; & this middle is not a point, but a very extended line, & of which it is quite difficult to determine the limits: one senses well that it is a certain number of probabilities which make moral certitude, but what is this number? & can we expect to determine it so precisely as the one by which we come to represent the physical certitude?

After having reflected, I have thought that of all the possible moral probabilities, that which affects most men in general, is the fear of death, & I have sensed consequently that all fear or all expectation, of which the probability would be equal to that which produces the fear of death, can in the moral be taken for unity to which one must report the measure of the other fears; & I report likewise that of the expectations, because there is no difference between the expectation & the fear, than that of the positive to the negative; & the probabilities of both must be measured in the same manner. I seek therefore what is really the probability that a man who carries himself well, & who consequently has no fear of death, dies nonetheless in twenty-four hours: In consulting the Tables of mortality, I see that one can deduce from it, that there are only odds of ten thousand one hundred eighty-nine against one, that a man of fifty-six years, will live more than a day.[2] Now as each man of this age, where reason has acquired all its maturity & experience all its force, has nonetheless no fear of death in the twenty-four hours, although he has only odds of ten thousand one hundred eighty-nine against one, that he will not die in this short interval of time; I conclude from it, that each probability equal or smaller, must be regarded as null, & that each fear or each expectation which is found below ten thousand, must neither affect us, or even occupy us a single instant the heart or the head.[3]

In order to make me better understood, suppose that in a lottery where there is only one lot & ten thousand tickets, a man takes only one ticket, I say that the probability to obtain the lot being only one against ten thousand, his expectation is null, since there is no more probability, that is to say, by reason of the expectation of the lot, than he has to fear death in twenty-four hours; & that this fear affecting it in no fashion, the expectation of the lot must not affect it further, & even still much less, since the intensity of the fear of death is quite greater than the intensity of all other fears or of all other expectation. If in spite of the evidence of this demonstration, this man persisted in wishing to hope, & that a similar lottery is drawn every day, he took each day a new ticket, counting always to obtain the lot, one could, in order to undeceive him, to wager with him end to end, that he would die before having won the lot.

Thus in all games, the wagers, the risks, the chances; in all the cases, in a word, where the probability is smaller than $\frac{1}{10000}$, it must be, & it is in effect for us absolutely null; & by the same reason in all the cases where this probability is greater than 10000, it makes for us the most complete moral certitude.

IX.

Thence we can conclude that the physical certitude is to the moral certitude as $2^{2189999}$: 10000; & that all the time that an effect, of which we are absolutely ignorant of the cause, arrives in the same fashion, thirteen or fourteen times in sequence, we are morally certain that it will arrive again likewise a fifteenth time, because $2^{13} = 8192$, & $2^{14} = 16384$, & consequently when this effect is arrived thirteen times, there are odds of 8192 against

[2] See above the result of the Tables of mortality.

[3] Having communicated this idea to Mr. Daniel Bernoulli, one of the greatest Geometers of our century, & the most versed of all in the science of probabilities; here is the response which he has made to me by his letter, dated at Basel 19 March 1762.

"I approve strongly, Sir, your manner to estimate the limits of the moral probabilities; you consult the nature of man by his actions, & you suppose in fact, that a person is not worried the morning if he will die this day; this being, as he dies, according to you, one out of ten thousand, you conclude that one ten-thousandth of probability must make no impression on the mind of man, & consequently that this ten-thousandth must be regarded as an absolute nothing. It is without doubt to argue Philosophy in Mathematics; but this ingenious principle seems to lead to a smaller quantity, because the exemption from fear is not assuredly in those who are already sick persons. I do not combat your principle, but it appears rather to lead to $\frac{1}{100000}$ than to $\frac{1}{10000}$."

I confess to Mr. Bernoulli, that as the ten-thousandth is taken from the Tables of mortality which never represent but the *average man*, that is to say, men in general, in good health or sick, sane or infirm, vigorous or feeble, there is perhaps a little more than odds of ten thousand against one, that a man in good health, sane & vigorous will not die in twenty-four hours; but it is quite necessary that this probability must be increased to one hundred thousand. Moreover, this difference, although very great, changes nothing to the leading consequences which I draw from my principle.

1, that it will arrive a fourteenth time; & when it is arrived fourteen times, there are odds of 16384 against 1, that it will arrive likewise a fifteenth time, this which is a greater probability than that of 10000 against 1, that is to say, greater than the probability which makes moral certitude.

One can perhaps say to me, that although we have no dread or fear of sudden death, it is quite necessary that the probability of sudden death be zero, & that its influence on our conduct be null morally. A man of whom the soul is good, when he loves someone, would he not reproach himself to retard by one day the measures which must assure the happiness of the loved person? If a friend entrusts to us a considerable deposit, do we not put the same day a note to this deposit? we act therefore in these cases, as if the probability of the sudden death were some thing, & we have reason to act thus. Therefore one must not regard the probability of sudden death as null in general.

This kind of objection will vanish, if one considers that one makes often more for the others, than one would not make for oneself! when one puts a note at the same moment that one receives a deposit, it is uniquely by honesty for the propriety of the deposit, for his tranquility, & not at all by the fear of our death in twenty-four hours; it is likewise of the readiness that one sets to make the happiness of someone or ours, it is not the sentiment of the fear of a death so near which guides us, it is our proper satisfaction which animates us, we seek to enjoy in all as soon as possible that it is possible to us.

A reasoning which could appear more founded, is that all men are carried to flatter themselves; that hope seems to be born in a lesser degree of probability than fear; & that consequently one is not in the right to substitute the measure of the one by the measure of the other: fear & hope are of sentiments & not of determinations; it is possible, it is even more than possible that these sentiments are not measured out of the precise degree of probability; & consequently must one give to them an equal measure, or even assign to them any measure?

To this I respond, that the measure of which there is question is not carried on the sentiments, but on the reasons which must give birth to them, & that all wise men must estimate the value of these sentiments of fear or of hope only by the degree of probability; because when even Nature, for the happiness of man, would have given to him more slope towards hope than towards fear, it is not less true of it that the probability is the true measure & of the one & of the other. It is likewise only by the application of this measure that one can undeceive oneself out of his false hopes, or reassure oneself out of his ill-based fears.

Before ending this article, I must observe that it is necessary to take guard to be mistaken out of that which I have said of the effects of which we do not know the cause; because I intend only the effects of which the causes, although unknown, must be supposed constants, such as those of natural effects; each new discovery in physics established by thirteen or fourteen experiences, which all confirm it, have already a degree of certitude equal to the one of moral certitude, & this degree of certitude increases with the double of each new experience; so that by multiplying them, one approaches more and more physical certitude. But it is not necessary to conclude from this reasoning, that the effects of chance follow the same law; it is true that in a sense these effects are of the number of those of which we are ignorant of the immediate causes; but we know that in general these causes to be able to be supposed constants quite remotely, are to the contrary necessarily variables & inconstant as much as is possible. Thus by the notion even of chance, it is evident that there is no liaison, no dependence among these effects; that consequently the past can influence nothing on the future, & one would be much & even completely mistaken, if one wished to infer from anterior events, some reason for or against posterior events. That one card, for example, has won three times in sequence, it is not less probable that it will win a fourth time, & one can wager equally that it will win or that it will lose, any number of times that it has won or lost, as soon as the law of the game is such that the chances are equal. To presume or to believe the contrary, as certain players do, is to go contrary to the principle even of chance, or not to remember the conventions of the game, it is always equally apportioned.

X.

In the effects of which we see the causes, a single evidence suffices to bring about physical certitude; for example, I see that in a clock the weights make the wheels turn, & that the wheels make the pendulum go, I am certain consequently, without having need of repeated experiences, that the pendulum will always go the same, as long as the weights will make the wheels turn; this is a necessary consequence of an arrangement which we have made ourselves in constructing the machine; but when we see a new phenomenon, an effect in Nature yet unknown, as we are ignorant of the causes, & as they can be constants or variables, permanent or intermittent, natural or accidental, we have no other ways to acquire certitude, but the experience repeated as often as it is necessary; here nothing depends on us, & we know only that we experiment; we are assured only by the same effect & by the repetition of the effect. As soon as it will be arrived thirteen or fourteen times in the same manner, we have already a degree of probability equal to moral certitude that it will arrive likewise a fifteenth time, & from this point we have soon to cross over an immense interval, & to conclude by analogy that this effect depends on the general laws of Nature, that it is consequently as ancient as all the other effects, & that there is physical certitude that it will arrive always as it is always arrived, & that there is lacking to it only to have been observed.

In chances that we have arranged, balanced & calculated ourselves, one must not say that we are ignorant of the causes of the effects: we are ignorant in truth of the immediate cause of each effect in particular; but we see clearly the first & general cause of all the effects. I am ignorant, for example, & I can not even imagine in any fashion, what is the difference of the movements of the hand, in order to pass or not pass ten with three dice, that which nevertheless is the immediate cause of the event, but I see evidently by the number & the mark of the dice which are here the first & general causes that the chances are absolutely equal, that it is indifferent to wager that one will pass or that will not pass ten; I see moreover, that these same events, when they succeed themselves, have no liaison, since at each coup of dice the chance is always the same, & nevertheless always new; that the past coup can have no influence on the coup to come; that one can always wager equally for or against, that finally the more long time one will play, the more the number of the effects for, & the number of the effects against, will approach equality. So that each experience gives here a product entirely opposed to the one of the experiences out of the natural effects, I wish to say, the certitude of the variability instead of that of the constancy of the causes; in those each evidence increases in the double the probability of the return of the effect, that is to say, the certitude of the constancy of the cause; in the effects of chance each evidence to the contrary increases the certitude of the variability of the cause; by demonstrating to us always more and more that it is absolutely inconstant and totally indifferent to produce one or the other of these effects.

When a game of chance is by its nature perfectly equal, the player has no reason to determine himself to such or such part; because finally, from the supposed equality of this game, there results necessarily that there is no good reason at all to prefer the one or the other part; & consequently if one deliberated, one could be determined only by some wrong reasons; thus the logic of the players has appeared to me entirely vicious, & even the good minds who permit themselves to play, fall in capacity of players, into some absurdities of which they are ashamed soon in capacity of reasonable men.

XI.

Moreover, all this supposes that after having balanced the chances & having rendered them equal, as in the game of *passe-dix* with three dice, these same dice which are the instruments of chance, are so perfect that it is impossible, that is to say, that they are exactly cubical, that the material of them is homogeneous, that the numbers are painted & not marked in hollows, in order that they not weigh more on one face than on another; but as it is not given to man to make anything perfect, & that there are no dice at all worked with this rigorous precision, it is often possible to recognize by observation, on which side the imperfection of the instruments of the sort made the chance lean. It is necessary for this only to observe attentively & long time the sequence of events, to count them exactly, to compare the relative numbers; & if of these two numbers the one exceeds by much the other, one can conclude from them, with good reason, that the imperfection of the instruments of the sort, destroyed the perfect equality of chance, & gives to it really a tendency stronger to one side than the other. For example, I suppose that after playing at *passe-dix*, one of the players was rather cunning, or to say better, rather rascally in order to have cast in advance one thousand times the three dice of which one must serve oneself, & to have recognized that in these one thousand evidences there had been six hundred which have passed ten, there will be consequently a very great advantage against his adversary in wagering to pass, since by experience the probability to pass ten with these same dice, will be to the probability to not pass ten:: $600 : 400 :: 3 : 2$. This difference which results from the imperfection of the instruments can therefore be recognized by observation, & it is by this reason that the players change often the dice & cards, when their fortune is contrary.

Thus however obscure that the destinies be, however impenetrable that the future appears to us, we could nevertheless by some repeated experiences, become, in some cases, as clear on the future events, as could be some beings or rather some superior natures who could deduce immediately the effects of their causes. And in the same things which appear to be pure chance, as the games & lotteries, one can still know the tendency of chance. For example, in a lottery which is drawn all fifteen days, & of which one publishes the winning numbers, if one observes those which have most often won during a year, two years, three years in sequence, one can deduce from it, with reason, that these same numbers will win again more often than the others; because in some manner that one can vary the movement & the position of the instruments of the lot, it is impossible to render them perfect enough in order to maintain the absolute equality of chance; there is a certain routine to make, to place, to mix the tickets, which in the breast even from the confusion produces a certain order, & makes that certain tickets must exit more often than the others; it is likewise of the arrangement of the cards to play, they have a kind of sequence of which one can grasp some terms by force of observations; because in assembling them by the worker one follows a certain routine, the player himself in shuffling them has a routine; the whole makes itself in a certain fashion more often than another, & consequently the observer attentive to the results collected in great number, will wager always with great advantage that one such card, for example, will follow another such card. I say that this observer will have a great advantage, because the chances before being absolutely equal, the least inequality, that is to say, the least degree of probability more, has very great influences in the game, which is in itself only a wager multiplied & always repeated. If this difference recognized by experience of the tendency of chance was only of a hundredth, it is evident that in one hundred coups, the observer would win his stake, that is to say, the sum which he has chanced at each time; so that a player supplied with these dishonest observations, can not fail to ruin at length all his adversaries. But we are going to give a powerful antidote against bad epidemic of the passion of the game, & at the same time some preservatives against the illusion of this dangerous art.

XII.

One knows in general that the game is an avid passion, of which the practice is ruinous, but this truth has perhaps never been demonstrated but by a sad experience on which one has not enough reflection in order to correct oneself by the conviction. A player, of which the fortune exposed each day to the coups of chance, undermines himself little by little & finds himself finally necessarily destroyed, attributes his loses only to this same chance which he accuses of injustice; he regrets equally both that which he has lost and that which he has not won; the greed & the false hope made to him some rights on the wealth of others; also humility to be found in the necessity that afflicted to have no longer means to satisfy his cupidity; in his despair he takes himself to his unlucky star, he does not imagine that this blind power, the fortune of the game, marches to the truth of an indifferent & uncertain step, but that to each walk it tends nevertheless to an end, & draws in a certain term what is the ruin of those who attempt it; he sees not that the apparent indifference which it has for good or for ill, produces with time the necessity of the bad, that a long sequence of chances is a fatal chain, of which the elongation brings misfortune; he senses not that independently of the harsh tax of the cards & of the tribute yet more harsh which he has paid to the knavery of some adversaries, he has passed his life to make some ruinous agreements; that finally the game by its same nature is a vicious contract as far as in its principle, a hurtful contract to each contractor in particular, & contrary to the good of all society.

This is not at all a discourse on vague morals, they are some precise truths of metaphysics which I submit to the calculus or rather to the force of reason; some truths which I claim to demonstrate mathematically to all those who have the mind sharp enough, & the imagination strong enough to combine without geometry & to calculate without algebra.

I will speak not at all of those games invented by artifice & computed by greed, where chance loses a part of his rights, where fortune can never balance, because it is invincibly carried away & always constrained to tend to one side, I wish to say all these games where the chances unequally apportioned, offer a gain so assured as dishonest to one, & leaves to the other only a certain & shameful loss, as in *Pharaon*, where the banker is only an avowed knave, & the punter a dupe, of whom one is agreed not to mock.

It is in the game in general, in the game most equal, & consequently the most honest that I find a vicious essence, I understand even under the name of game, all the agreements, all the wagers where one puts to chance a part of his wealth in order to obtain a similar part of the wealth of another; & I say that in general the game is an ill-understood pact, a disadvantageous contract to the two parties, of which the effect is to render the loss always more great than the gain; & to subtract from the good in order to add to the harm. The demonstration of it is as easy as evident.

XIII.

Take two men of equal fortune, who, for example, have each one hundred thousand livres of wealth, & suppose that these two men play in one or many coups of dice fifty thousand livres, that is to say, the half of their wealth; it is certain that the one who wins increases his wealth only by one third, & the one who loses, diminishes his by half; because each of them have one hundred thousand livres before the game, but after the event of the game, one will have one hundred fifty thousand livres, that is to say, a third more than he had, & the other has no more than fifty thousand livres, that is to say, half less than he had; therefore the loss is a sixth part greater than the gain; because there is this difference between the third & the half; therefore the agreement is harmful to both, & consequently essentially vicious.

This reasoning is not at all specious, it is true & exact, because although one of the players has lost precisely only that which the other has won; this numerical equality of the sum, does not prevent the true inequality of the loss & of the gain; the equality is only apparent, & the inequality very real. The pact that these two men make by playing the half of their wealth, is equal for the effect to another pact that never is a person advised to make, which would be to agree to cast into the sea each the twelfth part of his wealth. Because one can demonstrate to them, before they chance this half of their wealth, that the loss being necessarily a sixth greater than the gain, this sixth must be regarded as a real loss, which can fall indifferently either on the one or on the other, must consequently be equally shared.

If two men ventured to play all their wealth, what would be the effect of this agreement? one would do only to double his fortune, & the other would reduce his to zero; now what proportion has he here between the loss & the gain? the same as between all & nothing; the gain of one is only equal to a moderate enough sum, & the loss of the other is numerically infinite, & morally so great, that the work of all his life suffices not perhaps to regain his wealth.

The loss is therefore infinitely greater than the gain when one plays all his wealth; it is greater by one sixth part when one plays the half of his wealth, it is greater by one twentieth part when one plays the fourth of his wealth; in a word, some small portion of his fortune that one chances in a game, there is always more loss than of gain; thus the pact of the game is a vicious contract, & which tends to the ruin of the two contractors. New truth, but very useful, & as I desire what is known of all those who, by cupidity or by idleness, pass their life to tempt chance.

One has often demanded why one is more sensitive to the loss than to the gain; one was not able to make a fully satisfying response to this question, so long as one has doubt of the truth I just presented; however the response is easy: one is more sensitive to the loss than to the gain, because indeed, by supposing them numerically equal, the loss is nonetheless always & necessarily greater than the gain; the sentiment is in general only a reasoning implicitly less clear, but often more refined, & always more sure as the direct product of reason. One sensed well that the gain did not give us as much pleasure as the loss caused us pain; this sentiment is only the implicit result of the reasoning I just presented.

XIV.

Silver must not be estimated by its numerical quantity: if the metal, which is only the sign of the wealth, were the wealth itself, that is to say, if the good luck or the advantages which result from the wealth, were proportional to the quantity of silver, men would have reason to estimate numerically & by its quantity, but it is quite necessary that the advantages which one draws from silver, is in just proportion to its quantity; a man rich to one hundred thousand écus of pension, is not ten times happier than the man who has only ten thousand écus; there is more, it is that silver, as soon as one passes from certain boundaries, has nearly no longer real value, & wealth can not be increased of the one who possesses it; a man who would discover a mountain of gold, would not be more rich than the one who would find only a cubic toise of it.

Silver has two values both arbitrary, both of agreement, of which the one is the measure of the advantage of the particular, & of which the other makes the tariff of the wealth of the society; the first of these values has never been estimated but in a quite vague manner; the second is susceptible to a just estimation by the comparison of the quantity of silver with the product of the earth & of the work of men.

In order to arrive to give some precise rules on the value of silver, I will examine some particular cases of which the mind grasps easily the combinations, & which, as of the examples, will lead us by induction to the general estimation of the value of silver for the poor, for the rich, & even for the man more or less wise.

For the man who in his state, whatever it be, has only the necessary, silver is of an infinite value; for the man who in his state abounds in superfluous, silver has nearly no more value. But what is the necessary, what is the superfluous? I intend by the necessary *the expense that one is obliged to make in order to live as one has always lived*, with this necessary one can have his comforts & even some pleasures; but soon habit has made some needs; thus in the definition of the superfluous, I will count for nothing the pleasures to which we are accustomed, & I say that *the superfluous is the expense which can procure us some new pleasures*; the loss of the necessary is a loss which makes itself infinitely felt, & when one chances a considerable part of this necessary, the risk can be compensated by no hope, however great that one supposes it; on the contrary the loss of the superfluous has some limited effects; & if in the superfluous even one is still more sensitive to the loss than to the gain, it is because in effect the loss being in general always greater than the gain, this sentiment is found based on this principle, that the reasoning was not developed, because the ordinary sentiments are based on some common notions or on some easy inductions; but the delicate sentiments depend on exquisite & lofty ideas, & are indeed only the results of many combinations often too fine to be perceived clearly & nearly always too complicated in order to be reduced to a reasoning which can demonstrate them.

XV.

Petersburg problem

Mathematicians who have calculated the games of chance, & of whom the researches in this genre merit some praise, have considered silver only as a quantity susceptible to increase & to decrease, without other value than that of number; they have estimated by the numerical quantity of silver, the relationships of the gain & of the loss; they have calculated the risk & the expectation relatively to this same numerical quantity. We consider here the value of the silver from a different point of view, & by our principles we will give the solution of some cases embarrassing for the ordinary calculus. This question, for example, of the game of heads & tails, where one supposes that two men (Pierre & Paul) play one against the other, on these conditions that Pierre will cast into the air a piece of coinage as many times as it will be necessary in order that it present heads, & that if his arrives on the first toss, Paul will give to him an écu; if this arrives only on the second toss, Paul will give to him two écus; if this arrives only on the third toss, he will give to him four écus; if this arrives only on the fourth toss, Paul will give eight écus; if this arrives only on the fifth toss, he will give sixteen écus, & thus in sequence by doubling always the number of écus: it is clear that by this condition Pierre can only win, & that his gain will be at least an écu, perhaps two écus, perhaps four écus, perhaps eight écus, perhaps sixteen écus, perhaps thirty-two écus, &c. perhaps five hundred twelve écus, &c. perhaps sixteen thousand three hundred eighty-four écus, &c. perhaps five hundred twenty-four thousand four hundred forty-eight écus[4], &c. perhaps even ten million, one hundred million, one hundred thousand million écus, perhaps finally an infinity of écus. Because it is not impossible to cast five times, ten times, fifteen times, twenty times, a thousand times,

[4]*Translator's note:* Buffon is in error here. It should be 524288.

one hundred thousand times the coin without that it present heads. One demands therefore how much Pierre must give to Paul in order to indemnify him, or that which reverts to the same, what is the sum equivalent to the expectation of Paul who can only win.

This question had been proposed to me for the first time by the late Mr. Cramer, celebrated Professor of Mathematics at Geneva, during a tour that I made in that city in the year 1730; he said to me, that it had been proposed previously by Mr. Nicolas Bernoulli to Mr. de Montmont, as indeed one finds it *pages 402 & 407* of the Analyse of the games of chance, of this Author: I dreamt some times on this question without finding the knot of it; I did not see that it was possible to accord the mathematical calculus with good sense, without making some moral considerations enter; & having made part of my ideas to Mr. Cramer,[5] he said to me that I had reason, & that he had also resolved this question by a similar way; he indicated to me next the solution very nearly like that one has printed since in the Mémoires de l'Académie de Pétersbourg, in 1738, behind an excellent memoir of

[5]Here is that which I left of it then by writing to Mr. Cramer, & of which I have preserved the original copy. "Mr. de Montmort is content to respond to Mr. Nic. Bernoulli, that the equivalent is equal to the sum of the sequence $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, &c. écus continued to infinity, that is to say, $= \frac{\infty}{2}$, & I do not believe that indeed one can contest his mathematical calculation; however far to give an equivalent infinity, there is no man at all of good sense who wished to give twenty écus, nor even ten."

"The reason of this contradiction between the mathematical calculus & good sense, seems to me to consist in the little proportion that there is between money & the advantage which results from it. A Mathematician in his calculus, estimates money only by its quantity, that is to say, by his numerical value; but the moral man must estimate otherwise & uniquely by the advantages or the pleasure which he can procure; it is certain that he must be led to this view, & to estimate money only in proportion of the advantages which result from it, & not relatively to the quantity which, past of certain limits, could not at all increase his happiness; he could be, for example, scarcely more happy with a thousand millions than he would be with one hundred, nor with one hundred thousand millions, more than with one thousand millions; thus past certain limits, he would be very much wrong to chance his money. If, for example, ten thousand écus were all his wealth, he would be infinitely wrong to chance them, & the more these ten thousand écus will be an object with respect to him, the more it will be wrong; I believe therefore that his wrong would be infinite, as long as these ten thousand écus will make a part of his necessary, that is to say, as long as these ten thousand écus will be to him absolutely necessary for life, as he has been raised & as he has always lived; if these ten thousand écus are of his superfluous, his wrong diminishes, & the more they will be a small part of his superfluous & the more his wrong will diminish: but it will never be null, unless he can regard this part of his superfluous as indifferent, or else unless he regard the expected sum as necessary in order to succeed in a design which will give to him in proportion, as much pleasure as this same sum is greater than that which he chances, & it is in this fashion to envision a happiness to come, that one can not give at all rules, there are some people for whom the expectation itself is a pleasure greater than those which they could be able to procure themselves by the enjoyment of their stake; in order to reason more certainly on all these things, it would be necessary to establish some principles; I would say, for example, that the necessary is equal to the sum which one is obliged to expense in order to continue to live as one has always lived; the necessary of a King will be, for example, ten millions of pension (because a King who would have less, would be a poor King); the necessary of a man of condition, will be ten thousand livres of pension (because a man of condition who would have less, would be a poor lord); the necessary of a peasant will be five hundred livres, because unless to be in misery, he can expense no less to live & nourish his family. I would suppose that the necessary can procure us some new pleasures, or in order to speak more exactly, I would count for nothing the pleasures or advantages which we have already had, & after this, I would define the superfluous, that which would be able to procure us other pleasures or some new advantages; I would say more, that the loss of the necessary makes itself felt infinitely; that thus it can be compensated by no expectation, that to the contrary the sentiment of the loss of the superfluous is limited, & that consequently it can be compensated; I believe that one senses oneself this truth when one plays, because the loss, for little as it is considerable, always gives us more pain than an equal gain gives us pleasure, & this without that one can make enter properly mortified passion, since I suppose the game of entire & pure chance. I would say that the quantity of money in the necessary, is proportional to that which comes back to us of it, but that in the superfluous this proportion begins to diminish, & diminish so much more as the superfluous becomes greater."

"I leave you, Sir, to judge these ideas, &c. Geneva, this 3 October 1730. *Signed* Le Clerc de Buffon."

Mr. Daniel Bernoulli, on *the measure of the lot*, where I have seen the greater part of the ideas of Mr. Dan. Bernoulli accord themselves with mine, that which has given me great pleasure, because I have always, independently of his great talents in Geometry, regarded & recognized Mr. Dan. Bernoulli as one of the better minds of this century. I found also the idea of Mr. Cramer very just, & worthy of a man who has given to us proofs of his ability in all the Mathematical sciences, & to the memoir of which I render this justice, with so much more pleasure as it is to the commerce & to the friendship of this Scholar that I have due a part of the first knowledge that I have acquired in this genre. Mr. de Montmort gives the solution of this problem by the ordinary rules, & he says, that the sum equivalent to the expectation of the one who can only win, is equal to the sum of the sequence $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ écu, &c. continued to infinity, & that consequently this sum is equivalent to a sum of infinite silver. The reason on which this calculation is based, is that there is a half probability that Pierre who can only win, will have an écu; a fourth of probability that he will have two of them; an eighth probability that he will have four of them; a sixteenth probability that he will have eight of them; a thirty-second probability that he will have sixteen of them, &c. to infinity; & that consequently his expectation for the first case is a half-écu, because the expectation is measured by the probability multiplied by the sum which is to obtain; now the probability is a half, & the sum to obtain for the first coup is an écu; therefore the expectation is a half-écu: likewise his expectation for the second case is again a half-écu, because the probability is a fourth, & the sum to obtain is two écus; now a fourth multiplied by two écus, gives again a half-écu. One will find likewise that his expectation for the third case is again a half-écu; for the fourth case a half-écu, in a word for all the cases to infinity always a half-écu for each, since the number of écus increases in the same proportion as the number of the probabilities diminishes; therefore the sum of all these expectations is a sum of infinite silver, & consequently it is necessary that Pierre give to Paul for equivalent, the half of an infinity of écus.

This is mathematically true, & one can not contest this calculation; thus Mr. de Montmort & the other Geometers have regarded this question as well resolved; however this solution is so far from being true, that instead of giving an infinite sum, or even a very great sum, that which is already quite different, there is no man of good sense who wished to give twenty écus nor even ten, in order to buy this expectation by putting himself in the place of the one who can only win.

XVI.

The reason for this extraordinary contradiction of good sense & the calculus, comes from two causes, the first is that the probability must be regarded as null, as soon as it is very small, that is to say, below $\frac{1}{10000}$; the second cause is the small proportion that there is between the quantity of silver & the advantages which result from it; the Mathematician in his calculation, estimates the silver by its quantity, but the moral man must estimate it otherwise; for example, if one would propose to a man of a mediocre fortune to put one hundred thousand livres to a lottery, because there are odds of one hundred thousand against one, that he will win one hundred thousand times one hundred thousand livres; he is certain that the probability to obtain one hundred thousand times one hundred thousand livres, being one against one hundred thousand, it is certain, I say, mathematically speaking, that his expectation will be worth his stake of one hundred thousand livres; however this man would have very greater wrong to chance this sum, & as much great wrong, as the probability to win would be smaller, although the silver to win increased in proportion, & that because with one hundred thousand times one hundred thousand livres, he will not have the double of the advantages that he would have with fifty thousand times one hundred thousand livres, nor ten times as much advantage as he would have with ten thousand times one hundred thousand livres; & as the value of the silver, with respect to the moral man, is not proportional to its quantity, but rather to the advantages that the silver can procure; it is clear that this man must chance only in proportion to the expectation of these advantages, that he must not calculate on the numerical quantity of the sums which he could obtain, since the quantity of silver, beyond certain limits, could no longer increase his happiness, & since he would not be happier with one hundred thousand millions of pension, than with one thousand millions.

XVII.

In order to make sense the liaison & the truth of all that which I come to advance, we examine more closely only what the Geometers have done, the question that one comes to propose; since the ordinary calculus can not resolve it because of the morals which are found complicated with the mathematics, we see if we can by other rules, arrive to a solution which does not knock good sense, & that is at the same time conformed to experience; this research will not be useless, & we will furnish some sure ways to estimate to the just the price of silver & the value of the expectation in all cases. The first thing that I remark, is that in the mathematical calculus which gives for equivalent to the expectation of Pierre an infinite sum of silver; this infinite sum of silver, is the sum of a sequence composed of an infinite number of terms which are worth each one half-écu, & I see that this sequence which mathematically must have an infinity of terms, can not morally have more of it than thirty, since if the game endured to this thirtieth term, that is to say, if *heads* would present itself only after twenty-nine coups, there would be due to Pierre as sum of 520 million 870 thousand 912 écus, that is to say, as much silver as there exists of it perhaps in the entire realm of France. An infinite sum of silver is a being of reason which exists not, & all the expectations based on the terms in the infinite which are above thirty, exist no longer. There is here a moral impossibility which destroys the mathematical possibility; because it is mathematically & even physically possible to cast thirty times, fifty, one hundred times in sequence, &c. without the piece of coinage presenting heads; but it is impossible to satisfy the condition of the problem[6], that is to say, to pay a number of écus which would be due, in the case where this would arrive; because all the silver which is on the earth, would not suffice to make the sum which would be due, only to the fortieth coup, since this would suppose one thousand twenty-four times more silver than there exists in the entire realm of France, & that it is necessary although that out of all the earth there is one thousand twenty-four realms as rich as France.

Now the Mathematician has found this infinite sum of silver for the equivalent to the expectation of Pierre, only because the first case gives to him a half-écu, the second case a half-écu, & each case to infinity always a half-écu; therefore the moral man, by counting first likewise, will find twenty écus instead of the infinite sum, since all the terms which are above the fortieth, give some sums of silver so great, that they do not exist; so that it is necessary to count only a half-écu for the first case, a half-écu for the second, a half-écu for the third, &c. to the fortieth, that which makes in all twenty écus for the equivalent of the expectation of Pierre, a sum already quite reduced & quite different from the infinite sum. This sum of twenty écus will be reduced still more by considering that the twenty-first term would give more than one thousand million écus, that is to say, it would suppose that Pierre

[6]It is for this reason that one of our most able Geometers, the late Mr. Fontaine, has made enter into the solution what he has given to us of this problem, the declaration of the wealth of Pierre, because indeed he can give for equivalent only the totality of the wealth which he possesses. See this solution in the mathematical Memoirs of Mr. Fontaine, *in-4. Paris, 1764.*

would have so much more silver as there is in the richest realm of Europe, an impossible thing to suppose, both as soon as the terms from thirty to forty are again imaginaries, and the expectations based on these terms must be regarded as nulls, thus the equivalent of the expectation of Pierre, is already reduced to fifteen écus.

One will reduce it again by considering that the value of the silver must not be estimated by its quantity, Pierre must not count that one thousand million écus, it will serve him to the double of five hundred million écus, not to the quadruple of two hundred fifty million écus, &c. & that consequently the expectation of the thirtieth term is not a half-écu, no more than the expectation of the twenty-ninth, of the twenty-eighth, &c. the value of this expectation which, mathematically is found to be a half-écu for each term, must be diminished from the second term, & always diminished to the last term of the sequence; because one must not estimate the value of the silver by its numerical quantity.

XVIII.

But how therefore to estimate it, how to find the proportion of this value according to the different quantities? what therefore is two million of silver, if this is not the double of one million of the same metal? can we give some precise & general rules for this estimation? it appears that each must judge his state, & next to estimate his lot & the quantity of silver proportionally to that state & to the usage that he can make of it; but this manner is too vague & too particular in order that it can serve as principle, & I believe that one can find some more general & more sure ways to make this estimation; the first way that presents itself is to compare the mathematical calculus with experience; because in many cases, we can by some repeated experiences, arrive, as I have said, to know the effect of chance, as surely as if we deduced it immediately from causes.

I have therefore made two thousand forty-eight experiments on this question, that is to say, I have played two thousand forty-eight times this game by making a child cast the coin into the air; the two thousand forty-eight game matches, have produced ten thousand fifty-seven écus in total, thus the sum equivalent to the expectation of the one who can only win, is very nearly five écus for each match. In this experiment there have been one thousand sixty-one matches which have produced only one écu, four hundred ninety-four which have produced two écus, two hundred thirty-two matches which have produced four, one hundred thirty-seven matches which have produced eight écus, fifty-six matches which have produced sixteen, twenty-nine matches which have produced thirty-two écus, twenty-five matches which have produced sixty-four, eight matches which have produced one hundred twenty-eight, & finally six matches which have produced two hundred fifty-six. I retain this general result for good, because it is founded on a great number of experiments, & that besides it accords itself with another mathematical & incontestable reasoning, by which one finds very nearly this same equivalent of five écus. Here is this reasoning. If one plays two thousand forty-eight matches, there must be naturally one thousand twenty-four matches which will produce only one écu each, five hundred twelve matches which will produce two of them, two hundred fifty-six matches which will produce four of them, one hundred twenty-eight matches which will produce eight of them, sixty-four matches which will produce sixteen of them, thirty-two matches which will produce thirty-two of them, sixteen matches which will produce sixty-four of them, eight matches which will produce one hundred twenty-eight of them, four matches which will produce two hundred fifty-six of them, two matches which will produce five hundred twelve of them, one match which will produce one thousand twenty-four of them; & finally one match which one cannot estimate, but which one can neglect without sensible error, because I can assume, without harming but very slightly the equality of chance, that there would be one thousand

Trials

twenty-five instead of one thousand twenty-four matches which would produce only one écu, besides the equivalent of this match being put all the more, it can not be more than five écus, since one has seen that for a match of this game, all the terms beyond the thirtieth term of the sequence, give some sums of silver so great, that they do not exist, & that consequently the greatest equivalent that one can assume is five écus. Adding together all these écus, that I must naturally expect by the indifference of chance, I have eleven thousand two hundred sixty-five écus for two thousand forty-eight matches. Thus this reasoning gives very nearly five écus & half for the equivalent, this which accords itself with the experiment to nearly $\frac{1}{11}$. I sense well that one can object to me that this kind of calculation which gives five & a half écus of equivalent when one plays two thousand forty-eight matches, will give a greater equivalent, if one added a much greater number of matches; because, for example, there is found that if instead of playing two thousand forty-eight matches, one plays only one thousand twenty-four of them, the equivalent is very nearly five écus; that if one plays only five hundred twelve matches, the equivalent is no more than four & a half écus very nearly; that if one plays only two hundred fifty-six of them, it is no more than four écus, & thus always by diminishing; but the reason for it is that the toss that one can not estimate, is then a match considerable in the total, & so much more considerable, as one plays fewer matches, & that consequently it is necessary a great number of matches, as one thousand twenty-four or two thousand forty-eight in order that this toss can be regarded as of little value, or even as null. By following the same step, one will find that if one plays one million forty-eight thousand five hundred seventy-six matches, the equivalent by this reasoning would be found to be nearly ten écus; but one must consider all in the moral, & thence one will see that it is not possible to play one million forty-eight thousand five hundred seventy-six matches in this game, because to suppose only two minutes of time for the duration of each match, including in it the time that it is necessary to pay, &c. one will find that it would be necessary to play during two million ninety-seven thousand one hundred fifty-two minutes, that is to say, more than thirteen years[7] in sequence, six hours per day, this which is a convention morally impossible. And if one pays attention, one will find that between playing only one match & playing a great number of matches morally possible, this reasoning which gives some different equivalents for all the different numbers of matches, give for the mean equivalent five écus. Thus I persist to say that the sum equivalent to the expectation of the one who can only win is five écus, instead of half of an infinite sum of écus, as the mathematicians have said, & as their calculus requires it.

XIX.

We see now if after this determination, it would be possible to deduce the proportion of the value of the silver by ratio to the advantages which result from them.

The progression of probabilities is	$\frac{1}{2}$,	$\frac{1}{4}$,	$\frac{1}{8}$,	$\frac{1}{16}$,	$\frac{1}{32}$,	$\frac{1}{64}$,	$\frac{1}{128}$,	$\frac{1}{256}$,	$\frac{1}{512}$,	$\cdots$	$\frac{1}{2\cdot\infty}$,
The progression of sums of silver to obtain is	1,	2,	4,	8,	16,	32,	64,	128,	256,	$\cdots$	2^{∞}

The sum of all these probabilities, multiplied by those of all the sums of silver to obtain is $\frac{\infty}{2}$, which is the equivalent given by the mathematical calculation, for the expectation of the one who can only win. but we have seen that this sum $\frac{\infty}{2}$ can, in reality, be only five écus; it is necessary therefore to seek a sequence, such that the sum multiplied by the

[7]Translator's note. I compute nearly 16 years.

sequence of probabilities, is equal to five écus, & this sequence being geometric as that of the probabilities, one will find

that it is	$\cdots$	$1,$	$\frac{9}{5},$	$\frac{81}{25},$	$\frac{729}{125},$	$\frac{6561}{625},$	$\frac{59049}{3125},$
instead of &$\cdots$	$1,$	$2,$	$4,$	$8,$	$16,$	$32,$	

Now this sequence $1, 2, 4, 8, 16, 32,$ &c. represents the quantity of silver, & consequently the numerical & mathematical value.

And the other sequence $1,\ \frac{9}{5}, \frac{81}{25}, \frac{729}{125}, \frac{6561}{625}, \frac{59049}{3125},$ represents the geometrical quantity of silver given by the experiment, & consequently its moral & real value.

Here is therefore a general estimation, & correct enough for the value of the silver in all the possible cases, & independently of any assumption. For example, one sees, by comparing the two sequences, that two thousand livres does not produce the double advantage of one thousand livres, that it itself must be $\frac{1}{5}$ of it, & that two thousand livres is in the moral & in the real only $\frac{9}{5}$ of two thousand livres, that is to say, eighteen hundred livres. A man who has twenty thousand livres of wealth, must not estimate it as the double of the wealth of another who has ten thousand livres, because it is really only eighteen thousand livres of silver of this same money, of which the value is computed by the advantages which result from it; & likewise a man who has four thousand livres, is not four times richer than the one who has ten thousand livres, because he is in comparison really rich only of 32 thousand 400 livres; a man who has 80 thousand livres, has, by the same rule, only 58 thousand 300 livres; the one who has 160 thousand livres, must count only 104 thousand 900 livres, that is to say, that although he has sixteen times more wealth than the first, he has scarcely only ten times as much of our true money; likewise again a man who has thirty-two times as much silver as another, for example 320 thousand livres in comparison to a man who has 10 thousand livres, is rich in reality only by 188 thousand livres, that is to say, eighteen times or nineteen times more rich, instead of thirty-two times, &c.

The Miser is as the Mathematician; both estimate silver by its numerical quantity, the sane man considers neither the mass nor the number of it, he sees only the advantages which he can draw from it, he reasons better than the Miser, & senses better than the Mathematician. The écu that the poor has set apart to pay a tax of necessity, & the écu which completes the purse of a financier, have for the miser & for the Mathematician only the same value, these will count them for two united equals, the other will appropriate them himself with an equal pleasure, instead as the sane man will count the écu of the poor for a louis, & the écu of the financier as a liard[8].

XX.

Another consideration which comes to the support of this estimation of the moral value of silver, is that a probability must be regarded as null as soon as it is only $\frac{1}{10000}$, that is to say, as soon as it is as small as the fear not felt of death in twenty-four hours. One can even say, that awaiting the intensity of this fear of death which is much greater than the intensity of all other sentiments of fear or expectation, one must regard as near null, a fear or an expectation which would be only $\frac{1}{1000}$ of probability. The most feeble man draws to the lot without any emotion, if the ticket of death were mixed with ten thousand tickets of life; & the strong man must draw without fear, if this ticket is mixed out of one thousand; thus in all the cases where the probability is below one thousandth, one must regard it as near null. Now, in our question, the probability finding itself to be $\frac{1}{1024}$ from the tenth term of the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024},$ it follows that morally thinking, we must ignore all the following terms, & limit all our expectations to

[8] Translator's note. This is essentially a half-cent.

this tenth term; this which produces again five écus for the equivalent that we have sought, & confirms consequently the justice of our determination.

In reforming & abridging thus all the calculations where the probability becomes smaller than a thousandth, there will no longer result contradiction between the mathematical calculus & good sense. All the difficulties of this kind disappear. The man impressed by this truth will deliver himself no longer to some vain expectations or to some false fears; he will not give readily his écu in order to obtain a thousand, unless he sees clearly only the probability is greater than a thousandth. Finally he will correct himself from the frivolous hope to make a great fortune with some small means.

XXI.

Until here I have reasoned & calculated only for the truly wise man, who is determined only by the weights of reason; but must we not pay some attention to that great number of men who the illusion or the passion deceive, & who often are so easy to be deceived? Is it not the same to lose by presenting all the things such as they are? Expectation, however small that the probability be, is it not a good for all men, & the sole good of the unhappy? After having calculated for the Sage, we calculate therefore also for the man much less rare, who enjoys from his errors often more than from his reason. Independently of the cases where want of all means, a glow of hope is a sovereign good; independently of those circumstances where the agitated heart can rest itself only on the objects of its illusion, & enjoy only some desires; are there not thousand & thousand occasions where the same wisdom must cast before a volume of expectation instead of a mass of real wealth? For example, the will to make wealth, recognized in those who hold the reins of the Government, was it without budget, spills out onto all the people a sum of happiness which one can not estimate; was the expectation vain, is there a real good, of which the enjoyment is taken by anticipation on all the other goods. I am forced to avow that the full wisdom does not make the full happiness of man, that unfortunately the sole reason had in all time only a small number of cold listeners, & was never enthusiastic; that men full of goods, would not be found still happy if he hoped for it anew; that the superfluous becomes with time a very necessary thing, & that the sole difference that there is here between the Sage & the non-Sage, is that this last, at the same moment that there arrives to him a superabundance of wealth, converts this beautiful superfluous to sad necessity, & raises his state to the equal of his new fortune; while the wise man using of this superabundance only to take back some benefits & to procure himself some new pleasures, takes care of the consumption of this superfluous at the same time as he multiplies the enjoyment of it.

XXII.

The display of the expectation is the lure of all the cheats of silver. The great art of the maker of a lottery, is to present gross sums with very small probabilities, soon swollen by the spring of cupidity. These cheats swell again this ideal product by sharing it, & giving for a very small silver, of which everyone can be undone, an expectation which, although quite smaller, appears to participate in the magnitude of the total sum. One knows not that when the probability is under a thousandth, the expectation becomes null, however great that the promised sum be, since each thing, however great that it can be, is reduced to nothing as soon as it is necessarily multiplied by nothing, as is here the gross sum of silver multiplied by the null probability, as is in general each number which, multiplied by zero, is always zero. One is ignorant again that independently of this reduction of the probabilities to nothing, as soon as they are below a thousandth, the expectation suffers a successive & proportional loss to the moral value of silver, always less than its numerical value, so that the one of which the numerical expectation appears double of that of another, has nonetheless only $\frac{9}{5}$ of the real expectation instead of 2; & that likewise the one of which the numerical expectation is 4, has only $3\frac{6}{25}$ of this moral expectation, of which the product is the only real. That instead of 8, this product is only $5\frac{104}{125}$; that instead of 16, it is only $10\frac{311}{625}$; instead of 32, $18\frac{2799}{3125}$; instead of 64, $34\frac{191}{15625}$; instead of 128, $61\frac{17342}{78125}$; instead of 256, $110\frac{77971}{390625}$; instead of 512, $198\frac{701739}{1953125}$; instead of 1024, $357\frac{456276}{9765625}$, &c. whence one sees how much the moral expectation differs in all the cases of the numerical expectations for the real product which results from it; the wise man must therefore reject as false all the propositions, although demonstrated by the calculus, where the very great quantity of silver seems to compensate the very small probability; & if he wishes to risk with less disadvantage, he must never put his funds to the high risk,[9] it is necessary to share them. To chance one hundred thousand francs on a single vessel, or twenty-five thousand francs on four vessels, is not the same thing; because one will have one hundred for the product of the moral expectation in this last case, while one will have only eighty-one for this same product in the first case. It is by this same reason that the most surely lucrative commerces, are those where the common fund of the debit is divided by a great number of Creditors. The proprietor of the mass can attempt only slight bankruptcy, instead that it is necessary only one in order to ruin it, if this common fund of his commerce can pass only through a single hand, or even to be shared only among a small number of debtors. To play a big game in the moral sense, is to play a bad game; a *Punter in Pharaon*, who would put himself into the head to push all his cards to *quinze* & *và*, would lose nearly a fourth on the product of his moral expectation, because when his numerical expectation is to draw 16, the moral expectation is only $13\frac{104}{125}$. It is likewise of an infinity of other examples that one could give; & of all there will result always that the wise man must put to chance the least that is possible, & that the prudent man who, by his position or his commerce, is forced to risk gross funds, must share them, & subtract from his speculations all the expectations of which the probability is very small, although the sum to obtain is proportionally as great.

XXIII.

Geometric probability

Analysis is the only instrument by which one is served until this day in the science of probabilities, to determine & to fix the ratios of risk; Geometry appeared ill-suited to a work so delicate; however if one considers it closely, it will be easy to recognize that this advantage of Analysis on Geometry, is completely accidental, & that risk according as it is modified & composed, is found as a result of geometry as well as that of analysis; in order to be assured of it, it will be sufficient to pay attention that the games & the questions of conjecture turn customarily only on the ratios of discrete quantities; the human spirit more familiar with numbers than with measures of extent have always preferred them; the games are one proof of it, because their laws are one continual arithmetic; therefore to put Geometry in possession of its rights on the science of risk, the concern was only to invent some games which turn on size & on their ratios, or to reckon the small number of those of that nature which are already found; the game of franc-carreau is able to serve us for example: here are its conditions which are quite simple.

Franc-carreau

In a room floored or paved with equal tiles, of any figure, one throws a coin into the air; one of the players wagers that this coin after its fall will be found in free-tile, that is to say, on a single tile; the second wagers that this coin will be found on two tiles, that is to say, that it will cover one of the joints that separate them; a third player wagers that the coin will be found on two joints; a fourth wagers that the coin will be found on three, four or six joints: one requires the lot of each of these players.

[9] *gross aventure*: a loan made a high interest at the risk of total loss of capital in case of shipwreck.

I choose to begin with the lot of the first player & of the second; to find it, I inscribe in one a similar figure, holding back from the sides of the tile, to the length of the semi-diameter of the coin; the lot of the first player will be to that of the second, as the area of the circumscribing ring is to the area of the inscribed figure; one is able to demonstrate it easily, because as long as the center of the coin is in the inscribed figure, this coin is only able to be on a single tile, since by construction this inscribed figure is everywhere held back from the edge of the tile, by a distance equal to the radius of the coin; & to the contrary as soon as the center of the coin falls on the outside of the inscribed figure, the coin is necessarily on two or more tiles, since then its radius is greater than the distance from the edge of this inscribed figure to the edge of the tile; now, all the points where this center of the coin is able to fall, are represented in the first case by the area of the ring which makes the remainder of the tile; therefore the lot of the first player is to the lot of the second, as this first area is to the second, thus to render equal the lot of these two players, it is necessary that the area of the inscribed figure be equal to that of the ring, or what is the same thing, that it be the half of the total surface of the tile.

I amused myself by making the calculation of it, & I have found that to play in a fair game on square tiles, the side of the square must be to the diameter of the coin, as $1 : 1 - \sqrt{\frac{1}{2}}$; that is to say, to nearly three and a half times[10] greater than the diameter of the coins with which one plays.

To play on equilateral triangular tiles, the side of the tile must be to the diameter of the coin, as $1 : \frac{\frac{1}{2}\sqrt{3}}{3+3\sqrt{\frac{1}{2}}}$, that is to say, nearly six times[11] greater than the diameter of the coin.

On diamond tiles, the side of the tile must be to the diameter of the coin, as $1 : \frac{\frac{1}{2}\sqrt{3}}{2+\sqrt{2}}$, that is to say, nearly four times[12] as great.

Finally on hexagonal tiles, the side of the tile must be to the diameter of the coin, as $1 : \frac{\frac{1}{2}\sqrt{3}}{1+\sqrt{\frac{1}{2}}}$, that is to say, nearly double[13].

I have not made the calculation for the other figures, because these are the only ones which one is able to fill a space without leaving some intervals between the other figures; & I did not believe that it is necessary to warn that the joints of the tiles having some width, they give advantage to the player who wagers for the joint, & that by consequence, one will be well, to render the game again more equal, to give to the square tiles a little more than three & a half times, to the triangles six times, to the diamonds four times, & to the hexagons two times the length of the diameter of the coin with which one plays.

I seek now the lot of the third player who wagers that the coin will be found on two joints; & to find it, I inscribe in one of the tiles, a similar figure as I have already made, next I extend the sides of the inscribed figure until they meet those of the tile, the lot of the third player will be to that of his adversary, as the sum of the spaces contained between the extension of these lines & the sides of the tile, is to the remainder of the surface of the tile. This has no need to be fully demonstrated, as being well understood.

[10] 3.414 times.
[11] 5.9136 times
[12] 3.9424 times
[13] 1.9712 times.

I have also made calculation of this case, & I have found that to play in a fair game on square tiles, the side of the square must be to the diameter of the coin, as $1{:}\frac{1}{\sqrt{2}}$, that is to say, greater than one little less than a third.[14]

On the equilateral triangular tiles, the side of the tile must be to the diameter of the coin, as $1 : \frac{1}{2}$, that is to say, double.

On the diamond tile, the side of the tile must be to the diameter of the coin, as $1 : \frac{\frac{1}{2}\sqrt{3}}{\sqrt{2}}$, that is to say, greater than about two-fifths.[15]

On the hexagonal tiles, the side of the tile must be to the diameter of the coin, as $1 : \frac{1}{2}\sqrt{3}$, that is to say, greater than a half-fourth.[16]

Now the fourth player wagers that on the equilateral triangular tiles, the coin will be found on six joints, that on the square tiles or on diamonds it will be found on four joints, & on the hexagonal tiles it will be found on three joints; to determine his lot, I describe from the point of an angle of the tile, a circle equal to the coin, & I say that on the equilateral triangular tiles, his lot will be to that of his adversary as half of the area of this circle to that of the rest of the tile; that on the square tiles or on diamonds, his lot will be to that of the other, as the entire area of the circle is to that of the rest of the tile; & that on the hexagonal tiles, his lot will be to that of his adversary, as the double of the area of the circle is to the rest of the tile. In supposing therefore that the circumference of the circle is to the diameter as 22 is to 7, one will find that to play a fair game, on the equilateral triangular tiles, the side of the tile must be to the diameter of the coin as $1 : \sqrt{\frac{7\sqrt{3}}{22}}$, that is to say, the greater of one little more than a quarter.[17]

On the diamond tiles, the lot will be the same as on the equilateral triangular tiles.

On the square tiles, the side of the tile must be to the diameter of the coin, as $1 : \sqrt{\frac{11}{7}}$, that is to say, the greater of about one-fifth.[18]

On the hexagonal tiles, the side of the tile must be to the diameter of the coin, as $1 : \sqrt{\frac{21\sqrt{3}}{44}}$, that is to say, greater than about a thirteenth.[19]

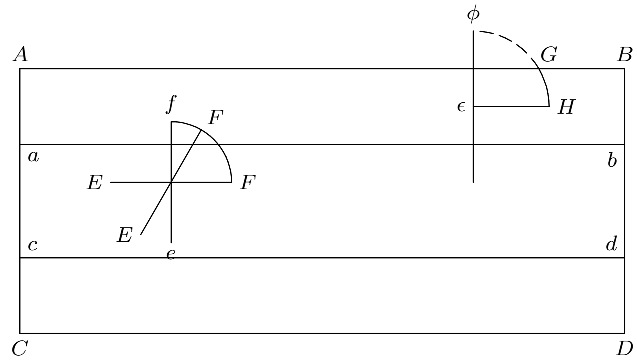
I omit here the solution of many other cases, as when one of the players wagers that the coin will fall only on a joint or on two, on three, &c. They are not more difficult than the preceding; & besides one plays this game rarely with conditions other than those of which we have made mention.

But if instead of throwing in the air a round piece, as a coin, one will throw a piece of another figure, as a square pistole of Spain, or a needle, a rod, &c. the problem will demand a little more geometry, although in general it is possible always to give the solution by the comparison of spaces, as we are going to demonstrate.

Needle problem

I suppose that in a room, of which the floor is simply divided by parallel joints, one throws into the air a rod, & that one of the players wagers that the rod will not cross any of the parallels of the floor, and that the other to the contrary wagers that the rod will cross some one of the parallels; one requires the lot of the two players. One is able to play this game on a draught board with a sewing needle or a headless pin.

[14] 1.4142 times.
[15] 1.623 times.
[16] 1.1547 times.
[17] 1.3470 times.
[18] 1.2536 times.
[19] 1.09986 times.



To find it, I draw first between the two parallel joints AB & CD of the floor, two other parallel lines ab & cd, holding back from the first ones by the half of the length of the rod EF, & I see evidently that as long as the middle of the rod will be between those two second parallels, it will never be able to cross the first ones in any position EF, ef, that it can find; and as all this which could occur above with ab occurs similarly below with cd, there is concern only to determine the one or the other; for this I remark that all the positions of the rod are able to be represented by the quarter of the circumference of the circle of which the length of the rod is the diameter; therefore calling $2a$ the distance CA of the joints of the floor, c the fourth of the circumference of the circle of which the length of the rod is the diameter, calling $2b$ the length of the rod, & f the length AB of the joints, I will have $f(a-b)c$ for the expression which represents the probability of not crossing the joint of the floor, or that which is the same thing, to the expression of all the cases where the middle of the rod falls below the line ab & above the line cd.

But since the middle of the rod falls out of the space $abdc$, contained between the second parallels, it can, according to its position, cross or not cross the joint; so that the middle of the rod being, for example, at ϵ, the arc ϕG will represent all the positions where it will cross the joint, & arc GH all those where it will not cross, and as it will be the same for all the points on the line $\epsilon\phi$, I call dx the small parts of this line, & y the arcs of circle ϕG, & I have $f(\int y\,dx)$ for the expression of all the cases where the rod will cross, & $f(bc - \int y\,dx)$ for those cases where it will not cross; I add this last expression to that found above $f(a-b)c$, in order to have the totality of cases where the rod will not cross, & since then I see that the lot of the first player is to that of the second, as $ac - \int y\,dx : \int y\,dx$.

If therefore one wishes that the game be fair, one will have $ac = 2\int y\,dx$ or $a = \frac{\int y\,dx}{\frac{1}{2}c}$, that is to say, to the area of one part of the cycloid, of which the generating circle has for diameter length $2b$ of the rod; now, one knows that this area of the cycloid is equal to the square of the radius, therefore $a = \frac{bb}{\frac{1}{2}c}$, that is to say, that the length of the rod must be made about three-fourths of the distance of the joints of the floor.

The solution of this first case leads us easily to that of another which at first would have seemed more difficult, which is to determine the lot of those two players in a room paved by square tiles, because by inscribing within one of the square tiles, a square held back from all the sides of the square by the length b, one will have at first $c(a-b)^2$ for the expression of one part of the cases where the rod does not cross the joint; next one will find $(2a-b)\int y\,dx$ for that of all the cases where it will cross, & finally $cb(2a-b) - (2a-b)\int y\,dx$ for the rest of the cases where it does not cross; the lot of the first player is to that of the second, as $c(a-b)^2 + cb(2a-b) - (ca-b)\int y\,dx : (2a-b)\int y\,dx$.

Therefore if one wishes that the game be equal, one will have $c(a-b)^2 + cb(2a-b) = (2a-b)^2 \int y\,dx$ or $\frac{\frac{1}{2}caa}{2a-b} = \int y\,dx$; but as we have seen above, $\int y\,dx = bb$; therefore $\frac{\frac{1}{2}caa}{2a-b} = bb$; thus the side of the square must be to the length of the rod, nearly as $\frac{41}{22} : 1$, that is to say, not quite double. Therefore, if one will play on a draught board with a needle for which the length will be the half of the length of the side of the square of the draught board, he will have the advantage to wager that the needle will cross the joints.

One will find by a similar calculation, that if one plays with a piece of square money, the sum of the lots will be to the lot of the player who wagers for the joint, as $aac : 4abb\sqrt{\frac{1}{2}} - b^3 - \frac{1}{2}Ab$, A marks here the excess of the area of the circle circumscribed on the square, & b the semi-diagonal of the square.

These examples suffice to give an idea of the games that one is able to imagine on the ratios of area; one can propose many other questions of this kind, which would not depart from being curious & even useful: if one would require, for example, how much one risks to pass a river on a board or less narrow; what must be the fear that one must have of lightning or the fall of a bomb, & a number of other problems of conjecture, where one must consider that the ratio of the area, & which by consequence concern Geometry all as much as Analysis.