Mémoire sur l'interpolation, ou Remarques sur les Remarques de Mr. Jules Bienaymé

Mr. Augustin Cauchy*

Comptes Rendus Hebd. Séances Acad. Sci. 37, 64-69 (1853)
OC I, 12 (522), 63-68.

The *Compte Rendu* of the last session contained a Memoir read at the last but one by Mr. Jules Bienaymé, at a moment where I was absent. This memoir is entitled: *Remarques sur les différences qui distinguent la méthode des moindres carrés de l'interpolation de* Mr. Cauchy, *et qui assurent la supériorité de cette méthode.* In reading this title, one could believe the method of least squares, always and under all relations, [65] preferable to the new method of interpolation that I have given in 1835. However, this conclusion would not be legitimate. In order to put the reader in reach to form himself an opinion in this regard, I have believed I must in my turn compare the two methods to each other. The algorithm of which I have made use in 1835 facilitates this comparison, by reducing the diverse methods proposed by the geometers, for the resolution of linear equations, to some general and very simple formulas, contained in the first pages of my Memoir, and that I am going to indicate.

We consider first m unknowns, $x, y, z, \dots, w$ linked to each other by m equations

$$\mathscr{A} = 0, \quad \mathscr{B} = 0, \quad \mathscr{C} = 0, \dots, \quad \mathscr{Z} = 0, \tag{1}$$

of which the first members are some linear functions of these unknowns. If the resultant of the table which has for terms the coefficients of $x, y, z, \dots, w$ in the functions $\mathscr{A}, \mathscr{B}, \mathscr{C}, \dots, \mathscr{Z}$ did not vanish, one will be able to draw from equations (1) the values of $x, y, z, \dots, w$, by eliminating one after the other these unknowns, arranged in a certain order, and by ascending from the last of the formulas thus obtained to those which precede it. If, in particular, one wishes to eliminate x from the second, from the third,..., from the last of equations (1), it will suffice to subtract from the function $\mathscr{B}$, or $\mathscr{C}$,..., or $\mathscr{Z}$ the product of $\mathscr{A}$ by the ratio of the coefficient of x in $\mathscr{B}$,or $\mathscr{C}$,..., or $\mathscr{Z}$ to the coefficient of x in $\mathscr{A}$. If one indicates, by aid of the characteristic letter Δ, the *differences of the first order* thus obtained, the elimination of x among the equations (1) will give the following:

$$\Delta\mathscr{B} = 0, \quad \Delta\mathscr{C} = 0, \dots, \quad \Delta\mathscr{Z} = 0, \tag{2}$$

Similarly, if one wishes to eliminate y from these, by aid of the equation $\Delta\mathscr{B} = 0$, it will suffice to subtract from the function $\Delta\mathscr{C}$,..., or $\Delta\mathscr{Z}$, the product of $\Delta\mathscr{B}$ by the ratio

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 6, 2010 of the coefficient of x in $\Delta\mathscr{C}$,..., or $\Delta\mathscr{Z}$, to the coefficient of x in $\Delta\mathscr{B}$. If one indicates, by aid of the characteristic Δ^2, the *differences of the second order* thus obtained, the elimination of y among the equations (2), will give the following:

$$\Delta^2\mathscr{C} = 0, \ldots, \quad \Delta^2\mathscr{Z} = 0, \tag{3}$$

By continuing thus, one will finish by joining to the equations (1) all the formulas contained with them in the following table: [66]

$$\left\{\begin{array}{rrrcr} \mathscr{A} = 0, & \mathscr{B} = 0, & \mathscr{C} = 0, & \ldots, & \mathscr{Z} = 0, \\ & \Delta\mathscr{B} = 0, & \Delta\mathscr{C} = 0, & \ldots, & \Delta\mathscr{Z} = 0, \\ & & \Delta^2\mathscr{C} = 0, & \ldots, & \Delta^2\mathscr{Z} = 0, \\ & & & & \\ & & & & \Delta^m\mathscr{Z} = 0, \end{array}\right. \tag{4}$$

and this table will permit not only calculating easily the values of $x, y, z, \ldots, w$, that one will be able to deduce from the single formulas

$$\mathscr{A} = 0, \quad \Delta\mathscr{B} = 0, \quad \Delta^2\mathscr{C} = 0, \ldots, \quad \Delta^m\mathscr{Z} = 0, \tag{5}$$

by arising from one to the other, after having drawn from the last the value of w, but yet to establish the correctness of the calculation by numerous verifications.

We suppose now the m unknowns $x, y, z, \ldots, w$ linked among them by n exact or approximate equations

$$\varepsilon_1 = 0, \quad \varepsilon_2 = 0, \ldots, \quad \varepsilon_n = 0, \tag{6}$$

n being equal or superior to m. In order to determine completely the values of the unknowns, it will suffice again to resolve m equations of the form (1), $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots, \mathscr{Z}$ designating m linear functions of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Besides, in the values of $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots, \mathscr{Z}$ expressed as functions of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ for some linear equations, that is to say of the form

$$\left\{\begin{array}{l} \mathscr{A} = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_n\varepsilon_n, \\ \mathscr{B} = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \cdots + \mu_n\varepsilon_n, \\ \mathscr{C} = \nu_1\varepsilon_1 + \nu_2\varepsilon_2 + \cdots + \nu_n\varepsilon_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \mathscr{Z} = \zeta_1\varepsilon_1 + \zeta_2\varepsilon_2 + \cdots + \zeta_n\varepsilon_n, \end{array}\right. \tag{7}$$

the factors $\lambda_1, \lambda_2, \ldots, \lambda_n; \mu_1, \mu_2, \ldots, \mu_n; \nu_1, \nu_2, \ldots, \nu_n; \ldots; \zeta_1, \zeta_2, \ldots, \zeta_n$;will be able to be arbitrarily chosen under a single condition, namely, that the values of $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots, \mathscr{Z}$ are not able themselves to satisfy any linear equation from which each of the unknowns $x, y, z, \ldots, w$ would be excluded. One must not be preoccupied with the case where this condition would not be able to be fulfilled; because this would be there an exceptional case, and in which equations (6) either would be mutually contradictory, or would become insufficient to determine the values of the unknown.

[67] It is good to observe that after having formed equations (1), one should substitute them into other equations of which one is able to easily draw the values of the

unknowns, for example equations (5). Besides one will be able to form directly these last ones, without passing through the equations (1). In fact, formulas (7) will give

$$
\tag{8}
\begin{cases}
\mathscr{A} = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_n\varepsilon_n, \\
\Delta\mathscr{B} = \mu_1\Delta\varepsilon_1 + \mu_2\Delta\varepsilon_2 + \cdots + \mu_n\Delta\varepsilon_n, \\
\Delta^2\mathscr{C} = \nu_1\Delta^2\varepsilon_1 + \nu_2\Delta^2\varepsilon_2 + \cdots + \nu_n\Delta^2\varepsilon_n, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\Delta^m\mathscr{Z} = \zeta_1\Delta^m\varepsilon_1 + \zeta_2\Delta^m\varepsilon_2 + \cdots + \zeta_n\Delta^m\varepsilon_n,
\end{cases}
$$

Now, in regard to equations (8), one will be able to determine successively the differences of the diverse orders comprised in the diverse horizontal lines of the table

$$
\tag{9}
\left\{
\begin{array}{cccc|c}
\varepsilon_1, & \varepsilon_2, & \dots, & \varepsilon_n, & \mathscr{A}, \\
\Delta\varepsilon_1, & \Delta\varepsilon_2, & \dots, & \Delta\varepsilon_n, & \Delta\mathscr{B}, \\
\Delta^2\varepsilon_1, & \Delta^2\varepsilon_2, & \dots, & \Delta^2\varepsilon_n, & \Delta^2\mathscr{C}, \\
\dots & \dots & \dots & \dots & \dots \\
\Delta^m\varepsilon_1, & \Delta^m\varepsilon_2, & \dots, & \Delta^m\varepsilon_n, & \Delta^m\mathscr{Z},
\end{array}
\right.
$$

by deducing, in the first horizontal line, the term $\mathscr{A}$ from the preceding combined with a first system of factors $\lambda_1, \lambda_2, \dots, \lambda_n$; next the second horizontal line of the first joined to a second system of factors $\mu_1, \mu_2, \dots, \mu_n$; next the third horizontal line of the second joined to a third system of factors $\nu_1, \nu_2, \dots, \nu_n; \dots$; etc. One is found thus to restore very simply, by the use of the characteristic letter Δ, in the proposition enunciated by Mr. Bienaymé, and relative to the independence in which they remain, in presence of one another, the diverse system of factors

$$\lambda_1, \lambda_2, \dots, \lambda_n; \quad \mu_1, \mu_2, \dots, \mu_n; \quad \nu_1, \nu_2, \dots, \nu_n; \dots$$

In reality, this proposition is able to be deduced from this simple observation, that two linear functions of $x, y, z, \dots, w$, identically equal among them, for example,

$$\mathscr{C}, \quad \text{and} \quad \nu_1\varepsilon_1 + \nu_2\varepsilon_2 + \cdots + \nu_n\varepsilon_n,$$

not ceasing to be identically equal, when one replaces one or many unknowns by their values drawn from certain linear equations, for example x and y, by their values drawn the two equations $\mathscr{A} = 0$, [68] $\Delta\mathscr{B} = 0$, this which reduces the two cited functions to the two following,

$$\Delta^2\mathscr{C}, \quad \nu_1\Delta^2\varepsilon_1 + \nu_2\Delta^2\varepsilon_2 + \cdots + \nu_n\Delta^2\varepsilon_n.$$

We imagine now, that after having determined the differences of the order m of the functions $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, one determines further their differences of order $n+1$, namely

$$\tag{10} \Delta^{m+1}\varepsilon_1, \quad \Delta^{m+1}\varepsilon_2, \cdots, \quad \Delta^{m+1}\varepsilon_n.$$

These last differences will be that which the preceding become when one eliminates the unknown w by aid of the equation $\Delta^m\mathscr{Z} = 0$, or else again that which the functions

$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ become, when one eliminates $x, y, z, \ldots, w$ by aid of equations (1) and (5). Hence, they will be reduced to zero, if one has $n = m$, or if the equations (6) are exact; and if, n being superior to m, the equations (6) are only approximate, to some constants so much smaller (setting aside the signs) as the approximation will be greater.

In being supported on the preceding considerations, one recognizes easily that the method of least squares and the new method of interpolation each have their advantages and their inconveniences; that the questions to which they are applied naturally are of two distinct kinds, the new method being especially employed in order to resolve some problems where the question is to fix at the same time both the value of the unknowns, and the number of those which must enter into the calculation; that, in order to render the method of least squares applicable to these problems, it would be necessary to borrow from another method the rule which makes it the merited principal; finally, that some results obtained by the new method one is able to deduce often, with a very great facility, those which the method of least squares would furnish. Such are the conclusions which are put into evidence in my Memoir, thus that I will explicate in more detail in a second article.

"Mr. Bienaymé demanded the floor and remarked that he has not attacked the use which has been able to be made of the method of *Mr. Cauchy* in certain cases; that he has made in this regard without any guarantees, twice, in the Note inserted in the *Compte Rendu* of the session of 4 July. His unique end was to caution on the differences which separate the process in question from the method of least squares, based on the theory of probabilities. This end will be more completely attained, since the examination that Mr. Cauchy has made of his Note, will call thus greater attention on the distinction which he has signalled. [69] Mr. Bienaymé has believed useful the warning which he gave, because there is in the first work of his scholarly colleague some phrases which would be able to occasion some scorn. At present, they will not be able to take place, and one will choose from inside information. But one had abused the method of least squares in more than one Memoir (not in this country, but in the beyond), and it has seemed that an analogous abuse was able to be more to fear yet with the method of Cauchy.

After the conclusions of the Memoir which Mr. Cauchy just read, which tends principally to justify the application of his process to a special class of problems, to convergent series especially, as Mr. Bienaymé has reserved expressedly this application, he believes he must add nothing to his remarks; but he maintains the entire correctness of it. Setting aside from the analysis, on which Mr. Cauchy has not been able to give a lecture, if Mr. Bienaymé has well seized the explications of Mr. Cauchy, he sees the confirmation of the differences which he has made understood, and he thinks not to have been so severe toward the method that Mr. Cauchy has found. He assesses, moreover, that the warning, that he had alone in view, being thus given, there is no place to occupy further the Academy on the subject to which he has well wished to lend his attention, neither to prolong a polemic on the words that Mr. Cauchy has believed to be able to raise, or to be able to use. As in the main, the persons who will have to be served of the process of which there is question, will be able to see very clearly, either after the new clarifications and rectifications of Mr. Cauchy, or after the Note of Mr. Bienaymé, the things that they will have to make according to the questions to resolve."