

Reflections on a singular Kind of Lottery named the Genoise Lottery

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An Italian once suggested a project of a kind of lottery which appeared much to the liking of most men, because of the very considerable gains that one could get from it without risking nearly anything. The plan was entirely different from ordinary lotteries, because each could determine not only his wager, but also the size of the gain to which he wished to aspire. There was rather some resemblance to the game of Pharaon, in consideration of the arbitrary wagers that one is able to put on any card that one wants; but it is however different with regard to the prizes that each is able to choose at will. The arrangement of this lottery depends uniquely on the probability calculus, and the entrepreneur, instead of drawing a fixed profit from it, risks very considerable loss, although according to the plan of which I come to speak, it would be probable that he earn a good portion of all the money that had been wagered. It is nearly as if I undertook to pay another 100 ecus for one <ecu> that he would have given me, in case that he throw with three dice, the first time, three sixes; it would be possible that I lose in this game 99 ecus. Now, the probability of winning an ecu being 215 times greater than that of losing 99 ecus, the advantage is on my side and has estimated value $\frac{29}{54}$ ecus, or a little more than a half ecu. That is to say, if I undertook in this manner towards 1000 persons of whom each would give to me one ecu, I would be able to estimate my advantage at $537\frac{1}{27}$ ecus, although it is possible that I lose 99000 ecus. It is on this basis that one will be able to evaluate the advantage of the one who would undertake the lottery mentioned, in comparing the wager of each with the probability that he will win.

DESCRIPTION OF THIS LOTTERY

This lottery consists of 90 tickets marked with the numbers 1, 2, 3, 4 etc. up to 90, from which one proposes to draw at random 5 at a fixed time; and then, these five numbers will earn those who will have chosen one or two or three of them beforehand, in order to bind their wagers to them. For one is able to participate in this lottery in many different ways.

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I. Either one chooses at will a number that does not exceed 90 and one pays also a sum of money that one will judge opportune. Then, when this number is found among the five which will be drawn, one will withdraw a prize which will be a certain multiple of the wager.

II. Or one chooses two numbers at a time, to which one attaches a certain wager, and in case that both are found afterwards among the five drawn, one will receive a prize quite considerable in proportion to the wager. Now, if only one of them is found among the five, one receives also a lesser prize.

III. Or one chooses three numbers at a time, in which one attaches at will a certain wager, and one is able to expect a prize some thousand times greater than the wager, in case that all three numbers are found among the five drawn; but the prize will be less, when two of the chosen numbers, or only one of them will be found.

I no longer remember the magnitude of the prize in detail that one pays for in each case, that which contributes nothing to the researches that I propose to make; but one understands easily that they are able to be very considerable for the case where three numbers that one will have chosen are encountered among the five drawn. And if one wanted to admit quadruples, the fixed prize for the case where all four numbers would be found among the five tickets extracted would be able to surpass beyond 100000 times the quantity of the wager.

It is evident that neither the number 90 of tickets, nor that of the 5 that one draws, is essential to the nature of this lottery and that one is absolutely free to establish any number of tickets and from them to draw finally more or less than five, this which leads me to some more general researches which could serve either to form some other plans of similar lotteries, or to examine those which will be suggested by some others.

Thus we put n for the number of all tickets marked by the 1, 2, 3, . . . , n and let one draw at random t ; and everything reverts to determine the probability that, of a certain number of <selected> numbers that one will have chosen, there is found either one or two or three or finally all in the t tickets which one is going to draw. Now, according to the number of <selected> numbers, the determination of the probability that one seeks is reduced to the following problems.

PROBLEM 1

1. *The number of all tickets being = n , from which one must draw at random t tickets, to find the probability that one number chosen at will be found.*

SOLUTION

It is evident, by the first rules of probability, that, in order that the chosen number is found among the t tickets which one is going to draw, the number of all tickets being = n , the probability is

$$= \frac{t}{n}$$

and in order that it is not found, the probability is

$$= \frac{n - t}{n}.$$

Thus, the solution provided:

$$\begin{array}{l|l} \text{that the chosen number} & \text{the probability is} \\ \text{is found among the drawn tickets} & \frac{t}{n} \\ \text{that it is not found there} & \frac{n-t}{n} \end{array}$$

COROLLARY 1

2. Thus, if the number of all tickets is 90 and if one draws 5, as in the case proposed at the beginning,

$$\begin{array}{l|l} \text{that a chosen number} & \text{the probability is} \\ \text{is found among the 5 tickets} & \frac{5}{90} = \frac{1}{18} \\ \text{and that it is not found there} & \frac{85}{90} = \frac{17}{18} \end{array}$$

COROLLARY 2

3. If one established 100 tickets and if one wants to draw 10, having chosen a number at will, then

$$\begin{array}{l|l} \text{that this number is found} & \text{the probability is} \\ \text{among the 10 tickets drawn} & \frac{10}{100} = \frac{1}{10} \\ \text{that it is not found there} & \frac{90}{100} = \frac{9}{10} \end{array}$$

PROBLEM 2

4. *The number of all tickets being = n, from which one is going to draw t tickets, if one has chosen two numbers, to find the probability either that both at this time, or that only one, or that none are found among the tickets drawn.*

SOLUTION

We distinguish the two chosen numbers, the one by *A*, the other by *B*; and let *A* be found among the *t* tickets drawn, the probability [§ 1] is

$$= \frac{t}{n}$$

and that it is not found there,

$$= \frac{n-t}{n}$$

We suppose that *A* is found there already, and in order to see if *B* is found there also, or not, it is necessary to consider that of the *n - 1* tickets one draws only *t - 1*; and that *B* is found there, the probability is

$$= \frac{t-1}{n-1}$$

and that it is not found there

$$= \frac{n-t}{n-1}.$$

Thus, that both numbers A and B are found at this time, the probability is

$$= \frac{t(t-1)}{n(n-1)},$$

and that the number A only is found, the probability is

$$= \frac{t(n-t)}{n(n-1)}.$$

The same probability is that the number B only is found; therefore, that the one or the other is found, without distinction, the probability is

$$= \frac{2t(n-t)}{n(n-1)}.$$

Now, that none are found or that both stay among the $n-t$ numbers not drawn, the probability will be

$$= \frac{(n-t)(n-t-1)}{n(n-1)},$$

whence we draw the following conclusions

that of two chosen numbers	the probability is
both are found there	$\frac{t(t-1)}{n(n-1)}$
that one only is found there	$\frac{2t(n-t)}{n(n-1)}$
that none are found there	$\frac{(n-t)(n-t-1)}{n(n-1)}$

PROBLEM 3

5. *The number of all tickets being = n , from which one is going to draw at random t tickets, if one has chosen three numbers, to determine the probability either that all three, or only two, or only one, or none are found among the tickets drawn.*

SOLUTION

We distinguish the three chosen numbers by the letters A, B, C ; and that two, A and B , are found in the drawn tickets, the probability [§ 4] is

$$= \frac{t(t-1)}{n(n-1)},$$

and that none are found there

$$= \frac{(n-t)(n-t-1)}{n(n-1)}.$$

We suppose that the two numbers A and B are found there; and we will have yet to consider $n-2$ tickets and to seek the probability that the number C is found in the $t-2$ tickets which will remain there; now, this probability [§ 1] is evidently

$$= \frac{t-2}{n-2},$$

and that C is not there, the probability is

$$= \frac{n-t}{n-2}.$$

Thus, in order that all three numbers A, B, C are found in the tickets drawn, the probability is

$$= \frac{t(t-1)(t-2)}{n(n-1)(n-2)};$$

now, that only A and B are found there, without C , the probability is

$$= \frac{t(t-1)(n-t)}{n(n-1)(n-2)}.$$

But it will be equally probable that one finds there only the two A and C , or the two B and C ; thus, in order that only two, without distinction, are found in the t tickets drawn, the probability is

$$= \frac{3t(t-1)(n-t)}{n(n-1)(n-2)}.$$

Next, that the number A is found there, the probability [§ 1] is $= \frac{t}{n}$; but that the other two, B and C , are found among the $n-t$ remaining tickets, the number of all ahead now to be regarded as $n-1$, the probability [§ 4] is

$$= \frac{(n-t)(n-t-1)}{(n-1)(n-2)};$$

thus, in order that the number A only is found there, the probability is

$$= \frac{t(n-t)(n-t-1)}{n(n-1)(n-2)};$$

and since each of the other two, B and C , are able to be found there also probably, the probability that one alone, which let this be, is found there, is

$$= \frac{3t(n-t)(n-t-1)}{n(n-1)(n-2)}.$$

Whence we conclude:

that of three chosen numbers	the probability is
all three are found there	$\frac{t(t-1)(t-2)}{n(n-1)(n-2)}$
that two alone are found there	$\frac{3t(t-1)(n-t)}{n(n-1)(n-2)}$
that one alone is found there	$\frac{3t(n-t)(n-t-1)}{n(n-1)(n-2)}$
that none are found there	$\frac{(n-t)(n-t-1)(n-t-2)}{n(n-1)(n-2)}$

PROBLEM 4

6. The number of all tickets being $= n$, from which one is going to draw t tickets, if one has chosen four numbers, to determine the probability either that all four, or that three, or only two, or one alone, or none are found in the tickets drawn.

SOLUTION

We designate the four chosen numbers by the letters A, B, C, D ; and having already determined the probability that of the three numbers A, B, C either all three, or two, or one alone, or none are found in the tickets drawn, we have only to combine with each of these cases the probability that the fourth number, D , is found also, or not; this which will clear the path for us to drive easily our researches to as many of the chosen numbers as one will want. Thus we take the formulas found for three numbers A, B, C and we join to them the probability that the fourth, D , is found there, or not:

that of the numbers A, B, C , there are found in the chosen numbers	the probability is	that the fourth, D , is found there	is not found there
all three	$\frac{t(t-1)(t-2)}{n(n-1)(n-2)}$	$\frac{t-3}{n-3}$	$\frac{n-t}{n-3}$
only two	$\frac{3t(t-1)(n-t)}{n(n-1)(n-2)}$	$\frac{t-2}{n-3}$	$\frac{n-t-1}{n-3}$
one alone	$\frac{3t(n-t)(n-t-1)}{n(n-1)(n-2)}$	$\frac{t-1}{n-3}$	$\frac{n-t-2}{n-3}$
none	$\frac{(n-t)(n-t-1)(n-t-2)}{n(n-1)(n-2)}$	$\frac{t}{n-3}$	$\frac{n-t-3}{n-3}$

Thence, we conclude the probability:

I. that the three numbers A, B, C with the fourth, D , are found there,

$$= \frac{t(t-1)(t-2)(t-3)}{n(n-1)(n-2)(n-3)};$$

II. that the three numbers A, B, C without the fourth, D , are found there, and that two of the numbers A, B, C with the number D are found there,

$$= \frac{t(t-1)(t-2)(n-t)}{n(n-1)(n-2)(n-3)} + \frac{3t(t-1)(t-2)(n-t)}{n(n-1)(n-2)(n-3)},$$

thus, that any three of the four numbers A, B, C, D are found, the probability is

$$= \frac{4t(t-1)(t-2)(n-t)}{n(n-1)(n-2)(n-3)};$$

III. that only two of the three numbers A, B, C without the fourth, D , are found there, the probability is

$$= \frac{3t(t-1)(n-t)(n-t-1)}{n(n-1)(n-2)(n-3)},$$

and that one alone of the three numbers A, B, C with the fourth, D , is found, the probability is

$$= \frac{3t(t-1)(n-t)(n-t-1)}{n(n-1)(n-2)(n-3)};$$

thus, that any two alone of the four numbers A, B, C, D are found in the tickets drawn, the probability will be

$$= \frac{6t(t-1)(n-t)(n-t-1)}{n(n-1)(n-2)(n-3)};$$

IV. that one alone of the three numbers A, B, C without the fourth, D , is found there, the probability is

$$= \frac{3t(n-t)(n-t-1)(n-t-2)}{n(n-1)(n-2)(n-3)},$$

and that none of the three numbers A, B, C , but the fourth D , alone is found there, the probability is

$$= \frac{t(n-t)(n-t-1)(n-t-2)}{n(n-1)(n-2)(n-3)};$$

thus, that any one alone of all four numbers A, B, C, D is found in the drawn tickets, the probability will be

$$= \frac{4t(n-t)(n-t-1)(n-t-2)}{n(n-1)(n-2)(n-3)};$$

V. finally, that none of the three numbers A, B, C nor the fourth, D , are found in the tickets drawn, the probability will be

$$= \frac{(n-t)(n-t-1)(n-t-2)(n-t-3)}{n(n-1)(n-2)(n-3)};$$

PROBLEM 5

7. *The number of all the tickets being = n , from which one is going to draw t tickets, if one has chosen as many of the numbers as one would wish, to determine the probability of all the possible cases which are able to take place.*

SOLUTION

The method that I have just explained in the solution of the preceding problem serves to uncover successively the probabilities of many chosen numbers, and the law of their progression being evident, we have only to put here before the eyes the formulas for each number of the <selected> numbers that one had chosen. But in order to shorten these formulas, since n marks the number of all tickets and t the number of those that one will be drawing from them, we designate by r the number of those which remain, such that

$$r = n - t \text{ or } n = t + r.$$

I. Having chosen one number alone:

that in the drawn tickets	the probability is
is found this number	$1 \cdot \frac{t}{n} = 1A^1$
that it is not found there	$1 \cdot \frac{r}{n} = 1A^0$

II. Having chosen two numbers:

that in the drawn tickets	the probability is
both are found there	$1 \cdot \frac{t(t-1)}{n(n-1)} = 1B^2$
one alone	$2 \cdot \frac{tr}{n(n-1)} = 2B^1$
none	$1 \cdot \frac{r(r-1)}{n(n-1)} = 1B^0$

III. Having chosen three numbers:

that in the drawn tickets	the probability is
all three are found there	$1 \cdot \frac{t(t-1)(t-2)}{n(n-1)(n-2)} = 1C^3$
only two	$3 \cdot \frac{t(t-1)r}{n(n-1)(n-2)} = 3C^2$
one alone	$3 \cdot \frac{tr(r-1)}{n(n-1)(n-2)} = 3C^1$
none	$1 \cdot \frac{r(r-1)(r-2)}{n(n-1)(n-2)} = 1C^0$

IV. Having chosen four numbers:

that in the drawn tickets	the probability is
all four are found there	$1 \cdot \frac{t(t-1)(t-2)(t-3)}{n(n-1)(n-2)(n-3)} = 1D^4$
only three	$4 \cdot \frac{t(t-1)(t-2)r}{n(n-1)(n-2)(n-3)} = 4D^3$
only two	$6 \cdot \frac{t(t-1)r(r-1)}{n(n-1)(n-2)(n-3)} = 6D^2$
one alone	$4 \cdot \frac{tr(r-1)(r-2)}{n(n-1)(n-2)(n-3)} = 4D^1$
none	$1 \cdot \frac{r(r-1)(r-2)(r-3)}{n(n-1)(n-2)(n-3)} = 1D^0$

V. Having chosen five numbers:

that in the drawn tickets	the probability is
all five are found there	$1 \cdot \frac{t(t-1)(t-2)(t-3)(t-4)}{n(n-1)(n-2)(n-3)(n-4)} = 1E^5$
only four	$5 \cdot \frac{t(t-1)(t-2)(t-3)r}{n(n-1)(n-2)(n-3)(n-4)} = 5E^4$
only three	$10 \cdot \frac{t(t-1)(t-2)r(r-1)}{n(n-1)(n-2)(n-3)(n-4)} = 10E^3$
only two	$10 \cdot \frac{t(t-1)r(r-1)(r-2)}{n(n-1)(n-2)(n-3)(n-4)} = 10E^2$
one alone	$5 \cdot \frac{tr(r-1)(r-2)(r-3)}{n(n-1)(n-2)(n-3)(n-4)} = 5E^1$
none	$1 \cdot \frac{r(r-1)(r-2)(r-3)(r-4)}{n(n-1)(n-2)(n-3)(n-4)} = 1E^0$

VI. Having chosen six numbers:

that in the drawn tickets	the probability is
all six are found there	$1 \cdot \frac{t(t-1)(t-2)(t-3)(t-4)(t-5)}{n(n-1)(n-2)(n-3)(n-4)(n-5)} = 1F^6$
only five	$6 \cdot \frac{t(t-1)(t-2)(t-3)(t-4)r}{n(n-1)(n-2)(n-3)(n-4)(n-5)} = 6F^5$
only four	$15 \cdot \frac{t(t-1)(t-2)(t-3)r(r-1)}{n(n-1)(n-2)(n-3)(n-4)(n-5)} = 15F^4$
only three	$20 \cdot \frac{t(t-1)(t-2)r(r-1)(r-2)}{n(n-1)(n-2)(n-3)(n-4)(n-5)} = 20F^3$
only two	$15 \cdot \frac{t(t-1)r(r-1)(r-2)(r-3)}{n(n-1)(n-2)(n-3)(n-4)(n-5)} = 15F^2$
one alone	$6 \cdot \frac{tr(r-1)(r-2)(r-3)(r-4)}{n(n-1)(n-2)(n-3)(n-4)(n-5)} = 6F^1$
none	$1 \cdot \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{n(n-1)(n-2)(n-3)(n-4)(n-5)} = 1F^0$
etc.	

In order to shorten, I have given to each of these formulas a certain mark by which I will serve myself in the following. Thus it is from these that it will be necessary always to draw the signification of these marks.

COROLLARY 1

8. It is evident how the values of the marks of each order are easily able to be found from the values of the marks of the order preceding. The following method seems the most simple:

$$\begin{aligned}
 B^2 &= \frac{t-1}{n-1}A^1, & B^1 &= \frac{r}{n-1}A^1, & B^0 &= \frac{r-1}{n-1}A^0; \\
 C^3 &= \frac{t-2}{n-2}B^2, & C^2 &= \frac{r}{n-2}B^2, & C^1 &= \frac{r-1}{n-2}B^1, & C^0 &= \frac{r-2}{n-2}B^0; \\
 D^4 &= \frac{t-3}{n-3}C^3, & D^3 &= \frac{r}{n-3}C^3, & D^2 &= \frac{r-1}{n-3}C^2, & D^1 &= \frac{r-2}{n-3}C^1, & D^0 &= \frac{r-3}{n-3}C^0; \\
 E^5 &= \frac{t-4}{n-4}D^4, & E^4 &= \frac{r}{n-4}D^4, & E^3 &= \frac{r-1}{n-4}D^3, & E^2 &= \frac{r-2}{n-4}D^2, \\
 & & E^1 &= \frac{r-3}{n-4}D^1, & E^0 &= \frac{r-4}{n-4}D^0; \\
 F^6 &= \frac{t-5}{n-5}E^5, & F^5 &= \frac{r}{n-5}E^5, & F^4 &= \frac{r-1}{n-5}E^4, & F^3 &= \frac{r-2}{n-5}E^3, \\
 & & F^2 &= \frac{r-3}{n-5}E^2, & F^1 &= \frac{r-4}{n-5}E^1, & F^0 &= \frac{r-5}{n-5}E^0
 \end{aligned}$$

etc.

COROLLARY 2

9. The calculation of the values of all these marks can therefore, in each case, easily be done by the calculus of logarithms. It is to this end that I have separated from each mark its numeric coefficient of which one will take into account easily, after having found the value of the mark.

COROLLARY 3

10. Thus it is very necessary to take guard that one does not take these marks for some powers, since the numbers which are held in the place of exponents are not true exponents of powers, but they mark only the number of <selected> numbers of which it is probable that they are found, in each case, in the tickets drawn.

COROLLARY 4

11. Since the probabilities, taken together, of all the possible cases of each order must give a complete certitude, their sum will always be equal to unity. Thus one will have:

$$\begin{aligned}
 1A^1 + 1A^0 &= 1 \\
 1B^2 + 2B^1 + 1B^0 &= 1 \\
 1C^3 + 3C^2 + 3C^1 + 1C^0 &= 1 \\
 1D^4 + 4D^3 + 6D^2 + 4D^1 + 1D^0 &= 1 \\
 &\text{etc.}
 \end{aligned}$$

PROBLEM 6

12. *Having established a similar of lottery of n tickets, of which one is going to draw t tickets, to determine the prize, consistent with the law of equality, that one is obliged to pay to the participants in each case, in ratio to their wager.*

SOLUTION

Since the prize is always proportional to the wager, we suppose the wager always of one ecu, so that, for each ecu that the participant will have paid, there will be returned to him the prize that we are going to determine consistent with the laws of equality. For this effect, it is necessary to consider separately the case where the participant will have chosen one or two or three or four etc. numbers, these which lead us to the following researches.

I. If the participant has chosen only one number and if he has paid one ecu,

in case that in the tickets drawn	the probability being	let the prize
this number is found	$1A^1$	a
that it is not found there	$1A^0$	0

The probability is thus A^1 that he receive a , and A^0 that he will receive nothing, whence his advantage is $= A^1a$, which, according to the rules of equality, to him ought to be worth as much as his wager 1. By which it is necessary that there be $A^1a = 1$, and hence the prize ought to be

$$a = \frac{1}{A^1}.$$

II. If the participant has chosen two numbers and if he has paid one ecu,

in case that in the tickets drawn	the probability being	let the prize
both are found	$1B^2$	a
one alone	$2B^1$	b
none	$1B^0$	0

The advantage to the participant will be thus $1B^2a + 2B^1b$, which ought to be equal to the wager 1, so that we have

$$1B^2a + 2B^1b = 1,$$

whence one is able to determine the two prizes a and b by an infinity of different ways, because, any value that one takes for the one, one will find that of the other. But since it is necessary to avoid the case where one would vanish or even become negative, one will fill this condition most conveniently by sharing unity between the two portions α and β , so that there is

$$\alpha + \beta = 1;$$

and then, one will have the prize in general

$$a = \frac{\alpha}{1B^2} \text{ and } b = \frac{\beta}{2B^1}.$$

III. If the participant had chosen three numbers and if he has paid one ecu,

in case that in the tickets drawn	the probability being	let the prize
all three are found	$1C^3$	a
only two	$3C^2$	b
one alone	$3C^1$	c
none	$1C^0$	0

The advantage of the participant being thus

$$1C^3a + 3C^2b + 3C^1c,$$

it is necessary that it be equivalent to the wager 1. For this effect, we apportion the wager 1 at will into three parts α, β, γ , so that there is

$$\alpha + \beta + \gamma = 1;$$

and thence, we will have in general the following determinations of the prizes

$$a = \frac{\alpha}{1C^3}, \quad b = \frac{\beta}{3C^2}, \quad c = \frac{\gamma}{3C^1},$$

whence one sees that these three prizes are able to be varied to infinity.

IV. If the participant had chosen four numbers and if he has paid one ecu,

in case that in the tickets drawn	the probability being	let the prize
all four are found	$1D^4$	a
only three	$4D^3$	b
only two	$6D^2$	c
one alone	$4D^1$	d
none	$1D^0$	0

We divide now unity into four parts equal or unequal, as one will judge opportune, or we put

$$1 = \alpha + \beta + \gamma + \delta;$$

and the values of the prizes are

$$a = \frac{\alpha}{1D^4}, \quad b = \frac{\beta}{4D^3}, \quad c = \frac{\gamma}{6D^2}, \quad d = \frac{\delta}{4D^1}.$$

V. If the participant had chosen five numbers and if he has paid one ecu,

in case that in the tickets drawn	the probability being	let the prize
all five are found	$1E^5$	a
only four	$5E^4$	b
only three	$10E^3$	c
only two	$10E^2$	d
one alone	$5E^1$	e
none	$1E^0$	0

Let one divide at will unity into five parts, so that there is

$$1 = \alpha + \beta + \gamma + \delta + \epsilon;$$

and one will have in general the just determination of these prizes

$$a = \frac{\alpha}{1E^5}, \quad b = \frac{\beta}{5E^4}, \quad c = \frac{\gamma}{10E^3}, \quad d = \frac{\delta}{10E^2}, \quad e = \frac{\epsilon}{5E^1}.$$

This determination is so easy that it would be superfluous to go further, and it is not probable that one ever makes use of more than 5 numbers, because of the too exorbitant prizes that one would have to grant.

COROLLARY 1

13. Thus it is only in the first case that the prize is determined, $a = \frac{1}{A^1}$; in the other cases, one is able to vary accordingly the prizes as much as the number of chosen tickets is great. This depends on the division of unity into as many parts as there are prizes in each case.

COROLLARY 2

14. The first way to divide unity is that which takes the parts equal, and then one will have

$$\text{for the second case} \quad \alpha = \beta = \frac{1}{2},$$

$$\text{for the third} \quad \alpha = \beta = \gamma = \frac{1}{3},$$

$$\text{for the fourth} \quad \alpha = \beta = \gamma = \delta = \frac{1}{4},$$

$$\text{for the fifth} \quad \alpha = \beta = \gamma = \delta = \epsilon = \frac{1}{5},$$

whence one will draw the determined prizes for each case.

COROLLARY 3

15. If one judges that, in this fashion, the highest prize of the superior cases becomes too great, the numeric coefficients provide us a similar way to divide which, rendering the formulas more simple, seem to produce a kind of equality. One could take

$$\begin{array}{ll}
 \text{for the IIInd case} & \alpha = \frac{1}{3}, \quad \beta = \frac{2}{3}, \\
 \text{for the IIIrd} & \alpha = \frac{1}{7}, \quad \beta = \frac{3}{7}, \quad \gamma = \frac{3}{7}, \\
 \text{for the IVth} & \alpha = \frac{1}{15}, \quad \beta = \frac{4}{15}, \quad \gamma = \frac{6}{15}, \quad \delta = \frac{4}{15}, \\
 \text{for the Vth} & \alpha = \frac{1}{31}, \quad \beta = \frac{5}{31}, \quad \gamma = \frac{10}{31}, \quad \delta = \frac{10}{31}, \quad \epsilon = \frac{5}{31}.
 \end{array}$$

COROLLARY 4

16. If one wanted to decrease the advantage of the highest prize in order to render the others more considerable, one could be served by the following divisions:

$$\begin{array}{ll}
 \text{for the IIInd case} & \alpha = \frac{1}{5}, \quad \beta = \frac{4}{5}, \\
 \text{for the IIIrd} & \alpha = \frac{1}{16}, \quad \beta = \frac{6}{16}, \quad \gamma = \frac{9}{16}, \\
 \text{for the IVth} & \alpha = \frac{1}{43}, \quad \beta = \frac{8}{43}, \quad \gamma = \frac{18}{43}, \quad \delta = \frac{16}{43}, \\
 \text{for the Vth} & \alpha = \frac{1}{106}, \quad \beta = \frac{10}{106}, \quad \gamma = \frac{30}{106}, \quad \delta = \frac{40}{106}, \quad \epsilon = \frac{25}{106}.
 \end{array}$$

SCHOLIUM

17. We represent at this time the prizes that will be provided by these three different methods to divide unity in each case.

Having chosen in advance	In case that in the drawn tickets are found	The prizes according to the		
		Ist method	IIInd method	IIIrd method
one number	the number or none	$\frac{1}{A^1}$ 0	$\frac{1}{A^1}$ 0	$\frac{1}{A^1}$ 0
two numbers	both	$\frac{2}{B^2}$	$\frac{3}{B^2}$	$\frac{5}{B^2}$
	one alone none	$\frac{4}{B^1}$ 0	$\frac{3}{B^1}$ 0	$\frac{5}{B^1}$ 0
three numbers	all three	$\frac{3}{C^3}$	$\frac{7}{C^3}$	$\frac{16}{C^3}$
	only two	$\frac{9}{C^2}$	$\frac{7}{C^2}$	$\frac{16}{C^2}$
	one alone	$\frac{9}{C^1}$	$\frac{7}{C^1}$	$\frac{16}{C^1}$
	none	0	0	0

Having chosen in advance	In case that in the drawn tickets are found	The prizes according to the		
		1st method	2nd method	3rd method
four numbers	all four	$\frac{1}{4D^4}$	$\frac{1}{15D^4}$	$\frac{1}{43D^4}$
	only three	$\frac{1}{16D^3}$	$\frac{1}{15D^3}$	$\frac{1}{43D^3}$
	only two	$\frac{1}{24D^2}$	$\frac{1}{15D^2}$	$\frac{1}{43D^2}$
	one alone	$\frac{1}{16D^1}$	$\frac{1}{15D^1}$	$\frac{1}{43D^1}$
	none	0	0	0
five numbers	all five	$\frac{1}{5E^5}$	$\frac{1}{31E^5}$	$\frac{1}{106E^5}$
	only four	$\frac{1}{25E^4}$	$\frac{1}{31E^4}$	$\frac{1}{106E^4}$
	only three	$\frac{1}{50E^3}$	$\frac{1}{31E^3}$	$\frac{1}{106E^3}$
	only two	$\frac{1}{50E^2}$	$\frac{1}{31E^2}$	$\frac{1}{106E^2}$
	one alone	$\frac{1}{25E^1}$	$\frac{1}{31E^1}$	$\frac{1}{106E^1}$
	none	0	0	0

PROBLEM 7

18. *Having fixed the prizes of a similar lottery according to the law of equality, to find the diminution of these prizes, so that the entrepreneur returns to himself a prescribed profit.*

SOLUTION

By reason of the expenses that the establishment of a similar lottery requires, it is necessary to hold back something of the prizes that the law of equality provided, as that practiced in the ordinary lotteries. Beyond this, a similar lottery would be authorized only for important needs, and in this regard, the diminution of the prizes ought to be more considerable. But, since the profit is not certain, as in the other lotteries, and since it could happen that the entrepreneur, despite all probability, loses quite considerably, it is very just that the hold back of the prizes be greater than the ordinary where one is contented with 10%. Certainly, as only the greatest prizes are those which would ruin the entrepreneur, it is reasonable that one increase the hold back only in the large prizes, and that one leaves those of the smaller prizes to ten percent. A greater hold back in the small prizes would also be only too apparent and disgust the participants, whereas, in the great prizes, one does not perceive nearly the diminution, seeing that few persons are in a condition to calculate the just value.

Now, in order to procure to the cash-box a profit of 10% on the lesser prizes, one has only to multiply by $\frac{9}{10}$; thus these will be the prizes in each case which correspond to a single number. For the prizes that correspond to two numbers, one could very well multiply them by $\frac{8}{10}$, these which will produce a profit of 20%, without one realizing it easily. And when three numbers are encountered in the tickets drawn, one could, with as much reason, multiply the prize by $\frac{7}{10}$, and those who agree to four numbers by $\frac{6}{10}$, and finally those who invite five numbers by $\frac{5}{10}$, that which is equivalent to a profit of 50%. But, in each case, one will adjust the diminution of the prizes as one will judge most opportune, and one will have principally in view to round the numbers as much as it is possible. Thus having made the plan on the prizes conform to the law of equality, it will be easy to apply the diminutions most properly which best fulfill the conditions that one will have in view.

PROBLEM 8

19. The number of all tickets being 90, of which one must draw in its time 5, to prepare the plan of the prizes which agree in all cases, according to the law of equality.

SOLUTION

Here is comprised the once intended lottery and of which I spoke in the beginning. We will see soon which prizes it is able to promise, in assigning those that the law of equality requires. Now, in order to apply in this case our general formulas, we have

$$n = 90, \quad t = 5 \quad \text{and remaining} \quad r = 85,$$

whence we draw first

$$A^1 = \frac{1}{18} \quad \text{and} \quad A^0 = \frac{17}{18},$$

and these two values lead us to those of all the remaining marks. But since we have need of these values reversed, I am going to express them in a way, just as their logarithms, in order to facilitate the calculation next.

$-\ln A^0 = 0.0248236$	$1 : A^0 =$	1.0588
$-\ln A^1 = 1.2552725$	$1 : A^1 =$	18.000
$-\ln B^0 = 0.0499343$	$1 : B^0 =$	1.1218
$-\ln B^1 = 1.2752436$	$1 : B^1 =$	18.8470
$-\ln B^2 = 2.6026025$	$1 : B^2 =$	400.5000
$-\ln C^0 = 0.0753389$	$1 : C^0 =$	1.1894
$-\ln C^1 = 1.2954470$	$1 : C^1 =$	19.7445
$-\ln C^2 = 2.6176663$	$1 : C^2 =$	414.6353 ¹
$-\ln C^3 = 4.0699639^2$	$1 : C^3 =$	11748.0001 ¹
$-\ln D^0 = 0.1010443$	$1 : D^0 =$	1.262
$-\ln D^1 = 1.3158882$	$1 : D^1 =$	20.696 ¹
$-\ln D^2 = 2.6329063$	$1 : D^2 =$	429.44
$-\ln D^3 = 4.0800643^2$	$1 : D^3 =$	12024.4 ¹
$-\ln D^4 = 5.7084532$	$1 : D^4 =$	511038
$-\ln E^0 = 0.1270578$	$1 : E^0 =$	1.340
$-\ln E^1 = 1.3365728$	$1 : E^1 =$	21.706 ¹
$-\ln E^2 = 2.6483267$	$1 : E^2 =$	444.96
$-\ln E^3 = 4.0902835^2$	$1 : E^3 =$	12310.7 ¹
$-\ln E^4 = 5.7135328^2$	$1 : E^4 =$	517050 ¹
$-\ln E^5 = 7.6429517$	$1 : E^5 =$	43949270 ¹

¹In the original edition, the marked numbers are 414.6354; 11748.054; 20.695; 12024.5; 21.705; 12310.5; 517051; 43949268.

²In the original edition, the marked numbers are 4.069959; 4.0800649; 4.0902841; 5.7135334.

Therefore here is the plan of this lottery according to the three different methods:

		Ist method	IInd method	IIIrd method
Ist case	1	18	18	18
IInd case	2	200.25	133.5	80.1
	1	4.712	6.282	7.539
IIIrd case	3	3916.00 ³	1678.29	734.25
	2	46.071	59.234	51.830
	1	2.1938	2.8206	3.7021
IVth case	4	127759.5	34069.2	11884.6 ⁴
	3	751.53	801.63	559.28
	2	17.893	28.629	29.961
	1	1.2934	1.3797 ⁵	1.9251
Vth case	5	8789854.0 ³	1417718.5 ⁵	414615.7
	4	20682.00 ³	16679.035 ⁵	9755.66 ⁴
	3	246.214 ³	397.119 ⁵	348.414 ⁴
	2	8.899	14.353	16.791
	1	0.8682	0.7002 ⁵	1.0238

At present, it is easy to diminish these prizes in proportion to the profit that one wants to procure for the entrepreneur

COROLLARY 1

20. Since these three methods are equally compliant to the law of equality, nothing prevents that one put into use all three at the same time and that one grant to the participants the freedom to submit their wager to those which will please them best. So, those who will choose two or more numbers will be the masters to determine themselves either for the first method or for the second method or for the third.

COROLLARY 2

21. Certainly, it will be in the interest of the entrepreneur to exclude entirely the first method, for the case where one chooses five numbers. Because, in case that one would catch precisely all five numbers which will be drawn in the five tickets, the prize of almost 9 millions, were it diminished up to half, it would ruin the bank.

SCHOLIUM

22. Now, in diminishing the prizes according to the rules explained above [§ 18] and in rounding the numbers, one is able to form the following plan which ought probably to bring to the entrepreneur a very considerable profit, without appearing disadvantageous to the interested parties.

³In the original edition, the marked numbers are 3916.02; 8789853.6; 20682.04; 246.210.

⁴In the original edition, the marked numbers are 11844.6; 9755.68; 348.410.

⁵In the original edition, the marked numbers are 1.3791; 1417718.3; 16679.06; 397.113; 0.7001.

Plan of such a lottery of 90 tickets of which one must draw 5.

Having chosen	in case that in the drawn tickets are found	one will receive, for each ecu which one will wager, one of the following prizes		
		ecus	ecus	ecus
1 number	this number	18	18	18
2 numbers	both	160	106	64
	one alone	4	$5\frac{1}{2}$	$6\frac{1}{2}$
3 numbers	all three	2741	1174	513
	only 2	36	47	41
	one alone	2	$2\frac{1}{2}$	$3\frac{1}{2}$
4 numbers	all four	76655	20441	7130
	only 3	526	561	391
	only 2	14	$22\frac{1}{2}$	24
	one alone	1	$1\frac{1}{4}$	$1\frac{3}{4}$
5 numbers	all five	4394927 ⁶	708859	207307 ⁶
	only 4	12409	10007	5853
	only 3	172	278	243
	only 2	7	$11\frac{1}{2}$	$13\frac{1}{2}$
	one alone	$\frac{3}{4}$	$\frac{1}{2}$	1

Still one could round these numbers better and augment, by this means, insensibly to advantage to the profit of the enterprise.

PROBLEM 9

23. *The number of all tickets being 100, of which one must draw at one time 9, to prepare the plan of the prizes which agree in all cases, according to the law of equality.*

SOLUTION

The number of tickets that one must draw being here greater than before, the prizes for the case of 5 chosen numbers will no longer become so exorbitant as in the preceding plan, that which will render the execution less dangerous.

Thus having here

$$n = 100, \quad t = 9, \quad \text{and the remaining } r = 91,$$

we have first

$$A^1 = \frac{9}{100} \quad \text{and} \quad A^0 = \frac{91}{100},$$

whence we will draw the values of the marks as follows.

⁶Original edition: 4394925; 207305.

$-\ln A^0 = 0.0409586$	$1 : A^0 =$	1.0989
$-\ln A^1 = 1.0457575$	$1 : A^1 =$	11.1111
$-\ln B^0 = 0.0823513$	$1 : B^0 =$	1.2088 ⁷
$-\ln B^1 = 1.0823513$	$1 : B^1 =$	12.0879
$-\ln B^2 = 2.1383027$	$1 : B^2 =$	137.50
$-\ln C^0 = 0.1241874$	$1 : C^0 =$	1.3310
$-\ln C^1 = 1.1193349$	$1 : C^1 =$	13.1624
$-\ln C^2 = 2.1704874$	$1 : C^2 =$	148.077
$-\ln C^3 = 3.2844308$	$1 : C^3 =$	1925.0
$-\ln D^0 = 0.1664764$	$1 : D^0 =$	1.4671
$-\ln D^1 = 1.1567166$	$1 : D^1 =$	14.3455
$-\ln D^2 = 2.2030166$	$1 : D^2 =$	159.594
$-\ln D^3 = 3.3121611$	$1 : D^3 =$	2051.92
$-\ln D^4 = 4.4930512$	$1 : D^4 =$	31120.833
$-\ln E^0 = 0.2092283$	$1 : E^0 =$	1.6189
$-\ln E^1 = 1.1945051$	$1 : E^1 =$	15.6497
$-\ln E^2 = 2.2358978$	$1 : E^2 =$	172.146
$-\ln E^3 = 3.3401898$	$1 : E^3 =$	2188.72
$-\ln E^4 = 4.5162810$	$1 : E^4 =$	32830.8
$-\ln E^5 = 5.7763524$	$1 : E^5 =$	597520

From these values one will form, according to the three methods, the following plan:

Case	Winning	Ist method	IInd method	IIIRD method
I	1	11.111	11.111	11.111
II	2	68.75	45.83	27.50
	1	3.022	4.029	4.835
III	3	641.66	275.00	120.3125
	2	16.453	21.154	18.509
	1	1.462	1.880	2.468
IV	4	7780.208	2074.722	723.740
	3	128.245 ⁸	136.795	95.438 ⁸
	2	6.649	10.639	11.134
	1	0.8966	0.9564	1.3344
V	5	119504	19274.84	5636.981
	4	1313.232	1059.06	619.450
	3	43.774	70.604	61.945
	2	3.443	5.553	6.496
	1	0.626	0.505	0.738

COROLLARY 1

24. Provided that such a lottery is not drawn until the funds are accrued beyond

⁷Original edition: 1.2087.

⁸Original edition: 128.240; 95.439.

some hundreds of thousand of ecus, the entrepreneur doesn't risk too much ruin, seeing that the highest prize, after being diminished, won't go up to 100000 ecus, supposing that the stake doesn't surpass one ecu.

COROLLARY 2

25. But in case that one would wish to form such a lottery in the small and the entire funds would not rise to 100000 ecus, one well must remove the first method in the case of five chosen tickets.

COROLLARY 3

26. It is necessary also to remark that such lotteries must be drawn with many repetitions, so that, if one had favored the participants too much, the others are able to compensate the entrepreneur. Now each time, one could draw the lottery immediately as the funds surpass a certain sum proportional to the prizes that one wants to admit.

SCHOLIUM 1

27. We diminish these prizes following the rules given above [§ 18], and we will obtain this

Plan of such lottery of 100 tickets of which one must draw 9

Having chosen	in case that in the drawn tickets are found	one will receive, for each ecu which one will wager, the one of the following prizes			in general
		ecu	ecu	ecu	
1 number	this number	10	10	10	<i>a</i>
2 numbers	both	55	36	22	<i>b</i>
	one alone	$2\frac{2}{3}$	$3\frac{1}{2}$	$4\frac{1}{3}^9$	<i>c</i>
3 numbers	all three	449	192	84	<i>d</i>
	only 2	13	17	15	<i>e</i>
	one alone	$1\frac{1}{3}$	$1\frac{2}{3}$	2	<i>f</i>
4 numbers	all four	4668	1245	434	<i>g</i>
	only 3	90	96	$66\frac{2}{3}$	<i>h</i>
	only 2	$5\frac{1}{3}$	$8\frac{1}{3}$	9	<i>i</i>
	one alone	$\frac{3}{4}$	$\frac{3}{4}$	1	<i>k</i>
5 numbers	all five	59752	9637	$2818\frac{1}{2}$	<i>l</i>
	only 4	788	635	$371\frac{2}{3}$	<i>m</i>
	only 3	$30\frac{1}{2}$	49	$43\frac{1}{3}$	<i>n</i>
	only 2	$2\frac{3}{4}$	$4\frac{1}{2}^{10}$	5	<i>p</i>
	one alone	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	<i>q</i>

⁹Original edition: $1\frac{1}{4}$.

¹⁰Original edition: $4\frac{1}{4}$.

SCHOLIUM 2

28. Besides these three methods, one is able to make for each case an infinity of other apportionments of the prizes that we marked in general in this plan, on the side, by the letters a, b, c, d etc. If one wants that these prizes be already diminished for the same reason as we have diminished the others, it is necessary to determine them by the following equations:

$$\text{I. } \frac{10}{9}a \cdot A^1 = 1,$$

or

$$0.1a = 1;$$

$$\text{II. } \frac{10}{8}b \cdot B^2 + \frac{10}{9}c \cdot 2B^1 = 1,$$

or

$$0.0090909b + 0.1838383c = 1;$$

$$\text{III. } \frac{10}{7}d \cdot C^2 + \frac{10}{8}e \cdot 3C^2 + \frac{10}{9}f \cdot 3C^1 = 1,$$

or

$$0.000742115d + 0.02532467e + 0.2532467f = 1;$$

$$\text{IV. } \frac{10}{6}g \cdot D^4 + \frac{10}{7}h \cdot 4D^3 + \frac{10}{8}i \cdot 6D^2 + \frac{10}{9}k \cdot 4D^1 = 1,$$

or

$$0.0000535547g + 0.002784845h + 0.04699425i + 0.3098139k = 1;$$

$$\text{V. } \frac{10}{5}l \cdot E^5 + \frac{10}{6}m \cdot 5E^4 + \frac{10}{7}n \cdot 10E^3 + \frac{10}{8}p \cdot 10E^2 + \frac{10}{9}q \cdot 5E^1 = 1,$$

or

$$0.000003347168l + 0.000253827m + 0.00652698n + 0.07261263p + 0.3549952q = 1.$$

From these formulas, one is able to draw the following prizes which seem convenient for the pragmatic:

$$\begin{aligned} a &= 10; \\ b &= 50, \quad c = 3; \\ d &= 200, \quad e = 20, \quad f = 1; \\ g &= 1000, \quad h = 100, \quad i = 10, \quad k = \frac{2}{3}; \\ l &= 5000, \quad m = 500, \quad n = 50, \quad p = 5, \quad q = \frac{1}{2}. \end{aligned}$$

Through such a plan, the bank would not risk as much as following the first or the second method.