RECHERCHES,
SUR UN PROBLME
DU
CALCUL DES PROBABILITÉS*

Nicolas Fuss

Acta Academiae Scientiarum Imperialis Petropolitanae 1779 pp. 81–92

§ 1.

The Problem, of which there is question, has been proposed & resolved by the celebrated *Jacques Bernoulli*, in his treatise, *De arte conjectandi*. Here is the enunciation of it:

> Two persons A & B play together with two dice, & agree that each recast the dice as often as he has brought forth points at the first cast, that the one there will win a sum 1, who will bring forth the most points in all the casts, & that, if both obtain the same number of points, they will divide equally the proposed sum; but soon B, one of the players, bored with the game, offers, instead of casting the dice, to take 12 points for his part, A consents to it, one demands who has the greatest expectation to win?

§2. I have not had the occasion to see the solution of the late Mr. *Bernoulli*; all that I know of this Problem is reduced to the simple notice that I have found in a memoir of Mr. *Mallet*, Professor of Astronomy at Geneva, *sur le calcul des Probabilités*, inserted in Volume VII of the *Acta Helvetica*, where it is shown that the expectation of player A is to that of B as 15295 to 15809. I have had the curiosity to seek myself the solution of this Problem, & I have found that the expectation to win of the first player is to that of the other, not as 15295 to 15809, but as 15295 to 15378. This difference in the results, although besides of very little consequence, shows that my solution differs also essentially from that of Mr. *Bernoulli*, & it is in this regard that it must merit some attention, when even the Problem would merit none of it.

§ 3. Now the question in itself is of the most singular in this genre, if not by its difficulty, at least by the Paralogisms which one risks to make & which one can avoid only with much circumspection in the reasoning. One wishes to judge the expectation of the two players by the number of points that A can expect in his casts, all the appearances are against player B with his twelve points. For because the number of all

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 2, 2009 the points of the dice is 21 & the number of its faces 6, the probability gives $\frac{21}{6} = \frac{7}{2}$ as a mean term which player A can expect at each cast, & it is easy to see that

if he casts 1,	there is 1 point	&	0 expectation	for the following
2	2		$\frac{7}{2}$	1
3	3		$\frac{14}{2}$	2
4	4		$\frac{21}{2}$	3
5	5		$\frac{28}{2}$	4
6	6		$\frac{35}{2}$	5
	21		$\frac{105}{2}$	

this which gives the number of the points that player A can expect from all the casts $\frac{1}{6}(21 + \frac{105}{2}) = 12\frac{1}{4}$, while his companion B has only 12 points to himself. However the expectation which this latter has, to win all fixed sum, or the half at least, is greater than that of the other player A, this which can appear quite the paradox.

§ 4. However by considering attentively all the conditions on which the two players are agreed, in being reminded: 1°. that A wins all the sum only by bringing forth beyond twelve points, either the less or the more; 2°. that he wins only the half by surpassing not at all, or by bringing forth exactly twelve points; 3°. that he loses all by bringing forth with all his casts any number whatsoever below 12: one senses well that these three circumstances change entirely the state of the question, & that, as it is not each point more cast by A, which contributes to make him win, his expectation quite depends on the number of those which he casts above the twelve points of B, which beyond that incur no risks of chance. It will be necessary therefore to examine separately all the cases which can take place, in order to be in a state to resolve the Problem, & to assign justly the ratio which there is between the expectations of the two players & which, as we just saw, can not be drawn from the plurality of the points; that the probability seems to promise to player A.

§ 5. But before touching to the solution of this Problem, it will be necessary to enter into some preliminary researches on the game of dice in general, by making the enumeration of all the cases which can take place, when with a number of given dice one wishes to produce any number of points whatever. Let for this effect the number of dice $= n$, & since the number of its faces is 6, we consider the hexanomial:

$$P = z + z^2 + z^3 + z^4 + z^5 + z^6,$$

raised to the nth power

$$P^n = (z + z^2 + z^3 + z^4 + z^5 + z^6)^n;$$

and one knows that by developing this power, each coefficient of z indicates in how many ways its exponent can be the sum of n numbers. Application made to our Problem the coefficient of any power whatever of z will be the number of cases, the exponent will be the number of points, & the one of all the Hexanomial P will be the one of the dice of which one serves oneself.

§ 6. It will be good to remark here, that the extreme terms of the developed power P^n, namely x^n and x^{6n}, have the same coefficient, likewise that the neighboring terms x^{n+1} & x^{6n-1}, & generally all the terms equally extended from the first & from the last. Thus the number of indefinite points $6n - \lambda$ can be brought forth in as many ways as the number $n + \lambda$, which is its complement to seven times the number of dice, that is to say to $7n$. If therefore, in the developed power, the member $x^{n+\lambda}$ is affected by the coefficient Λ, the letter Λ indicates, in how many ways the number of points $n + \lambda$ & $6n - \lambda$, can be brought forth with n dice. Now the number of all the possible cases, or else the sum of all the ways, in which each number of points can be cast with these n dice, is equal, as we know, to the sum of the members of the Hexanomial raised to the nth power, namely 6^n Therefore by dividing, by the sum of all the possible cases, the number of ways in which the number of points $n + \lambda$ or $6n - \lambda$ can be produced, one finds the probability that this number will be cast $= \frac{\Lambda}{6^n}$.

§ 7. All comes back here to determine the letter Λ for an indefinite exponent $n + \lambda$ of z. We develop for the effect the power P^n, this which is made most easily in the following manner. Since

$$P = z(1 + z + z^2 + z^3 + z^4 + z^5) = \frac{z(1 - x^6)}{1 - z},$$

there is

$$P^n = \frac{z^n(1 - x^6)^n}{(1 - z)^n},$$

We transform the fraction $\frac{1}{(1-z)^n}$ into a series, as well as the power $(1 - x^6)^n$, & by making use, for the coefficients, of the characters[1] introduced by Mr. *Euler*, we have

$$\frac{1}{(1-z)^n} = 1 + \left(\frac{n}{1}\right) z + \left(\frac{n+1}{2}\right) z^2 + \left(\frac{n+2}{3}\right) z^3 + \left(\frac{n+3}{4}\right) z^4 + \text{etc.}$$

& by multiplying by z^n,

$$\frac{z^n}{(1-z)^n} = z^n + \left(\frac{n}{1}\right) z^{n+1} + \left(\frac{n+1}{2}\right) z^{n+2} + \left(\frac{n+2}{3}\right) z^{n+3} + \text{etc. likewise}$$

$$(1 - z^6)^n = 1 - \left(\frac{n}{1}\right) z^6 + \left(\frac{n}{2}\right) z^{12} - \left(\frac{n}{3}\right) z^{18} + \left(\frac{n}{4}\right) z^{24} - \text{etc.}$$

By multiplying therefore one of these series by the other, one will find easily, for each exponent of z, the coefficient which corresponds to it.

[1] *Translator's note*: This refers to the notation for the binomial coefficient.

§ 8. In order to find the value of Λ without passing through the preceding coefficients, we take the series for $\frac{z^n}{(1-z)^n}$, by rejecting all the terms which can not produce the power $z^{n+\lambda}$, which is only affected by the coefficient Λ, & by reversing the order of its terms we will have

$$\frac{z^n}{(1-z)^n} = \cdots \left(\frac{n+\lambda-1}{\lambda}\right) z^{n+\lambda} \cdots + \left(\frac{n+\lambda-7}{\lambda-6}\right) z^{n+\lambda-6} \cdots + \left(\frac{n+\lambda-13}{\lambda-12}\right) z^{n+\lambda-12} + \text{etc.}$$

Therefore, because

$$(1-z^6)^n = \left(\frac{n}{0}\right) - \left(\frac{n}{1}\right) z^6 + \left(\frac{n}{2}\right) z^{12} - \left(\frac{n}{3}\right) z^{18} + \text{etc.}$$

multiplication making the terms which contain the power $z^{n+\lambda}$ will be,

$$z^{n+\lambda}\left(\left(\frac{n+\lambda-1}{\lambda}\right) - \left(\frac{n}{1}\right)\left(\frac{n+\lambda-7}{\lambda-6}\right) + \left(\frac{n}{2}\right)\left(\frac{n+\lambda-13}{\lambda-12}\right) - \&\text{c.}\right)$$

therefore

$$\Lambda = \left(\frac{n+\lambda-1}{\lambda}\right) - \left(\frac{n}{1}\right)\left(\frac{n+\lambda-7}{\lambda-6}\right) + \left(\frac{n}{2}\right)\left(\frac{n+\lambda-13}{\lambda-12}\right) - \left(\frac{n}{3}\right)\left(\frac{n+\lambda-19}{\lambda-18}\right) + \&\text{c.}$$

or else, one will have by virtue of the property of these characteristics, $\left(\frac{\alpha}{\beta}\right) = \left(\frac{\alpha}{\alpha-\beta}\right)$, demonstrated by Mr. *Euler*, & by putting $n+\lambda=\mu$,

$$\Lambda = \left(\frac{\mu-1}{n-1}\right) - \left(\frac{n}{1}\right)\left(\frac{\mu-7}{n-1}\right) + \left(\frac{n}{2}\right)\left(\frac{\mu-13}{n-1}\right) + \&\text{c.}$$

For this general expression we will be in a state to assign for each number of given dice, in how many ways one can bring forth each number of possible points, as one can see by the following examples.

§ 9. Case of two dice

$\Lambda = \left(\frac{\mu-1}{1}\right) - 2\left(\frac{\mu-7}{1}\right) + \left(\frac{\mu-13}{1}\right)$

For	2	points	1	case	For	8	points	$7-2.1=5$	cases
	3		2			9		$8-2.2=4$	
	4		3			10		$9-2.3=3$	
	5		4			11		$10-2.4=2$	
	6		5			12		$11-2.5=1$	
	7		6						

The sum of all the cases is $36=6^2$.

§ 10. Case of three dice.

$\Lambda = \binom{\mu-1}{2} - 3\binom{\mu-7}{2} + 3\binom{\mu-13}{2} - \binom{\mu-19}{2}$.

Points	Cases	Points	Cases	Points	Cases
3	1	9	$28-3.1{=}25$	15	$91-3.28+3.1{=}10$
4	3	10	$36-3.3{=}27$	16	$105-3.36+3.3{=}6$
5	6	11	$45-3.6{=}27$	17	$120-3.45+3.6{=}3$
6	10	12	$55-3.10{=}25$	18	$136-3.55+3.10{=}1$
7	15	13	$66-3.15{=}21$		
8	21	14	$78-3.21{=}15$		

The sum of all the cases is $216 = 6^3$.

§ 11. Case of four dice.

$\Lambda = \binom{\mu-1}{3} - 4\binom{\mu-7}{3} + 6\binom{\mu-13}{3} - 4\binom{\mu-19}{3} + \binom{\mu-25}{3}$.

μ	Λ	μ	Λ	μ	Λ
4	1	10	$84-4.1{=}80$	16	$455-4.84+6.1{=}125$
5	4	11	$120-4.4{=}104$	17	$560-4.120+6.4{=}104$
6	10	12	$165-4.10{=}125$	18	$680-4.165+6.10{=}80$
7	20	13	$220-4.20{=}140$	19	$816-4.220+6.20{=}56$
8	35	14	$286-4.35{=}146$	20	$969-4.286+6.35{=}35$
9	56	15	$364-4.56{=}140$	21	$1140-4.364+6.56{=}20$

μ	Λ
22	$1330-4.455+6.84-4.1{=}10$
23	$1540-4.560+6.120-4.4{=}4$
24	$1771-4.680+6.165-4.10{=}1$

The sum of all the cases is $1296 = 6^3$.

Solution of the proposed Problem.

§ 12. Player A, who has conceded to player B twelve points in advance, can bring forth at the first cast either an ace, or two, or three, — or finally six. The probability that he cast one or the other of these six number is $\frac{1}{6}$. But because he can next recast the die as many times as he has brought forth points on the first cast, in order to determine his expectation on the gain for all the casts,

let for the 1 cast his expectation	$= \dfrac{1}{6}a$
2	$\dfrac{1}{6}b$
3	$\dfrac{1}{6}c$
4	$\dfrac{1}{6}d$
5	$\dfrac{1}{6}e$
6	$\dfrac{1}{6}f$

& the entire expectation of player A becomes

$$s = \frac{1}{6}(a+b+c+d+e+f).$$

In order to determine the value of the letters a, b, c, d, e, f, it will be necessary to distinguish & to resolve the following cases.

§ 13. If player A brings forth only the Ace with the first cast, he can no longer recast the die, & having only a single point against 12, he loses, & hence his expectation is null, that is to say $a = 0$.

§ 14. If the first cast give the Deuce, A will have again a cast to make. But being able to expect on the second cast only 6 points at the most, he will have in all only 8 points against 12. Therefore he loses again in this case here; there is consequently $b = 0$.

§ 15. If A casts the Three on the first cast, he has still two casts in reserve, or else he can make the second cast with two dice all at once; and because he has had only three points on the first cast, it will be necessary that he bring forth beyond 9 on the second cast. If he casts only 9, he takes the half of the sum $2M$, being on par with B. This number 9 can be brought forth in 4 ways, with two dice, as we have seen § 9. Therefore, the sum of all the possible cases being 6^2, the probability that this number 9 will be cast, is $= \frac{4}{36} = \frac{1}{9}$, & hence the expectation on the half of the stake M will be $\frac{1}{9}M$. But if the second cast produces the number 10, or 11, or 12, A wins all. The table given above for the case of two dice shows that for the number 10 there are three cases, for 11 two cases & a single case for 12. It can therefore arrive in six ways that A wins, & hence the probability is $\frac{6}{6^2}$; & the expectation of the entire gain $c = \left(\frac{1}{3} + \frac{1}{9}\right) M = \frac{4}{9}M$.

§ 16. If he brings forth the Four, he has yet three casts to make, or, that which returns to the same, he will play again one time with three dice. In casting 8 points he is on par with B & wins the half M; but in bringing forth beyond 8, he wins all the sum $2M$. For the first case he has 21 & in general 6^3 possible cases. The probability of the cast is therefore $\frac{21}{6^3}$ & the expectation on the half, of the gain $\frac{21}{6^3}M$. The sum of the cases, where the player can bring forth beyond 8 points is $6^3 - 56$, & consequently his expectation $\frac{6^3-56}{6^3}.2M$. We will have therefore $d = \frac{21-2(6^3-56)}{6^3}.M$.

§ 17. By casting the Five our player A will make the second cast with four dice, & he will bring forth 7 points in order to be on par with B, & because this number of points can be produced in 20 different ways with four dice, the expectation which he has to win the half, or to not lose, will be $\frac{20}{6^4}M$. The number of cases, where he can win all the sum $2M$, is $6^4 - 35$, & the probability $\frac{6^4-35}{6^4}$, which multiplied by $2M$ & added to $\frac{20}{6^4}M$, gives the expectation

$$e = \frac{20 + 2(6^4 - 35)}{6^4}M.$$

§ 18. Finally the number Six, brought forth on the first cast, procures to player A the permission to play the second time with five dice. The probability that he will have still 6 points, because of the 5 cases which can take place, will be $\frac{5}{6^5}$. The sum of the possible cases, where A can cast beyond 6, being $6^5 - 6$, the probability will be $\frac{6^5-6}{6^5}$;

this here, multiplied by $2M$, & added to the other, multiplied by M, we furnish the letter f, namely

$$f = \frac{6 + 2(6^5 - 6)}{6^5} M.$$

§ 19. Having determined in this fashion the letters a, b, c, &c. it will be easy to assign the entire expectation to player A, which was $s = \frac{1}{6}(c + d + e + f)$ (because $a = b = 0$). For because

$$\begin{aligned} c &= \frac{4}{9} M \\ d &= \frac{21 - 2(6^3 - 56)}{6^3} M = \left(2 - \frac{91}{6^3}\right) M \\ e &= \frac{20 + 2(6^4 - 35)}{6^4} M = \left(2 - \frac{50}{6^4}\right) M \\ f &= \frac{6 + 2(6^5 - 6)}{6^5} M = \left(2 - \frac{7}{6^5}\right) M \end{aligned}$$

we will have

$$s = \frac{1}{6}\left(2 + 2 + 2 + \frac{4}{9} - \frac{91}{6^3} - \frac{50}{6^4} - \frac{7}{6^5}\right) M$$

or else

$$s = \frac{1}{6}\left(6 - \frac{127}{6^5}\right) M = \left(1 - \frac{127}{6^6}\right) M.$$

§ 20. We see therefore now that the expectation of A is effectively smaller than that of B, which is

$$2M - s = \left(1 + \frac{127}{6^6}\right) M.$$

So that A has expectation only on the half of the sum $2M$, diminished by the part $\frac{127}{6^6} = \frac{127}{46656}$, instead that B has expectation on the half of the game increased by this 368^{th} part. The expectation of A is therefore to that of B as $1 - \frac{127}{6^6} : 1 + \frac{127}{6^6}$, or else as

$$46529 : 46783 = 15295 : 15378.$$

A ratio which differs sensibly enough from that which the late Mr. *Bernoulli* must have found.