CORRESPONDENCE OF HUYGENS AND HUDDE REGARDING GAMES OF CHANCE

EXTRACTED FROM VOLUME V
OF THE
OEUVRES COMPLÈTES
OF
CHRISTIAAN HUYGENS

No. 1374[1]

Christiaan Huygens to J. Hudde
4 April 1665
Hudde responds in No. 1375.

Numbers of the two questions of chance found other than he, that is in stead of his numbers 232, 159, 104, I find 4. 6. 9. and instead of his 14 and 19 I find 35 and 64. I am assured that mine are good.

Propose to him the question of heads or tails. It is of another kind.

No. 1375[2]

J. Hudde to Christiaan Huygens
5 April 1665
Response to No. 1375. Chr. Huygens responds in No. 1384.

SIR

Your missive of the present 4 is come to me into hand today afternoon. I wish well to confess that its début seemed very unexpected to me, and bewildered me much. It is true, *homo sum & nihil humani a me alienum puto*,[3] but, agreeing to me in all the other results with yours and also with those of the Grand Pensioner[4] in the solution of this question,[5] as I see and I myself perceive afterwards, and to have lacked justly in the two, being the only ones where you had not added the solution, I was able with much difficulty to submit or to believe it, so much more as I am not accustomed to do the thing precipitously on my calculations, slightly or superficially, but to the contrary to bring all attention possible; also I have wished to seek, again this evening, where the thing would go wrong, in order to make you part at the same time of that which is arrived to me.

Date: 1665.

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH. Prepared February 8, 2009.

[1]T. V. page 304.

[2]T. V. page 305-308.

[3]I am man & nothing human can be alien to me.

[4]Jan de Witt

[5]Here follow in the Dutch text some words of which the translator was not able to comprehend the sense.

After therefore having reviewed my calculations on these two questions, although only in haste, as it sufficed for the moment, and without having found any fault, I dared not however accuse you of error, so much less as you write very expressly to be certain on your side of not having a bad calculation. I think then if there would not be found any double sense in the enunciation of the questions, and if, you having interpreted them in a certain sense and I in another, we had each resolved some different questions and thus instead of two, two pairs. And it is also arrived following this thought because as for these questions, for the solution of which I myself have given the numbers 232, 159, 104, and to you on the contrary these 9, 6, 4, namely:

"Three players A, B, C, taking 12 tokens of which 4 are white and 8 are black, playing under the condition, that the one who, blindly, will have drawn from it the first one white token will win, and that A will take *the first*, B *the second*, *and then* C, *and then anew* A, and thus the rest by turn. One demands the mutual proportion of their chances."[6]

You understand with these words: *that* A *will take the first*, in this sense that in drawing a black token, he will return it with the 11 others before B draws, so that always one draws a token from a number of 12; then this question demands much less calculation than in the sense where I myself have interpreted it, that is that the drawn tokens are not at all returned, but each keeps his own: as it happens in the drawing of the beans at Hoorn and in Frise, during the election of the Magistrate.

In the other question, for which I myself have found the numbers 14 and 19, and you 35 and 64, namely

"Having taken, as above, 12 tokens, 4 whites and 8 blacks, A wagers against B the he will draw blindly 7 tokens, *among which he will have* 3 *whites*. One demands what is the ratio of the chance of A to that of B.[7]

You take with the words, *among which he will have* 3 *whites*, without exclusion, and myself with inclusion of more of whites than 3: because when among the 7 tokens there are found four whites, there are also three whites.

This which made me believe that you have intended these questions thus, it is that I have found that your numbers are good in that sense; but mine acceptable in the other sense, the first and the only which are come to me in thought, and perhaps also the most natural sense of these words. But, although it is, this matters little to us; always is it that your numbers agree with the one of the calculations, mine with the other; and that yours are a little easier to find than mine, as it results from the calculus of the one and the others.

Next, for that which regards the question which you propose to me as appearing easy and simple, but demanding rather meditation, namely:

"A and B casting in turn heads or tails, under the condition that the one who brings tails,[8] will take all that which is set; and A casts the first, when one has yet set nothing. One demands how much A loses if he accepts this game, or how much he should give to B in order to be able to end?"

I have wished to think at the same time, and I find that B under these conditions will profit $\frac{1}{6}$ of a ducat. At least, this is true in the sense where I interpret the words: but who knows, if we make from it the same two or many questions, so that it would be able rightly to be your turn, in case of difference, to discover the double sense. I am curious to

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[6]This is the second problem proposed by Huygens at the end of his treatise: "van Rekeningh in spelen van Gheluck."

[7]This is the fourth problem of the treatise of Huygens.

[8]Hudde has omitted inadvertently the words "must put each time a ducat, except the one who brings forth heads." See Letter 1405.

understand if we ourselves agree, although I doubt not, at least if we intend the words in a different sense. . .

SIR — Your very humble servant

J. HUDDE

No. 1384[9]

Christiaan Huygens to J. Hudde

10 April 1665

Response to No. 1375. J. Hudde responds in Nos. 1392 and 1404.

HUDDE

10 April

It is true that we have both calculated well, and that because of the double sense which we are able to give to the questions. In the question of heads or tails his solution, that A would lose $\frac{1}{6}$ of a ducat, does not agree with mine, which is $\frac{4}{27}$ of a ducat. . .

No. 1392[10]

J. Hudde to Christiaan Huygens

17 April 1665

Response to No. 1384. Chr. Huygens responds in No. 1404.

Amsterdam, 17 April 1665

SIR

In the instant that I thought to put feather to paper in order to respond to you to yours of the present 10, I received your second,[11] in which you have well conjectured in great part the cause of my silence until now. Because as I do not leave willingly something without response, I had had the intention to write to you, according to your desire, on some observations of the new or returned comet, and to add at the same time that which I would have been able to have meditated on our different solutions of the question which you had proposed; but having been prevented until now by some joyous reunions and some visits of friends from the country, I have not owing to all been able to reach my end, nor been able properly and consequently also wished to think on this question. However against this habit I had resolved now to respond only in part, since these friends and I we think to remain together yet some days, and to exit from the city tomorrow: . . .

SIR

Your very humble servant

J. HUDDE

[9] T. V. page 318.

[10] T. V. page 330.

[11] This letter of Chr. Huygens to J. Hudde has not been found.

No. 1403[12]

J. Hudde to Christiaan Huygens

5 May 1665

Response to No. 1384. Chr. Huygens responds in No. 1404.

SIR

Being returned to the house some days ago, I have not wished to neglect, among some other mathematical speculations into which I am fallen since then, to return also on the question which you have proposed and to research, as in my preceding, what would be the cause of our different results.

This question, which you have proposed for the first time in a letter of 4 April,[13] is enunciated thus:

"A and B cast in turn heads or tails, under the condition that the one who casts tails *will put each time a ducat, but that the one who casts heads*, will take all that which is set, and A casts the first, during which nothing has been set yet. The question is, how much A loses, when he enters into the game, or how much he would give to B, in order to be able to finish?"

In my response[14] of 5 April I find all these same words, so that the underlined words, which you write[15] not having found in my letter, must have been mistaken in the haste of copying, this which has been able to arrive here so much more easily than the same word "werpt" preceding and immediately following. I find also to have responded to this question, *that with this condition B will profit $\frac{1}{6}$ of a ducat, at least that this will be true in the sense that I attributed to the words, but that perhaps to this question we could be able to make yet two or more*. Because, as one had not specified here the number of casts or of the times, that one must put in a ducat, nor indicated expressly its indetermination, it seemed to me that there resulted still reason to doubt, if perhaps in the question some thing would be able to have been omitted on the determination of the *times*, or, otherwise, if one could explicate it by *one time on all sides*, or else to take it for undetermined. This last assumption will have appeared to me the most probable, and I had learned also next by your missive[16] of 10 April that you understood it thus, as there results from the words: *now, in order to avoid in the following all double sense, I would add yet that I intend that each time that A or B cast tails, he must put a ducat, so that sometimes there is found many ducats set, before the first time one casts heads, that is before one takes all that which has been set*. But as in your preceding letter[17] one finds on the subject in question yet these words: *"You occupy yourself so much more easily to examine this, that it appears that this not require much calculation, but that only it is necessary to find the path in order to attain that which is desired*; and as it was necessary to calculate more in the indeterminate sense than in the aforesaid determined sense, I choose provisionally this one in preference to the other, so that my solution regards the question thus proposed: A and B cast by turn tails or heads, under the condition that the one who casts tails, *but only for the* 1^{st} *time*, will set a ducat &c. And that you take the question in this sense here: A and B cast by turn heads or tails, under the condition that the one who casts tails, *always without ceasing*, will put a ducat, &c. But however, although no more than yourself I can see only there remains yet

[12] T. V. page 348-351.
[13] See Letter No. 1374.
[14] See Letter No. 1375
[15] This letter is lost. See Letter No. 1392.
[16] See Letter No. 1384.
[17] See Letter No. 1374.

some uncertainty in the terms of the question, nonetheless the results that we find do not agree, because according to your calculation A would lose $\frac{4}{27}$ of a ducat, and according to mine $\frac{2}{9}$.

Finally, Sir, in order to thank you in proportion to a question so ingenious, I will end by another question, of which I can assure you the same thing that you said to me of yours, that is that it is necessary to calculate very little (according to my method), and that it is necessary only to find the path in order to attain the desired end: it is enunciated thus: "A and B draw blindly in turn. A always 1 of 3 tokens, of which three there are two white and one black. B likewise always from a certain number of white and black tokens, of which the *ratio* remains invariable; under condition that the one who draws a white token will enjoy all that which is set, but that to the contrary the one who draws a black will always add a ducat: and A will draw the first before anything has been set. One demands, when one wishes to have equivalent conditions on both sides, so that, A commencing to draw, there is no advantage for any of the two, what *ratio* should be found between the aforesaid white and black tokens?"

In ending, I will remain, with my sincere complements.

SIR

Your very humble servant
J. HUDDE

No. 1404[18]
Christiaan Huygens to J. Hudde
10 May 1665
Response to No. 1392. J. Hudde responds in No. 1422.

10 May 1665

SIR

The reason for which I proposed to you the question of heads or tails, was only that, in indicating to me that which you had written with respect to the calculus in the games of chance, you added, that you did not think that one could suppose still something in particular in this matter. Because, the aforesaid question coming to mind to me a little after, it seemed to me good of you to propose it as subject of new speculation, if you had desired. I believe that now you will have perceived, as well as me, that this question is of another genre than all those that one finds in my printed treatise,[19] and that one could imagine yet many others, all distinct among them, and requiring more meditation. But the utility is not great enough, in order to employ much time. As for the question that you have well wished to propose as conclusion of your last, it appears to me first difficult rather, but it is terminated more easily than I had believed. And I find that the proportion of white and black tokens of B is between equals, that is that B must have an equal number of white and black tokens, in order to make the conditions of A and B become equivalent, as you have proposed. I would like well to know if you have found the same result, and that so much more as it seems that our calculations follow some different paths. Because, if it was otherwise, you would have found also in the question of heads or tails the true result of $\frac{4}{27}$, seeing that this is much easier than your question and that however one finds it nearly in the same manner according to my method; as also when one gives to B a certain number of white and black tokens, and when one demands how much A wins or loses then. Thus I

[18] T. V. page 352-354.
[19] Van Rekeningh in spelen van Geluck

find that when B has 1 white token and 2 black, the remainder being supposed as above, A wins $\frac{207}{343}$ of a ducat.

Since in my question of heads and tails, the condition of A is worse, because he casts first, at that time as nothing has yet been set, one could demand how much A and B would have to put at the beginning (that is each an equal sum) in order that since the beginning, *ceteris positis ut prius*,[20] their conditions are equivalent. I know not yet, to what point this question would be difficult or not, seeing that I have not yet reflected. Also I have not posed it for you to demand the solution, but only because it comes to me in mind, as resulting from the question that you have lastly proposed. I ask you only to let me know, in regard to this, if you have obtained an equal result; in finishing I am

SIR

Your very humble servant

No. 1422[21]

J. Hudde to Christiaan Huygens

29 June 1665

Response to No. 1404. Chr. Huygens responds in No. 1427.

Amsterdam 29 June 1665

SIR

Two consecutive voyages to Texel,[22] of which the first is made when your last of 10 May is arrived to me only many days after; next the arrival of my brother;[23] thus some other obstructions are cause of this tardy response, which would have dragged much too long, if you treated together things which are not able to permit delay. But as our letters speak only of solutions of questions proposed only as subjects of speculation and under condition that we have desired (because for you, as also for me, the proposition of some question has no other end), I have awaited still a long time, and wish to make use of my sleeplessness maxim, according to which I never put myself, or at least so rarely, to calculate, that when I am not employed my time more usefully or more agreeably. You demand therefore, in your last, but to know if our calculations give the same result, as much in regard to the question of an equivalent game, that I have proposed to you, as of the following, which differs little from it.

A and B draw blindly in turn, A always 1 of 3 tokens, of which two are white and 1 black, B always 1 of 3 tokens, of which to the contrary two are black and one white; under the condition that the one who brings forth a white token will enjoy all that which is set, but that the one who takes a black, will add always a ducat, and A will draw the first, when one will have nothing yet set. One demands that which A wins or loses in this case?

You find for mine that *the proportion of the white and black tokens of B is between equals, or that B must have a like number of white and black tokens, in order to render the conditions of A and B equivalents; and as for the other, that A wins* $\frac{207}{343}$ *of a ducat.*

But I, for mine, I find that the ratio of the black tokens to the whites is not at all that between equals, but that of 3 to 2; and for the other, that A will not win $\frac{207}{343}$, but $\frac{9}{245}$ of a ducat.

Since in that which regards this following question, on which you had not meditated, namely:

[20]Translator's note: The rest supposed as previously.

[21]T. V. page 380-384.

[22]Texel is both a city and part of the Frisian Island group of the Netherlands.

[23]Hendrik Hudde, born 1616 in Amsterdam.

A and B cast by turns at heads or tails, with the condition that the one who casts tails will set a ducat, but that the one who casts heads will take all that which is set; and A will cast first. One demands how much A and B must set since the beginning, that is each an equal sum, in order to make that the condition of A and B become the same;

I find for the solution of this question $1\frac{1}{3}$ ducat for the stake to them two, or $\frac{2}{3}$ for each. And I can not believe that I myself would be deceived in these calculations, seeing that I have calculated each by two different ways and that the two methods, the one by and the other without Algebra, are the same as I have employed for the solution of three questions the most difficult which are contained in your Treatise. However, as these questions do not seem to you (nor to me more) to be of such utility that one uses much time, I do not wish to assure you absolutely to have well calculated in all this; it would be possible thus that I myself am deceived by this down thence in some small letter, as it is arrived in the second of the 5 questions that you have proposed at the end of your Treatise. Because, in the calculation of the chance of A, I find, in revising it, that I have taken an o, which resembled a little to an a, being around written of the form α, for an a, whence it is resulted that for the chance or the portion of A, which was only $\frac{231}{495}$ of the stake, I found $\frac{232}{435}$, and how, following the same method, I had well calculated the chance of B, which was $\frac{159}{495}$ of the stake, there remained necessarily for the portion of C $\frac{104}{495}$ of the stake: in consequence, instead of the previously given numbers **232**, 159, **104**, you wish to put the correct numbers **231**, 159, **105**, or else these here 77, 53, 35. By this example you will be perhaps carried, when you have the occasion and the desire of it, to review again one time your calculations, seeing that in a letter of 10 April[24] you have written that you had also found good these numbers originally given and inexact **232**, 159, **104**. I find also that an error, that I have committed in the night, being fatigued by the calculation, has escaped me three times during the day, whence it results that instead of $\frac{1}{9}$, the true answer[25] of your question of heads or tails in the sense where I had taken it first, I have put $\frac{1}{6}$, and that consequently also, for the answer to the question in the sense where you have explicated it next, I have given $\frac{2}{9}$ instead of the exact number $\frac{4}{27}$, such as you have calculated. . . .

SIR

Your very humble servant
J. HUDDE

No. 1423[26]

J. Hudde to Christiaan Huygens
[29 June 1665]
Appendix to No. 1422.

SIR

As, in rereading your last,[27] I see that the question that I had proposed (and of which I wrote that there was little calculation necessary according to my method) had appeared to you first rather difficult, I will add yet here that, according to my method, it is easier to calculate and to resolve, than your first of heads or tails; and that one will have always $ca + cb = ad$, that is if for the chances that

[24]See No. 1384.
[25]The text has "facit." Latin meaning "It makes" and in this sense used to introduce the answer to problems.
[26]T. V. page 385-386.
[27]See Letter No. 1404.

$$\text{he has}\left\{\begin{array}{l} \left.\begin{array}{ll}\text{to take all, one names} & a\\ \text{to set 1} & b\end{array}\right\}\text{that of A, who casts first.}\\ \left.\begin{array}{ll}\text{to take all,} & c\\ \text{to set 1} & d\end{array}\right\}\text{that of B, who casts after him.}\end{array}\right.$$

As in the equation one has $a = 2$, $b = 1$, one finds $3 = 2d$, and consequently the proportion of the white tokens to the blacks is as 2 to 3, as you have said above.[28]

No. 1427[29]

Christiaan Huygens to J. Hudde
7 July 1665
Response to Nos. 1422 and 1423. J. Hudde responds in No. 1431.

HUDDE

7 July 1665.

SIR

Having, since my last, let rest all calculation relative to your questions, I have had now some trouble before arriving to the point reached previously seeing that I had made my preceding calculations without adding the necessary explication. Nevertheless, revived and brought back in spirit by the different results that you and I have found in the last 2 questions, I have resumed with pleasure this meditation, and, myself being reminded all that which I had forgotten, I have found it as follows. First, as for the question which I have proposed, where A and B draw in turn from 3 tokens. A always from 2 whites and 1 black, but B from 1 white and 2 blacks; under the condition that the one who draws a white token will take all that which is in the game, but that the one who draws a black will set always a ducat, and that A will draw first, when there is yet nothing in the game. In this question I do not find error in my preceding calculation, according to which A wins $\frac{207}{343}$ of a ducat, instead as you have found $\frac{9}{245}$ of a ducat. And I doubt not at all that you found my result exact, when you will check your calculation, as it will take place for you, in this which I believe, in seeing that which follows.

In your question, where anew A chooses the first of 2 white tokens and 1 black, and where one demanded what number of white and black tokens B must choose in order to make that their chances to both become equivalents — I find that your solution and mine are both lacking, yours indicating that the white and black tokens of B must be between them as 2 to 3, and mine that the number of white and black tokens must be equal. But according to my corrected calculation (by which I acknowledged to have set a $+$ for a $-$) the true proportion is the following: if one supposes that the white tokens are to the blacks as c to d, there comes $cc = \frac{1}{6}d + \frac{1}{6}\sqrt{73dd}$; so that one can not give the proportion of the white and black tokens in rational numbers, but only by approximate; and if B had 11 white tokens and 7 blacks, he was yet a little disadvantage; while, on the contrary, you give to him less white tokens than blacks.

Next, when with you one supposes that the white and black tokens of A, who casts first, are between them as a to b; and the white and black tokens of B as c to d, then I find that

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[28] According to this rule of Hudde, if a were $= 10$ and $b = 1$, it would become $11c = 10d$, that is that for B the proportion of the white tokens to the black would be of 10 to 11, while A would have 10 white tokens and 1 black; this which would render the chance of A at such point better than that of B, as B would lose quite all his money. So that the chances are not at all equals, and that therefore the rule is false. [Chr. Huygens]

[29] T. V. page 391-395.

in order to render the chances equivalents, the general rule is $cc = -dc + \frac{aadc}{ab+bb} + \frac{add}{b}$. Instead of which you find this here: $ca + cb = ad$ or $a = \frac{ad}{a+b}$. This great difference, and also that which you write that this question of equivalent chances, proposed by you, is easier to resolve according to your method than my first of heads or tails, assures me that we follow some totally different paths. Remains to see who has chosen the good. And as for me, I trust mine as far as the preceding rule that I have supposed here, that I would dare well to risk a chance according to it, by taking the part of A or of B, that one would wish me to give. But if you offered to me to make of it likewise as for your rule, I myself would sense certain of gain; because taking the part of A and leaving you that of B, and giving to A 10 white tokens and one black, there comes following your rule $11c = 10d$. That is for B 10 white tokens and 11 blacks; consequently the chance of A will be so much better than that of B, this which appears clearly, and still better when one takes the proportion of white tokens to the blacks of A still greater. Here A wins according to my rule, which serves me here, $\frac{105}{131}$ from that which one stakes each time. As for my last question of heads or tails, in order to render equals the chances of A and of B, here I find the same result as you, namely that each must set at first $\frac{2}{3}$ of a ducat.

In my papers I do not find calculation relative to the 2$^{\text{nd}}$ of the 5 questions at the end of my treatise, but the reason for which I have let pass as good the numbers 232, 159, 104, although the second alone was such, will be perhaps this one, that having calculated the chance of B (because according to my manner each chance is calculated apart), and finding that it agrees with your calculation, I have not believed necessary to take yet more trouble, as yet now I have calculated immediately the portion of C, which is $\frac{7}{33}$ of that which is found set. And here one has need of so little calculation, that I would not wish of the good or bad result to draw a conclusion in regard to that which had been able to arrive to me in the aforesaid more difficult questions.. . .

SIR

Your very humble servant

No. 1431[30]

J. Hudde to Christiaan Huygens

20 July 1665

Response to No. 1427. Chr. Huygens responds in No. 1434.

SIR

In my last[31] I believed to be so sure to have well calculated all (pushing me, as I wrote it to you, on two diverse calculations made according to two different paths and which were not new but the same by which I had calculated the principal questions contained in your small treatise of "Rekening in spelen van geluk," and obtained the same results as you), that I have not foreseen correction of my obtained results, but that, quite to the contrary, I myself was awaiting to that which you have also discovered the error of your different results, as to me I had found the better in regard to the 1$^{\text{st}}$ question and you had made part of it, and that, by according to us one time, we would have made by your response an end to these calculations of games of chance. But I confess that never anything has struck me more, nor appeared more unexpected, as your last of 7 present which is the response to my preceding; in which I see that you yourself are given the trouble to resume with ardor your meditations, which to you were more or less exited from memory after some time of relaxation, and

[30] T. V. page 400-417.

[31] See Letter No. 1422.

which however at the end you have found the same result as before of $\frac{207}{343}$, instead of mine $\frac{9}{245}$; you added that you doubted not at all that me, in revising my calculations, I would find that your result was true. Further, that in the question of an equivalent game, supposed by me, you had well found that the first announced result, to give to B a like number of white tokens and of black, was not just, but also that mine, giving to B 3 black tokens against 2 whites, was no true longer; but that in supposing the proportion of white tokens to the blacks as of c to d, you obtained this equation $c = \frac{1}{6}d + \frac{1}{6}\sqrt{37dd}$, so that in this case here the proportion of the white and black tokens could not be found in rational numbers. Since, in supposing with me that for A, who casts first, the ratio of the white and black tokens were as a to b and for B as c to d, you had found, as general rule in order to render the chances equivalents, this equation $cc = \frac{-dc+aadc}{ab+bb} + \frac{add}{b}$, instead of the better $c = \frac{ad}{a+b}$. And finally this great difference, joined to that which I had written that my question of equivalent game, according to my method, was more easy to resolve than your first of heads or tails — made you believe fervently that we followed some entirely different paths; but that you had so much confidence in your aforesaid rule, that you would dare well to risk a chance according to that there in taking the part of A or of B which one would wish to give you; but that you would be sure to gain, if likewise I offered you to make it according to my rule; because by taking the part of A and leaving me that of B, and by supposing that one gives to A 10 white tokens and 1 black, there would come to B, according to my rule, 10 white tokens and 11 blacks: thence (add you) there results evidently that the chance of A is better than that of B, and that it would become yet better, if one increased the proportion of the white tokens to the blacks of A. And that A, in this case of 10 against 1, according to the rule that you had for this, would win $\frac{105}{131}$ of that which each time had been set. But that however in your last question of heads or tails, in order to render equal the chances of A and of B, you had found the same result as me namely that since the beginning each must put $\frac{2}{3}$ of a ducat.

What could well be, do you believe, my thoughts, when I came to read all that for the 1st time? Because I saw well immediately that we ourselves can no longer be counted to agree in any of our 4 questions, although in the first and in the last the results are the same on both sides; seeing that the general rule which we had for the similar questions would not know how to conform to mine, since otherwise your two results $\frac{207}{343}$ and this last $\frac{105}{131}$, should have agreed with my $\frac{9}{245}$ and 0; but that probably the accord in these first and last questions were born of the equality of the aforesaid letters a, b, c, d, which in the others were supposed unequal.

My first thoughts fell on myself. Would I myself be deceived perhaps anew in my calculation? However there is little probability to this, seeing that I have each calculation by 2 different methods and since they had given to me the true results in other questions, and since I have found them to agree. But that which is arrived to you in the 1st question, can this not be encountered in the others? yes truly; but I know also that the foundation of this error has been posed in the night, while you half slept, and that to me I myself have been present to my other calculations with some much more awakened sense. Is it that the gentleman of Zuilichem, — being now in possession of my general rule which, as of others, is of such nature that a single example among an infinite number, and of usual an easy example to determine, can indicate its falsity when it is not good, — would itself be thus still mistaken? And that when, having nearly totally forgotten his previous reasonings, he has anew resumed the affair with ardor? Here is what seemed to me still less probable, in particular if I took account of this habit and finesse of ideas which you have acquired, to a higher degree than others, in matter of games of chance: and especially when at the same time I came to consider the rank that now you occupy among the scholars and the most excellent Mathematicians of this century. Certainly, if then I had been held to risk a chance with you under the aforesaid conditions, I would have well wished to lose something in order to be dispensed of it. I say: “something,” because being then not at all attentive to my reasonings, and nonetheless myself remembering quite well to the attention which I had given, I trusted myself a little to my proper forces. However I let at this moment the thing *in medio* and deferred my opinion until new examination. I was near to render myself to the country, in order to seek some relaxation and, outside of the teeming and the agitation of the townsmen, reassemble my ideas, which since some time had been quite distracted and dispersed by the misfortunes of the Republic; and even in order to make an experience out of myself, namely to some point, in these troubled times, I could hold myself tranquil and exempt from all fear. But I see that the highest mountains are found between the done and the said, that nothing is easier than to discover the passage which leads to tranquility and nothing more difficult than to follow it:

Rex est qui metuit nihil,
Rex est quique cupit nihil.

Thus is he, but you uproot a little the fear and desire. *Hic opus, hic labor est*. This therefore, Sir, as also some other amusing exercise, has occupied me here in the country during 5 or 6 days, and has not permitted that I sent to you sooner this response.

In rereading your letter, my thoughts are first fallen on this example of which it follows *evidently*, say you, that the chance A (according to my rule) is better than that of B, and still more when one supposes yet greater the proportion of the white tokens to the blacks of A, which chance however must be equal according to the conditions. So that thus the defectiveness of my rule was demonstrated by the evidence. By reflecting a little, I can not find that you had reason; well to the contrary, not only the justice of my rule appeared in the evidence in this example, and even still more when one took the proportion of the white tokens to the blacks of A still greater, but also the defectiveness of your rule stood out in a demonstrative manner. This rendered to me a little courage and made me find my better chance, to such point that, if we had to play according our rules, I would have judged the game fair following my rule, but very advantageous for me according to yours, if you had taken the chance of A, me that of B. So that according to my ancient faith (because at this hour I had not yet reviewed my calculations and my reasonings on this question) I had well dared and wished to risk 2 chances to the conditions offered by you.

That with the chance of B, and leaving to you that of A, I had profit in playing according to your rule, this is manifest, if only one pays attention that neither A nor B can lose, at least that A has not put firstly a ducat. *Because when A drawing from it the 1st time brings forth a white token, the game is ended.*[32] and a person neither loses nor wins anything in this case, but each remains in its entirety, seeing that at the beginning of the game nothing has been set, and that one has supposed an equal game: that is, supposed that before the beginning of the game the conditions of A are as good as those of B, and reciprocally those of B as those of A. So that it is indifferent that A takes the first, or that he sets a ducat and lets B draw. If now B has greater chance, or only equal chance, to draw this ducat by putting to it anew, or, that which reverts to the same, *if there are more, or only as many, white tokens as blacks*, then certainly the condition of B must necessarily be the better, and so much better , as B has more white tokens in proportion to the blacks, since the whites make to win the game. Because, we give only to B as many white tokens as blacks; as A

[32] I take this otherwise. [Chr. Huygens]

has put a ducat and as B must draw, it is manifest, in an equal game, that B in drawing a black token must necessarily lose a ducat, that A would pocket then with his; but if, after the drawing of a black token, these two ducats remain in the game, A in drawing the 2^{nd} time, and falling on a white token, will not have profit from it, but will enjoy only of that which came back to him already according to the equal game; but bringing forth a black token, he must anew set a ducat, so that B not only conserves a turn of equal chance on this new stake of A, but yet on the 2 ducats which already belonged to A; this which makes that if B comes to draw a white token, he would win already 2 ducats more than according to the equal game, but if he draws a black token, it is necessary to him only to set a ducat, which now already returns to A.

Whence there results that B would keep still always some portion of that which he had already lost according to the equal game, and that, consequently, he would have advantage in the same proportion. How much more advantage B would have therefore, if one gave him more white tokens than black.

When now one gives to A 10 white tokens and 1 black, there will come for B (following your little rule) more than 9 times as many white tokens than black, and if one gives to A 100 white tokens and 1 black, there will come for B more than 99 times as many white tokens than black; and thus in succession, the proportion of the white tokens to the blacks of B increasing still if one takes the proportion of the white tokens to the blacks of A yet greater. So that from this there results demonstratively that your aforesaid rule would not know how to be good.

Next, that the justice of my rule is set in *evidence* in the same examples, it is what is evident, first, from what, according to them, B *would never know how to have as many white tokens as blacks*; and since in the good proportion that one observes in the examples, which indicate to you, as ratio, that the more the proportion of white tokens to the black of A is great, the more also that of the white tokens to the blacks of B tend towards unity. Because if one gives to A 10 white tokens and 1 black, there comes for B 10 white tokens and 11 blacks; if one gives to A 100 tokens and 1 black, there comes for B 100 white tokens and 101 blacks; if one gives to A 1000 white tokens and 1 black, there comes for B 1000 tokens and 1001 blacks. And one sees easily that reason teaches us the same thing, if only one remembers from that which precedes, namely, that he can have neither gain nor loss before A has put first a ducat; because B, drawing next, must approach so much more of the equal chance to win or to set a ducat, as A has a greater chance to win, when B draws to false.

As for that which regards the other question, for which you give the answer $\frac{207}{343}$, you will see also clearly to have failed, if only you wish to calculate that which A would win from B in the case where all the conditions of the question would remain the same, to that near that A was not at all held to ever set something, but only B, who then was also the only one who would lose. And a good calculation will indicate to you that the condition of A in this circumstance, where A could even lose nothing and where B remains engaged as previously, is not worth $\frac{207}{343}$ of a ducat at all, but only $\frac{12}{49}$. Equally in this last example, where A has 10 white tokens and 1 black, and B 10 white tokens and 11 blacks, and where, according to the rule, that you have for this, A would win $\frac{105}{131}$ of that which has been set each time, you will see that A, when even that, as previous, he would not be held to ever set something, could only win from B $\frac{21}{440}$ of that which each time must be set; and consequently it is manifest that A must win even less, if, besides, he is held to set, because this can not bring gain to A, but only loss.

Now, as your general rule which is adapted to this last question must necessarily be applicable also to the preceding, thus as to your first question of heads or tails, it follows clearly that according to it you can only find by chance some concordant results in the answer of the first question, and consequently also of the last, which emanates from it with very little change.

And now I believe that you are at least also astonished, as I was myself; because, to what it seems to me, you did not expect correction to your corrections, although you said the contrary at the end of your letter, but only in jest. And perhaps you will smile now anew with me, in seeing that we have exchanged so many letters on both sides on these questions of games of chance, and that we are not more advanced, but that quite rather we have retrograded, seeing that now there is not left a single question, in which we can be assured to be completely in agreement. But it seems to me however that there begins to become times to make an end to this affair, which has endured quite long time enough. The shortest path in order to arrive there will be rightly that I show you one of my Methods, by which I had resolved all the questions of the games of chance which we have proposed, which in their order are the following 4; and thus I will be able to demonstrate at the same time that which I had said above in regard to your results of $\frac{207}{343}$ and $\frac{105}{131}$.

1st Question proposed by you.

A and B cast in turn at heads or tails, under the condition that the one who casts tails, will set a ducat, but that the one who casts heads will sweep off all that which has been set. And A casts the first, when nothing yet has been put. One demands, what is the disadvantage of A when he engages this game, or how much he must give to B in order to be able to end it?

We have both given for response, that A would lose thus $\frac{4}{27}$ of a ducat.

2nd Question proposed by me.

A and B draw by turn blindly, A always one of 3 tokens, of which two are white and one black; B equally always a certain number of white and black tokens, of which the ratio remains invariable; under the condition that the one who draws a white token will enjoy all that which has been set, but that the one to the contrary who brings forth a black token, will add always a ducat; and A will draw first, when nothing has yet been set. If now one wishes to have the equivalent condition on both sides, of such sort that, A beginning to draw, there was neither advantage for the one nor for the other, one demands what ratio there must exist between the aforesaid white and black tokens?

You said that this required ratio is as c to d, c being $= \frac{1}{6}d + \frac{1}{6}\sqrt{37dd}$; or generally $cc = \frac{-dc+aadc}{ab+bb} + \frac{add}{b}$, when one supposes that the ratio of the white tokens to the blacks of A, who casts[33] first, is as a to b. And to me I find it as 2 to 3, or generally $c = \frac{ad}{a+b}$.

3rd Question proposed by you.

When in this second question B has 2 black tokens and 1 white token, the rest being supposed as preceding; how much is it that A wins or loses then? You said that A would win then $\frac{207}{343}$ of a ducat, and I not more than $\frac{9}{245}$.

4th Question proposed by you.

A and B cast by turn at heads or tails, under the condition that the one who casts tails will set a ducat, but that the one who casts heads will sweep off all that which has been set; and A will cast first. One demands, how much A and B must set at the beginning, that is each an equal sum, in order to make the conditions of A and of B become equals?

[33] Read: draws.

Here we give both the same solution, namely $\frac{2}{3}$ of a ducat for the stake of each apart.

Finally to show to you at the same time that the question of an equal game which I have proposed, is easier to calculate, according to my method, than your first of tails or heads, thus as I had represented it, I will begin by this, in posing the ratio of the white and black tokens as above, as much of A than of B, and by calling r that which must be set after one has drawn a black token. Now, I consider 1° *That in this equal game one can not win nor lose on both sides unless A, who casts first, comes to draw a black token*[34] and thus to set r; and consequently that there is each one, which A casts[35] first, or else that he sets r and lets B draw. 2° I calculate that which returns to B from the possible stake of A, and reciprocally that which returns to A from the possible stake of B; that is that which the condition of B would be worth, if A alone was obliged to put r in drawing a black token; and again reciprocally that which the condition of A would be worth, if B alone were obliged to set. 3° I consider that these values must be equals between them. And finally 4°, as these values on both sides are expressed by two infinite progressions, of which the ratio of the terms is of the same form, as it must be necessarily, because of the equality of their sums, that their first terms on both sides are equal. So that

$$\begin{array}{ccc} 1^{\text{st}} \text{ term of B} & & 1^{\text{st}} \text{ term of A} \\ \dfrac{bcr}{ca+cb+ad} & = & \dfrac{adbr}{(a+b)\times(ca+cb+ad)} \end{array}$$

$$\text{and consequently } c = \frac{ad}{a+b},$$

as I had given it for the general answer; or else, by applying the rule to this case here, where a is equal to 2, $b = 1$, one obtains $c = \frac{2d}{3}$ or $3c = 2d$, and consequently also, for the ratio demanded of the white and black tokens of B, as above.

Next, in order to resolve your 1st question generally, it is necessary for me to consider, according to this method, in the 1st place the sum of these two aforesaid progressions, and then in second place still the 1st coup of A, as much as he can cast heads, and to end thus the game, that is to make lose one who had advantage in this game lose, all this advantage; and finally that, if one names x the better or worse chance of B, this value of the 1st coup of A, for as much as he can cast heads, the more that which could return to him from the possible stake of B, being subtracted together from that which would return to B from the possible stake of A, the difference must be equal to x, that is

$$\frac{bcr\times(ad+bd+ac+bc)}{(ca+cb+ad)^2} - \frac{addbr+abcdr}{(ca+cb+ad)^2} - \frac{a}{a+b}x = x$$

and consequently

$$\frac{bcr\times(bd+ac+bc-\frac{add}{c})}{(ca+cb+ad)^2} = \overline{1+\frac{a}{a+b}}\times x$$

Such is therefore my general rule[36] for the similar questions as your 1st and 3rd; thus, if one supposed $a,\ b,\ c,\ d$ each $= 1$, according to the content of the 1st, one obtains $\frac{2}{9}r = \frac{3}{2}x$ and $x = \frac{4}{27}r$. And likewise, when one takes $a = 2,\ b = 1,\ c = 1,\ d = 2$, following the content of the 3rd, one obtains $-\frac{3}{49}r = \frac{5}{3}x$ and $x = -\frac{9}{245}r$.

[34]Not of all, because A wins notably enough, when at the beginning he draws a white token. [Chr. Huygens.]
[35]Read: draws
[36]This rule is false. [Chr. Huygens]

So that according to the 1st question B would win $\frac{4}{27}$ of a ducat, and that according to the other he would lose $\frac{9}{245}$ of a ducat; as I had written to you.

And now one can see also at the same time the truth of that which I have said above with respect to the results $\frac{207}{343}$ and $\frac{105}{131}$ found by you: Because if one considered all in the same manner as in the questions, except that A will not be obligated to put something if he draws a black token, one will find that this will be worth to A only $\frac{addbr+abcdr}{(ca+cb+aa)^2}$, entirely so much as, according to the calculation above, there would come back to A from the possible stake of B, that which in this case of the 3rd question is not more than $\frac{12}{49}r$, or $\frac{12}{49}$ of a ducat. And when one takes $a = 10,\ b = 1,\ c = 10,$ and $d = 11$, as in the other case, where you find according to your rule that A would win $\frac{105}{131}$ of a ducat, that will not be worth to A more than $\frac{21}{440}r$, the whole in accord with that which we have said above.

In last place and finally, for that which regards the 4th question, one deduces it so easily from this previous that it is not necessary to add here the 25 letters which I employ. Consequently we will end here, and we believe that thus we are also arrived to the term of our questions of games of chance. I remain

SIR

Your very humble servant

This 20 July 1665 in the country
justly outside of the fumes of Amsterdam.

No. 1434[37]

Christiaan Huygens to J. Hudde
28 July 1665
Response to Nos. 1431. J. Hudde responds in No. 1445.

The Hague 18 July 1665.

SIR

I believe that now we will have soon an end to our questions of games of chance, but until then we have it not at all, and I see with astonishment the singular incidents, which you retain so long. You must not think that I jest in saying at the end of my last[38] that I awaited anew some correction, because I have never had one such confidence in myself as to believe that in the calculus and even in the reasonings I would not be subject to error, and now I am still much more timid than before, in seeing that Mister Hudde, after having revised his calculation to 2 or 3 times, and after having corrected in an awaken mind that which he had dragged up in drowsiness, and having found all by 2 different paths, which gave to him the same result — in seeing, say I, that nonetheless all this, he can deceive himself. He will be without doubt strangely surprised to understand this, and still more when I will dare to affirm that there has not been a fault in my calculations and that, when to he and I we have obtained a like result for one same question, I myself have calculated justly and he badly. All this I will show here, however barring correction. In beginning by your general rule

$$\frac{bcr \times (bd + ac + bc - \frac{add}{c})}{(ca + cb + ad)^2} = \overline{1 + \frac{a}{a+b}} \times x,$$

I say that if you played according to this rule and take the part of A, you would lose apparently your money. Because giving to A 1 white token and 1 black, that is posing

[37] T. V. page 419-424.

[38] See Letter No. 1427.

$a = 1$ and $b = 1$, but giving to B 1000 white tokens and 1 black, there comes according to this rule $\frac{2000999r}{4004001} = \frac{3}{2}x$, that is $\frac{4001998}{12012003}r = x$, so that x or the gain of B in each game will be less than $\frac{1}{3}r$, that is less than $\frac{1}{3}$ of a ducat, or of that which is set each time when one draws a black token. If B gave therefore to A at the beginning of each game $\frac{1}{3}$ ducat you would think that this would be sufficient wealth to A. But as since first A has one chance against one to remain at the same point or he must put a ducat, that is to lose almost this ducat afterwards – that this is 1000 against 1 to wager that B will snap it up, it follows from the problem of my little treatise on the calculus of games of chance, that A loses very little less than a half ducat, so that in accepting $\frac{1}{3}$ of a ducat, there will make in each game a remission of around $\frac{1}{6}$ of a ducat. You can understand thence that your rule is not good, but as far as the true source of the fault, I can research it according to that which you have wished to explicate only to half of your method.

But by mine, which is quite simple and which can with difficulty be unknown to you, since it is necessary only the 3rd problem of my aforesaid little treatise, with the knowledge of infinite progressions, — by mine I find that the true rule is

$$\frac{bcr \times (bd + ac + bc - \frac{add}{c})}{(ca + cb + ad)^2} = x.$$

according to whcih the loss of A in my first question of heads and tails is not $\frac{4}{27}r$, but $\frac{2}{9}r$. And in the 3rd question the gain of A is not $\frac{9}{245}$ but $\frac{3}{49}$.

But I have also found in the first question $\frac{4}{27}r$ as yourself, and in the third $\frac{207}{343}r$, it remains to say now, as I have calculated well. You must therefore know that in posing my questions I have omitted inadvertently to add at the end that *I understood that the game must not end before something had been set on one side or another*. It is followed from it that you have supposed that if A at the beginning cast heads, or else drew a white token, the game would end; and I confess that my casualness has been the first cause of it. But your fault in the calculation has prevented me from remarking that there resulted from it some misunderstanding, because myself being perceived a short time after having mailed my last[39] that this omission could give place to another interpretation of my problems, I could not however presume that this arrived indeed, since I saw that you had found the same result as I, of $\frac{4}{27}r$, in the question of heads or tails:[40] which agreement is certainly quite singular. By addition, to that which I believed, I calculated this same question according to the interpretation in which I saw to have given occasion; but finding in this case $\frac{2}{9}r$ and not $\frac{4}{27}r$, I kept myself fully assured that you had taken it in the same sense as me, and consequently I did not judge necessary to make you know something of it. And thence it is arrived also that I have not sought to resolve the 2nd question, that of the equal game, similarly to the other manner; if I had done it, I would have found that in this sense your rule of $c = \frac{ad}{a+b}$ was good, as it is indeed, and thus the idea would have come to me of double interpretation. I have therefore always considered all our questions with the aforesaid little clause, believing firmly, by the reason mentioned, that you made the same. You will find that my rule of $cc = -dc + \frac{aadc}{ab+bb} + \frac{add}{b}$ is good, according to the sense that I have imagined, and that good also are the numbers which I have obtained in the other questions, namely $\frac{4}{27}r$, $\frac{207}{343}r$ and $\frac{105}{131}r$, if you wish well to take the trouble to verify them; and you will see, at the same time, that the general solution of these questions, in this manner, gives a little more trouble than in the manner of which you have taken them.

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[39]See Letter No. 1427.

[40]See Letter No. 1422.

Finally, as for that which regards the 4th question, here it is certain that you understand the proposition in the same manner. And calculating in it by a method analogous to that of the 2nd, you have arrived also to the good solution, which is the same as mine.

So that being obliged to write again much on this matter, I will finish here, and in awaiting the end of our exercise, I remain

SIR Your obedient servant

HUYGENS DE Z.

No. 1446[41]

J. Hudde to Christiaan Huygens

21 August 1665

Response to No. 1434.

SIR

Being returned only the day before yesterday in the evening from a small trip to the interior, it is the cause that I have not responded sooner to your last: in which I see rather that I have reason to believe that in the end we will have found the complete issue of our questions of games of chance. In this it is arrived to us, as it arrives ordinarily when one disputes, and when after a half-dozen hours of disputations having found finally, all in disputing, the true *status Quaestionis*, one comes in a moment to be understood.

But I am astonished with you from all the singular circumstances which we have encountered here. And as you do not yet know them yet totally, and because at the same time you can see that never have I calculated from irregularity, not even in that which I have changed once and named an error, and finally that you can at the same time know my other and first method; — I will take still the trouble to say to you, as from the first I have considered and calculated the thing. And in order to make it in good order, I must resume it *a principio*.

Therefore it is you who have been the first to write to me a letter[42] with respect to these questions of games of chance, in which you correct two results that I had given to you for the 2nd and 4th questions inserted at the end of your little Treatise of the games of chance, where they are found without answer, so that you can see if they agreed with yours; next you propose to me to resolve your first question of heads or tails, in these terms: *A and B cast in turn heads or tails, under the condition that the one who casts tails will set each time a ducat, but that the one who casts heads will take all that which is set. And A casts the first, while nothing has yet been set. The question is, how much A loses, when he enters into this game, or how much would he be able to give to B in order to be able to end it?*

After having shown in my response[43] that there was a double sense in these 2nd and 4th questions which you had supposed, and that your numbers had relationship to one of the senses, mine to the other, I will give $\frac{1}{6}$ of a ducat as the solution of this here, but I will add with premeditation these words: Now this is true in the sense where I interpret the words: *but who knows if we do not make likewise two or very many questions*; and consequently that this can well be your turn, in case of difference, to research the double sense. I am curious to learn if you yourself agree, although I doubt it not, at least if we understand the words in the same sense.

[41] T. V. page 441-462.

[42] See Letter No. 1375.

[43] See Letter No. 1375.

In your response[44] to this you put me completely outside of fault concerning the error of suspected calculation, and you attribute the cause of it only to yourself, as not having supposed these 2 questions appropriately and without ambiguity; adding that you yourself remembered still well that when you had the first idea of that of 3 players you had understood in the same sense as I myself had intended; that, as for the other, you can not say with certitude in what manner you had understood when you proposed it, since one of the interpretations fit as well as the other; but that it mattered little in what signification one understood it; and that you find my numbers to be good in my interpretation, as yours in the other. Next, as for the other question of heads or tails you dispel, as sharply as if you had known my thoughts when I calculated it, the double sense that I had encountered, — in saying: *Now in order to avoid in the following all double sense, I will add yet this, that I intend that each time that A or B cast tails, he must set a ducat, so that sometimes there can be found many ducats set, before the first time one casts heads, that is before one takes all that which has been set. I can not see that now there remains here some uncertainty, but I doubt if the question is taken by you in this respect, since your calculation, following which A would lose* $\frac{1}{6}$ *of a ducat, does not agree with mine, because I myself find that A loses* $\frac{4}{27}$ *of a ducat.*

To this I have responded[45] to you anew, that I had not taken the question in this indeterminate sense, although I had thought well, in adding my reason; but by provision in this sense: A and B cast in turn heads or tails, under the condition that the one who casts tails will set a ducat *but only for the* 1^{st} *time &c.*; but (continued I) although not more than yourself I can see only there remains still some uncertainty in the terms of the question, however the results that we find do not agree: because according to your calculation A *would lose* $\frac{4}{27}$ *of a ducat, and according to mine* $\frac{2}{9}$. And in this letter I have proposed to you for the first time my question of the equivalent game, formed with the words of your question, as much as the enunciation can permit it.

Now in your response[46] to this I find the reason why you had proposed to me this question of heads or tails; namely, because you judge that it was of another category of questions than all those which were found in your printed Treatise, and because, by showing you that which I had calculated with respect to some games of chance I would have added that I did not think that something of singular could still be proposed in this matter. Would I not say that I not think that in this matter something singular could still be proposed, of which the fundamentals were not contained in that which I had written in these sheets of paper? since I had resolved more questions, and also following in other ways, than it was found in your Treatise. Assuredly I believe to have wished to say then that alone, and remain still in the same opinion.

Next you respond to me, on my question of the equal game, which it appeared first to you rather difficult, but that it will be ended more easily than you had believed: that you found the proportion of the white and black tokens of B as equal to equal, and that you desired namely what result I had found, seeing that our calculations seemed to follow some different paths; you gave for reason that, if I followed the same as you, I would have found also in the question of heads or tails *the true result* $\frac{4}{27}$; because this was still much easier according to your method, which was nearly the same for these two questions.

[44] See Letter No. 1384.
[45] See Letter No. 1403.
[46] See Letter No. 1404.

And after this comes your 2nd question, to which you add the answer $\frac{207}{343}$ of a ducat that A would win under this condition, and you demand to know also my result for this question. And your question of equal game terminates this letter.

Now, to all this I had responded before your fleet put to sea for the 1st time; but having been prevented by a sudden trip towards the fleet to copy this letter and to mail it to you, the thing did not make it then. This response besides did not differ from that which long time after alone I have sent[47] to you, except in the change of the results of the two questions which you have proposed, but which are of like nature. Because first I had taken for the answer $\frac{3}{49}$ instead of your $\frac{207}{343}$, but later I have multiplied by $\frac{3}{5}$ and consequently gave $\frac{9}{245}$ instead of $\frac{3}{49}$. And thus it must necessarily to follow the change of the 2 previous results of your first, namely $\frac{1}{9}$ instead of $\frac{1}{6}$ and $\frac{4}{27}$ instead of $\frac{2}{9}$.

Now I will show you how I am come there; and it will suffice to make it in your 1st question,[48] since the 2nd is of like nature. I will suppose therefore that the condition of A would be worth when

A cast for the 1st time	a
B . . .	b
A cast for the 2nd time	c
B . . .	d
A cast for the 3rd time	e
B . . .	f
&c.	

For 1. Since then the stake would not know how to rise above 2 ducats, I put, when A cast for the 1st time, that A had

1 chance for	0
1 . . .	$-1+b$
that is	$-\dfrac{1}{2}+\dfrac{1}{2}b=a$

When B cast for the 1st time, that then A had

1 chance for	0
1 . . .	c
that is	$\dfrac{1}{2}c=b$

A becoming now to cast for the 2nd time, he casts for 2 ducats, which will no longer increase, so that then from these 2 there will return $\frac{4}{3}$ to A. Since then $c=\frac{4}{3}$,[49] there comes $b=\frac{2}{3}$, and $a=-\frac{1}{6}$. So that A would lose under this condition $\frac{1}{6}$ of a ducat, as I had written firstly. In 2nd place, considering the *undetermined* stake, as I made next according to your interpretation, I used the same Method, and I found then that the chance of A had the value of this infinite progression $-\frac{1}{2}+\frac{1}{8}+\frac{3}{32}+\frac{5}{128}+\frac{7}{512}$ &c. is together $=\frac{2}{9}$. So that A would lose in this sense $\frac{2}{9}$ of a ducat, as I had given also the 1st time. When I had calculated this, I calculated again by addition, in the same manner, the chance of B and found that B won in

[47]See Letter No. 1422.

[48]It is the question which one finds formulated at the beginning of Letter No. 1403.

[49]Indeed, the chance of A, to obtain the two ducats, is worth then $2\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots\right)=\frac{4}{3}$.

one sense $\frac{1}{6}$ and in another $\frac{2}{9}$ of a ducat, this which accords with that which precedes. The reason why I interpreted first the question in this 1st sense, rather than in the 2nd, appears here clearly, because since, in proposing this question, you had added that I would favor myself so much more easily in seeking the solution of it when it had been recognized that it was not necessary to calculate much, but that only the way was to find in order to attain the end, — I was not able to apply these words to this 2nd calculation, but well to the 1st. And as I was already drowsy and as it was time to sleep, it did not seem to me proper to make this calculation in uncertainty, but rather to make you understand, with my solution, that this question, likewise as the two preceding, were not exempt of ambiguity, and next to await over there your proper ulterior determination. But what can well be, think you already, the reason of this change? Returning from the trip to the lodgings,[50] I judged that the better would be, before sending to you this letter, to examine again one time if I would know how to find where you had your answer of $\frac{4}{27}$, which anew you had affirmed[51] to be true. And I swear that this research has cost me well 3 times as much time as all the rest. I thought no longer a double sense in these words, such as they were, since now you had already sought with reflection to remove all ambiguity, and since also I had not perceived another in it. I thought no longer of a fault in my calculation, since I could not believe to have committed it, after I had calculated the chance of A, and next that of B, and that following these two calculations I had found the same result, and that by some rather different numbers, since the infinite progressions were completely different on both sides. I could no longer doubt my reasoning, since it supported itself on an entirely simple Theorem, which was, in this which I believe, the 1st of your Treatise. And the one and the other were confirmed again more by this other Method,[52] that I have communicated to you and by which, being only it subtracted that which you will see later, I had everything calculated and found conformed. I held also for rather certain that you had calculated well, seeing that now you said for the 2nd time that your answer was good. I thought therefore to put you in agreement, and to understand the Text not so much according to the words, but according to the accessory circumstances; and it seemed to me very probable that your employed this 1st Method, or one which was little different from it, since you have rather made usage of it in some of the questions discussed in your Treatise. Then my thoughts fell on the 1st coup of A, and I sought the explication in this one. After having tested now this, now that, I thought finally that A, in casting heads, and thus finishing the game, would come to win as much in this manner as he loses by the condition of the game. Now, in this thought *where one can fall easily*,[53] there is a double sense evident in the word to *win*; because one can take it in such fashion that, A casting heads, B would have nothing to give to A, in which sense I had already found the answer of $\frac{1}{6}$ and of $\frac{2}{9}$, and you had copied it; or else in this sense that A, in casting heads, would win, that is, would receive, as much from B, as he would lose by the condition of the game. In consequence, I examined then if I would obtain your answer in this sense. And having found it,[54] I no longer doubted that you also had understood this question thus; because these words: *well understood that A in casting heads on the first coup will win as much as he loses by the conditions of the*

[50]See Letter No. 1422.

[51]See Letter No. 1404.

[52]It is the method mentioned in Letter No. 1431.

[53]This does not seem to me at all.

[54]Indeed, by posing $-z$ for the mathematical expectation of A at the beginning of the game under the assumption that A casting heads receives nothing, and $-x$ for this expectation in the interpretation, in addition rather singular, of Hudde, one obtains the equation $-z + \frac{1}{2}x = -x$, whence there results $x = \frac{2}{3}z$.

game being added to the question such as you have supposed, it can easily permit these two explications. And although in your second question, of like nature, I did not obtain the same result as you, of $\frac{207}{343}$, nevertheless it seemed to me probable that you yourself must be deceived, seeing that it is more difficult to calculate and that you had given the result only for the 1st time, while you had now given already 2 times the one of the 1st, namely $\frac{4}{27}$.

With this I considered again the other circumstances which I found in this regard, such as the solution which you have given of my question of an equal game, whence it seemed to me that I could conclude also something, as being strongly linked, and having the chances to draw or to add not equals but unequals, by which it resembled your second, which differed from the first only by this inequality. Moreover, I considered also that you affirmed that the solution of mine was more difficult than that of your first; and in none of your letters I found other circumstances of which it appeared possible to conclude something in order to discover the sense of the text. And as for the greatest difficulty of this solution, it was evident when, first all was calculated, as if it concerned not at all an equal game, and that next one posed $x = 0$. And it is also that which I had done in order to control my first general calculation of an equal game, seeing that then I had already made the general calculation of your question, for which I obtained $\frac{\frac{1+v}{1-v} \times \frac{crv}{d} - vr}{1-v} = x$, whence $v = \frac{bd}{(c+d)\times(a+b)}$, or else after reduction $\frac{bcr\times\left(bd+ac+bc-\frac{add}{c}\right)}{(ca+cb+ad)^2} = x$. Whence it is evident (as also from my 2nd general calculation[55]), which has not changed in this regard that, by conceiving $x = 0$, one will find $bd + ac + bc = \frac{add}{c}$, or $add - bcd - acc - bcc = 0$, this which, after division by $d + c = 0$, gives $ad - ac - bc = 0$ or $ad = ac + bc$, as I have written to you also[56] for the general answer of similar questions of an equal game, such as mine was. Next, in that which regards your solution of this question of mine, although it differed from mine, it seemed to me nevertheless that I saw enough probably the cause, seeing that you had set the proportion of c to d as equal to equal,[57] this which could have been also, if the preceding equation had been divided by $d - c = 0$; but being divided by $d + c = 0$, I judged that you had taken a $+$ for a $-$, this which is yet an accident as strange as the coincidence of our results of $\frac{4}{27}$, particularly when one considers that later you have corrected this given answer, by noting that your answer had also been produced by the use of one sign for the other.[58] To this is added yet, that I considered that a fault of the calculation itself was able to be slipped so much more easily as you wrote, in the same letter, as the utility of this question or of similar questions was not great enough in order to employ much time; and as I myself was of the same opinion, and that however also I rated to have already employed too much time, as much to calculate as to write the letters; especially as the unhappy news of the defeat of our Fleet came to be mixed with it, which so filled my thoughts, and I was not able, nor wished, to occupy them a long time to similar speculations; all this made me to change immediately, without more reflection, my first numbers according to the aforesaid assumption, judging me to moralize sufficiently assured of your opinion, this which was not little confirmed by these mentioned circumstances; so that next until the reception of your last I have not doubted it a single time. Also in this letter[59] I give the solution of my question proposed on an equal game, not

[55] See Letter No. 1431.
[56] See Letter No. 1423.
[57] See Letter No. 1404.
[58] See Letter No. 1427.
[59] It concerns Letter No. 1422.

finding as you for the desired ratio, that of equal to equal, but that of 3 to 2, and likewise finding, for the answer of your question of an equal game, $\frac{2}{3}$ of a ducat for the stake of each. Beyond this, in order to come sooner to an issue of the affair, I added[60] my general answer for all the similar questions of an equal game, as mine was, by posing to this effect $ac + bc = ad$. However, all that with this circumspection, that never until the present have I accused you of error in calculation, but that I have only given my results, adding however that I did not believe myself to be deceived in this calculation, seeing that I had calculated the whole by 2 different ways, and expressly, that these questions seemed to not be of great enough utility (as you also yourself had said before me) in order to employ much time, and that consequently I did not wish to assure you absolutely to have well calculated in all. And I wish readily to leave to you to judge if I have not employed sufficient time, and have enough probability in order to believe to have calculated well and to have interpreted the question according to your idea.

Now, the response[61] to this letter of mine contains only, in sum, that you had revised your preceding calculations and that you had not found fault in your given answer $\frac{207}{343}$; but that you had recognized that the two given solutions, of my question of an equal game, yours and mine, were both erroneous, and that now you found according to your corrected calculation $c = \frac{1}{6}d + \frac{1}{6}\sqrt{37dd}$; that your fault had arisen by having set a $+$ for a $-$; that your obtained for the general rule $cc = -dc + \frac{aadc}{ab+bb} + \frac{add}{b}$, instead of mine $c = \frac{ad}{a+b}$; that you held for certain that we followed different paths, as much the cause of this great diversity, as also because I had written that this question of mine was easier to resolve according to my method than your first of heads or tails: and next you added an example, in which the falsity of my rule would appear evidently; but that you had found the same result as I in your question of an equal game.

Thence, I saw therefore that we ourselves agree as for the solution of your 1st and of your last question; and with that that the fault of your answer first given for my question, was arisen by having taken one sign for another, in the same way as I had thought. But I saw also little as before why we were not in agreement on all the other points. I was therefore going to the country to seek my diversion in some much more useful speculations (as I have written also in my response) and I judged that then it was not worth the trouble to conjecture anew on our agreement; but, in order to arrive to an end to our affair, which in this conjuncture of time seemed to me of so little consideration that I believed to take again rather too much than too little trouble, I thought that it sufficed, to show, by retortion, that your example did not prove the falsity of my rule, but quite to the contrary; on the other side, that the given solutions, of $\frac{207}{343}$ and $\frac{105}{131}$, were no longer any good according to the sense of the words of the words of the Text; since finally, and principally, to add one of my methods that I had employed and by which, according to my opinion, all this was able to appear more easily. Because this, in that which I believed, was the shortest path, and should necessarily lead us to the end of the affair. Such was then the content of my response.[62]

But in your response[63] to that, I see that you had forgotten something in your questions, and although this is with respect to the 1st coup of A, where I myself also had sought the explication, nonetheless it is another thing that I had presumed on so many probabilities; because you said to have forgotten inadvertently to add at the end of the question, that you intended: *that the game is not finished unless on both sides one had set something*,

[60]See Letter No. 1423.
[61]See Letter No. 1427.
[62]See Letter No. 1431.
[63]See Letter No. 1438.

while the words of the Text, which alone could be presupposed to regard the end of the game, are only the following: *the one who casts heads to sweep off all that which has been set*; and consequently when there is nothing in the game, one could not draw anything, but nonetheless the game must be ended (when one supposes that the question lacks nothing, as I had to suppose it.) And in this sense I had first given for answer $\frac{1}{6}$, next, for a subsequent determination in the manner to put, $\frac{2}{9}$ of a ducat, for the loss of A in your first question; that which you rejected then, as not being the correct result, thus later also the solutions which I have given of the question that I had proposed, considered as much in particular as in general; and nonetheless you also have found now that these were the true results according to this proper sense of the Text. But here it is necessary for me to note anew a singular accident, because I did not take the trouble to write to you that lengthy letter, in order to show you the strange accidents which so long have prevented our agreement; namely that you have never responded the least thing on the first answer that I had given, of $\frac{1}{6}$; because if thereupon you had copied to me your result, or even had compared it to mine with the intention to research where would be the fault, we would have without any doubt, if it had pleased you, found the end immediately; seeing that the thing in this sense required so little calculation, that one would have in less than no time been able to examine and calculate the whole, and consequently to discover where the fault is found. I swear also that I have never been able to comprehend why you have never responded anything to me; but it must not be that I had been able to economize, and to employ in some things more useful, this trouble that in vain I have take in order to research our agreement. Finally, there remains to say only this, that I have verified all the results that you have given, namely $\frac{4}{27}$, $\frac{207}{343}$, $c = \frac{1}{6}d + \frac{1}{6}\sqrt{37dd}$, for the questions taken in your sense, or, to speak more exactly, for your questions newly proposed; except that for 37 it is necessary to put 73, this which apparently will have been caused by the copy.[64] Next I have judged useless to calculate the rest, namely $\frac{105}{131}$[65] and the general rule for your question of an equivalent game, as being of the same nature.

Now there remains only to say a word with respect to my last interpretation, according to which instead of the numbers $\frac{1}{6}$ and $\frac{2}{9}$, that I had given before, I have given others of them; it is that I have thought, by reason of all these things mentioned competitors, that you must have supposed that A, coming to cast heads for the 1st time, had the chance 1 for x. Therefore, in first place, I believe that now you understand well the first method[66] that I have given, and that you will perceive whence the difference is resulted of which you say not to having been able to find the original; because in omitting only $\frac{ax}{a+b}$, this which is entirely the difference arising from this assumption, one finds the answer which you have calculated, and which agrees with my first general result.

In second place, I swear that I have not spoken too properly, when in giving $\frac{1}{6}$ and $\frac{2}{9}$ instead of my preceding results $\frac{1}{9}$ and $\frac{4}{27}$, I added that in this I had committed a fault; because properly one can say only that there is a fault when, having well understood the words according to the literal sense, one comes to disregard the thought of the author, expressed either ambiguously or insufficiently. But as now I believed certainly to have discovered your thought, to such point that ever, until the arrival of your last letter, I have not doubted, I did not regard of so nearly at this time then. I say, in these times then, because if the times had been a little more cheerful for our Fatherland; if there had not been divergence of opinion and that to many resumptions, this which rendered me a little

[64] Indeed, in the draft of Letter No. 1427, one will find the correct number, that is 73.

[65] See Letter No. 1427.

[66] This from Letter No. 1431.

morose; if I had not begun some more useful speculations; perhaps then I would have been a little more exact and I would have taken account of this difference. But wishing now to speak with more precision, I would say, first, that none of us two have until here resolved the question in the sense that the words of the Text allow exactly, because since in the Text one does not find a word to terminate or to finish the game, one would need well rather to consider the game as without end than as ended. In 2nd place, when one wishes to suppose an end to the game, so that it has been made on both sides, I believe that my first interpretation, following which I have given $\frac{1}{6}$ and $\frac{2}{9}$, is the only one, mathematically speaking, that one can apply to the words of the Text, which was presupposed to regard the termination of the game; and consequently that none of your given solutions previously is good, excepting only that of the last question; seeing that it is necessary to suppose that the questions are presented well and exactly, especially after one has sought expressly to rid them of all ambiguity, and that one can not give another sense to the words, which returns themselves to the end of the game, as the one that I have given. And finally, supposing that some thing has been omitted for the complete determination of your thought, it seems to me that one will fall more easily into my interpretation, than into yours (at least that is arrived to me); because my addition introduces only one ambiguity in the words of the Text, by leaving to them their entire sense; but you remove it immediately, and suppose thus an entirely other question.

You see therefore that this question of heads or tails, which you have proposed, is much more fecund than are the 2nd and 4th placed at the end of your Treatise, which are only doubled, while this has engendered from it at least two others, as I had already presumed in my first response.

So that therefore if it results here that I am myself deceived in no part, not even in the case where for these two results $\frac{1}{6}$ and $\frac{2}{9}$ I have given others of them, nor when, going to retortion, I have defended my general rule and my results, and on the other hand accuse yours; since all this remains true when one regards the game as limited and when one supposes then, as one must, that the questions are announced to sufficiency and not imperfectly. No part, say I, excepting only your 2nd Question, at the end of your Treatise of games of chance, where I had given for the desired numbers 232, 159, 104, arisen, as I have written, [67] from that which I had taken for a and o which resemble slightly an a. And for that all our little adventures are here gathered, it is necessary still to note one which arrives rarely also, namely that you thought first that I myself was deceived with respect to these numbers, and that you gave in their stead the numbers 9, 6, 4; and that next, after I had shown that your numbers were correct in another sense than the one where I had taken for myself the question, you responded to me, in return, that you had also found mine good in my interpretation, although however of these three numbers there was only one which was good, and that from this number alone it was impossible to conclude the others.

Finally, Sir, you see what trouble I have taken in order to reconcile our thoughts, and that I have offered in all our questions of games of chance; but that nonetheless, for the three lasts, that has not yet wished to succeed results before, my general calculations being come by hand to you, it had pleased to explicate to you more precisely your opinion. To that therefore in the following (if anew some similar thing arrived) it would be necessary better to pay attention: because you can be assured that otherwise I myself would be engaged very easily in the same labyrinth, as being unhappy to guess your opinion, and nevertheless disposed to the highest degree to the harmony of our thoughts.

[67] See Letter No. 1422.

In ending I will remain
SIR

Your very humble servant
J. HUDDE