ENCYCLOPÉDIE MÈTHODIQUE
TOME 2, PP. 404–408

JEAN BERNOULLI III

MEAN *to take among observations*, (*Arith.*) This subject seems to me to have become one of those which is more of a province of a work such as the one here. The *Dictionnaire raisonné des Sciences*, &c. seems to promise in the *word* ARITHMÉTIQUE to treat it in the *word* MOYEN, but one does not find there his waiting satisfied; I will endeavor to remedy at least in part to this omission.

When one has made many observations of one same phenomenon, & when the results are not completely in accord with one another, one is certain that these observations are all, or at least in part approximate, from some source from which the error is able to arise; one has custom then to take the *mean* among all the results, because in this manner the different errors apportion themselves equally in all the observations, the error which can be found in the mean result will become thus mean among all the errors. There is no doubt that this practice is very useful in order to diminish the uncertitude which is born of the imperfection of instruments & of the inevitable errors of observations; but it is easy to realize that it does not diminish it as much as one would desire it, & that it is susceptible in more than one regard to be improved, because by taking simply the arithmetic *mean,* one does not take account of the more or less probability of the exactitude of the observations, of the different degrees of skill of the observers, &c. Different great geometers have undertaken this useful research, they have considered it under different points of vue, & have treated it more or less in detail; it is strongly wished that the astronomers, the physicians & generally all the observers, benefit from the results of these researches in the discussion of their observations.

Father Boscovich has been led to meditate on this matter, when he had sought to draw the mean ellipticity of the earth from all the known degrees, by proposing himself the solution of the following problem: *Being given a certain number of degrees, to find the correction which it is necessary to make in each of them, by observing these three conditions; the first, that their differences be proportional to the differences of the versine of a double latitude; the second, that the sum of the positive corrections be equal to the sum of the negative; the third, that the sum of all the corrections, as many positive as negative, be the least possible for the cases where the first two conditions are satisfied*. He has exposed the result of this solution in *Volume IV* of the *Mémoires de l'institut de Boulogne*; he has developed it in his *Supplémens de la Philosophie*, in Latin verse, composed by Mr. Benoît Stay, *Volume II*, p. 420; & the translator of his *Voyage astronomique & géographique* has made it the subject of a very interesting note which is found at the end of his translation; & in which one sees this solution applied to a table of measured degrees, more extended than the one of which Father Boscovich has made use in the supplement cited. I believe the reader to be able to return to these different sources who will wish to get an idea of this method.

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH .

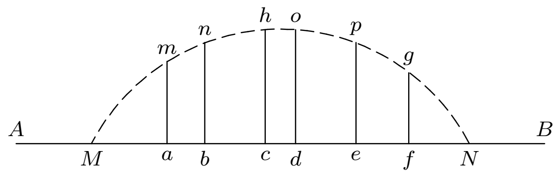

FIGURE 2

I will not stop longer at the theory which M. Lambert has given *on the degree of certitude of observations & experiences*, in the first volume of his German *Mémoires de mathématique*,[1] & which he has clarified by many examples: this work is known. One will find an extract of the memoir of which I speak, in the *Journal littéraire* which appears at Berlin; & without doubt that an able geometer who is charged to give in these supplements the substance of different interesting writings of Mr. Lambert, will not permit passing over this one here.

I will limit myself here to a summary of two memoirs which are not printed; & if one joins the lecture of this which one owes to Fr. Boscovich & to Mr. Lambert on the same matter, one can satisfy oneself on all the principle questions in which it can take place: I am ignorant if some other authors have treated it.

The first memoir of which I propose to give the extract, is a small Latin writing of Mr. Daniel Bernoulli, which he communicated to me, in 1769, & which he maintained for a long time among his manuscripts in the plan without doubt of extending it further. It has for title: *Dijudicatio maxime probabilis plurium observationum discrepentium; atque verisimillima inducio inde formanda*.

Mr. Bernoulli supposes that one represents by some portions $Aa,\ Ab,\ Ac,$ &c. of a straight line AB (*fig. 2, pl. I of Géométrie*), the results of a certain number n observations, & he remarks that, under this assumption, the ordinary practice would give for the mean, among these observations, a straight line $AC = \frac{Aa+Ab+Ad+\text{\&c.}}{n}$ but, he says, one does not take acccount in this fashion of the different degrees of probability of the observations, & however there is no doubt that the small errors take place no less often than the great. Consequently from this remark, he supposes that the number of observations which fall on the points $a,\ b,\ d, e,$ &c. are proportional to the perpendiculars $am,\ bn,\ do,\ ep,$ &c. & this hypothesis gives $AC = \frac{Aa\cdot am+Ab\cdot bn+Ad\cdot do+Ae\cdot ep}{am+bn+do+ep,\ \text{\&c.}}$, an expression which shows that the point C falls no longer on the center of gravity of the points $a,\ b,\ d,\ e,$ &c. but in the one of the lines $am,\ bn,\ do,\ ep,$ &c.

One can, by many considerations, adopt a half-ellipse or a semi-circle for the curve $MmnoN$, which passes through the points $m,\ n,\ p,$ &c. & the radius will indicate the greatest error, or a little beyond, that an observer can ever commit in making some observations such as those of which there will be question. It is therefore necessary that each observer judge himself impartially & with sagacity.

Mr. Bernoulli observes next that the analytic determination of the center of the regulating semicircle will be a very difficult applicatiion, because one attains a nearly intractible equation; this is why he prefers the method of approximation that we are going to show.

[1] *Translator's note*: That is, the *Beyträge*.

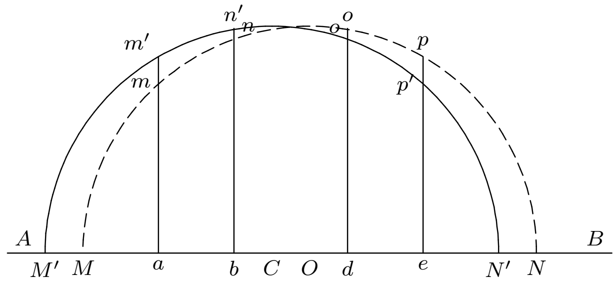

FIGURE 3

Let AB (fig. 3) be the line on which one reports the observations; let one adopt on this line a fixed point A, & let one suppose that the observations fall on the points $a,\ b,\ d,\ e,$ &c. in such a way that $AO = \frac{Aa+Ab+Ad+Ae+Af}{n}$, by seeking first by the ordinary rule the point O mean among the observed points $a,\ b,\ d,\ e,$ &c. & by understanding by n the number of observations. Let one describe next on the center O, & with the radius r, the semicircle $MmnopqN$, and let one take it for the first moderating semicircle, in such a way that perpendiculars $am,\ bn,\ do,\ ep,$, &c. on MN, express the different degrees of probability of analogous observations. Let after this one choose the center of gravity of all the lines $am,\ bn,\ do,\ ep,$, &c., it will fall near enough to the point C, by making $AC = \frac{Aa\cdot am+Ab\cdot bn+Ad\cdot do+Ae\cdot ep+\text{\&c.}}{am+bn+do+ep+\text{\&c.}}$; but if, from this point C, & with the radius r, one describes a second moderating semicircle $M'm'n'o'p'N'$, & if one repeats the same operation, one will find another point C' little distant from the first C, but more correct, & one can continue in the same manner until the difference is hardly sensible.

After this exposition of his method, Mr. Bernoulli observes that the line Aa being arbitrary & remaining invariable in all the operation, one can make $Aa = 0$, & suppose the beginning precisely at the extremity a, so that $aC = \frac{ab\cdot bn+ad\cdot do+ae\cdot ep+\text{\&c.}}{am+bn+do+ep+\text{\&c.}}$.

Passing next to an example, he supposes that one has made three observations which fall at the points $b,\ d,\ e,$ & he takes of 1000 parts the radius to which he wishes to compare the distances.

By admitting moreover, he says, that the greatest error be of $160''$, & that one has found bd, by example, of $120''$ or of $200''$, it is necessary to make $bd = 750$ or $= 1250$ parts. Thus, the distance of a point to the center of the moderating semicircle being given, one will find, without other calculation, its application, by searching in the tables the sine which correspond to this distance regarded as a cosine.

Let therefore $bd = 900$ parts & $be = 1200$ parts, one will have $bO = 700$ parts, & this will be, following the ordinary rule, the distance between the observed point b & the true position. One will have moreover $Od = 200$ parts, & $Oe = 500$ parts; therefore $bn = 866$ parts, & consequently $bC = \frac{900\cdot 980+1200\cdot 866}{714-930+866} = 750$ parts. Therefore since bC surpasses bO, it follows that the point C must be taken on the other side, or that it is necessary to place it between O & d, whence result $OC = -50$ parts for the first correction under the adopted hypothesis. By passing now to the second, that is to say, by choosing the point C', we take for center the pont C that one just found, & we will have at present $bC = 750$ parts, & $bn' = 661$; $Cd = 150$ & $dO' = 989$; $Ce = 450$ & $ep' = 893$; finally $bC' = \frac{900\cdot 989+1200\cdot 893}{661+989+893}$. This second correction differs yet enough sensibly from the first, one will seek a third by taking C' for the center of the semicircle, & the same process gives $bC'' = 780$, a distance which differs yet less from 771 than 771 differed from 750; the fourth correction gives 784; the fifth, 787, & one will find finally the truth expressed by 792: moreover, by making these operations, one will notice several expedients to the mean by which one can shorten them.

If one takes the moderating semicircle too great, continues Mr. Bernoulli, one could take off a great part of its utility: because, we suppose his radius of 1500 parts instead of 1000, all things equal besides, it will be necessary to change the 1500, 900 & 1200 parts which one had previously into 1000, 600 & 800 parts greater by half. The second correction bC will become nearly 471 parts, & it will be necessary to be taken, because one will never find one greater: now these 481 parts are worth only 721 parts, under the preceding assumption. Thus, the comparison of these two examples show how much it matters that each observer knows to appreciate his dexterity.

I just indicated the substance of the memoir of Mr. Daniel Bernoulli, I pass to the second memoir of which I have said I would give an extract; it is from Mr. de la Grange, & has for title: *Mémoire sur l'utilité de la méthode de prendre le milieu entre le résultat de plusieurs observations, dans lequel on examine les avantages de cette méthode par le calcul des probabilités, & où l'on résout différens problêmes relatifs à cette matière*. One will see that the ten problems which made the object comprehend all that which one can expect in the most delicate & most varied analysis in this matter.

Here is first the first problem that Mr. de la Grange proposes: one supposes that, in each observation, one can be deceived by one unit, as much greater as lesser; but that the number of cases which can give an exact result, is to the number of cases which can give an error of one unit as $a : 2b$; one demands what is the probability of having an exact result, by taking the mean among the particular results of a number n of observations?

The solution of this problem gives $\frac{A}{(a+2b)^n}$ for the sought probability, & Mr. de la Grange shows that one can determine in addition in a manner the coefficient A, which he finds $= a^n + n(n-1)a^{n-2}b + \frac{n(n-1)(n-3)a^{n-4}b^4}{2\cdot 2} + \frac{n(n-1)(n-2)\cdots(n-5)a^{n-6}b^6}{2\cdot 3\cdot 2\cdot 3}$ +&c. He draws next from his solution different corollaries, & he determines in a first remark, the law which the terms of the series $\frac{1}{5}$, $\frac{9}{25}$, $\frac{1}{5}$, $\frac{29}{125}$ follow, &c., which represents the probabilities which correspond to 1, 2, 3, &c. observations; this law is discovered by the expressions which follow, & in which A', A'', A''', &c. designate the values of A' which correspond to $n = 1$, 2, 3, &c.; one has

$$\begin{aligned} A' &= a \\ A'' &= \tfrac{3aA'+4b^2-a^2}{2} \\ A''' &= \tfrac{5aA''+2(4b^2-a^2)A'}{3} \\ A^{iv} &= \tfrac{7aA'''+3(4b^2-a^2)A''}{4}, \text{ \&c.} \end{aligned}$$

Some other similarly important remarks follow the first, & lead Mr. de la Grange to seek in the following problem the probability that by taking the *mean* among the results of n observations, the error will not surpass the fraction $\frac{m}{n}$, m being $< n$.

Mr. de la Grange considers here that by taking the *mean* among the result of n observations, the error can be either 0, or $\frac{\pm 1}{n}$, or $\frac{2}{n}$, or $\frac{\pm 3}{n}$, or, &c. to $\frac{\pm n}{n}$, namely, 1; that thus, the probability that the error is no greater than $\frac{\pm m}{n}$, will be the sum of the probabilities that the error will be null, or $\frac{\pm 1}{n}$, or $\frac{\pm 2}{n}$, or, &c. to $\frac{\pm m}{n}$, & consequently he seeks first what is the probability that the error will be $\frac{\pm \mu}{m}$.

He finds it $= \frac{2M}{(a+2b)^n}$, M is expressed by $\frac{n(n-1)\cdots(n-\mu+1)}{1\cdot 2\cdots\mu} a^{n-\mu} b^{\mu} + \frac{\mu-2}{1} \cdot \frac{n(n-1)\cdots(n-\mu-1)}{1\cdot 2\cdots\mu+2} a^{n-\mu-1} b^{\mu+2} + \frac{(\mu+4)(\mu+3)}{1\cdot 2} \cdot \frac{n(n-1)\cdots(n-\mu-3)}{1\cdot 2\cdots\mu+4} a^{n-\mu-4} b^{\mu+4} +$ &c.

He expresses next the same probability by a series, & draws from these results a great number of curious inductions; he proves, for example, that it is more advantageous to take the *mean* only among an even number of observations.

Mr. de la Grange indicates also, in a scholium, the changes that the two preceding solutions would demand: if, instead of assuming an equal number of cases in order to have a positive error & a negative error, one would admit the hypothesis that he considers after that more generally in Problem III, of which here is the enunciation.

Supposing that each observation is subject to an error of one unit to less, & to an error of r units to the plus, & that the number of cases which are able to give 0, -1, $+r$ of error, is respectively a, b, c, one demands what is the probability that the mean error of many observations will be contained within some given limits?

Solution. Let n be the number of observations of which one wishes to take the *mean*, one will have, for the probability, that the mean error is $\frac{\mu}{n}$ the quantity $\frac{\mu}{(a+b+c)^n}$; & the probability that the mean error will be contained between these limits $\frac{-p}{n}$, $+\frac{q}{n}$ will be expressed by the series $\frac{(-p+1)+\&c+(-1)+(0)(1)+\&c+(q-1)}{(a+b+c)^n}$.

Problem IV. Supposing all as the preceding problem, we demand what is the mean error for which the probability is the greatest?

Solution. This probability is expressed by $\frac{rc-b}{a+b+c}$, & we can regard this quantity as the error of the mean result, & consequently takes it for the correction of this result.

Problem V. We suppose that each observation is subject to some given errors any whatsoever, & that we know at the same time the number of cases where each error is able to take place, we demand the correction that it will be necessary to make to the mean result of many observations?

Solution. Let p, q, r, s &c. be the errors to which each observation is subject, & a, b, c, d, &c the cases which are able to give these errors, namely, a the number of cases which would give the error p, b the number of cases which would give the error q, & thus of the others, the correction that one seeks will be $= \frac{ap+bq+cr+\&c.}{a+b+c+\&c.}$

Mr. de la Grange does not lack, no more than the other geometers who have treated this matter, to bring back also the solution of this problem to the determination of the center of gravity of a certain number of weights. Here are two corollaries that he draws from it.

First corollary. If one regards, he says, the quantities a, b, c, &c. as some weights applied to an indefinite straight line at some distances equal to p, q, r, &c. from a fixed point on this line, & if one seeks the center of gravity of these weights, the distance from the center to the fixed point will be the correction, that it will be necessary to make to the mean result of many observations; this follows evidently from the formula that we have found above for the value of this correction.

Second corollary. Therefore, if one supposes that each observation is subject to all the possible errors which are able to be comprehended between some given limits, & if one knows the curve of the facility of errors in which the abscissas being supposed to represent the errors, the ordinates represent the facilities of these errors, it will have only to seek the center of gravity of the total area of this curve, & the abscissa corresponding to the center, will express the correction of the mean result. Thence we see that, if the curve of which there is concern is equal & similar on one side & the other of the ordinate which passes through the origin of the abscissas, so that this ordinate is a diameter of the curve of which there is concern, then the correction will be null, the center of gravity falling necessarily on the diameter. This case has place every time that the errors are able to be equally positive & negative.

Problem VI. Mr. de la Grange supposes actually that we have verified an instrument any whatsoever, & that having reiterated many times the same verification, we have found different errors of which each is found repeated a certain number of times, & he seeks the error which it will be necessary to take for the correction of the instrument. He names p, q, r, &c. the found errors; & α, β, γ, &c. the numbers which mark how many times each error is found repeated in making n verifications, & his solution, which is based on the method of *maximis* & *minimis*, gives to him for the sought correction the quantity $\frac{\alpha p+\beta q+\gamma r}{n}+$ &c. where the mean error among all the particular errors that the n verifications have given.

Mr. de la Grange remarks next how one is able to know *a posteriori* the law of the facility of each of the errors to which an instrument is able to be subject; for, if we wished, says he, to take account also, at least in an approximate manner, of the intermediate errors to which the instrument would be able to be subject, there would only be to take, in a straight line VX (*fig.* 4),[2] some abscissas AP, AQ, AR, &c. proportionals to the found errors p, q, r, &c. & having applied there some ordinates Pp, Qq, Rr, &c. proportionals to the quantities α, β, γ, &c. we would make pass through the extremities p, q, r, &c. a parabolic line $u\ q\ a\ p\ r\ x$, we would seek next the center of gravity of the area of all the curve & the perpendicular dropped from this center onto the axis, would cut an abscissa which would be the correction of the instrument.

I will not stop to some lengthy remarks that Mr. de la Grange makes immediately on this corollary, & I pass to a proposition which gives place to the development of certain artifices of profound & particular calculation.

Problem VII. One has many observations, in each of which we suppose that we have been able to be deceived equally in any one of these quantities $-\alpha \ldots -2,-1,0,1,2,-\beta$, we demand what is the probability that the error of the mean result of n observations will be $\frac{\mu}{n}$, or what will be contained between these limits $\frac{-p}{n}$ & $\frac{+q}{n}$?

Mr. de la Grange seeks first the response to the first of these two questions, it is contained in the general expression which follows: $\frac{1}{1.2.3\ldots(n-2)\varsigma^n}\big((\pi+1)(\pi+2)\cdots(\pi+n-1)-n(\pi+1-\varsigma)(\pi+2-\varsigma)\cdots(\pi+n-1-\varsigma)+\frac{n(n-1)}{2}(\pi+1-2\varsigma)(\pi+2-2\varsigma)\cdots(\pi+n-1-2\varsigma)-\text{\&c.}$

We continue this series until this that some one of the factors $\pi+1$, $\pi+1-\varsigma$, &c. becomes negative; & it is necessary to remark that $\pi = n\alpha+\mu$ & $\varsigma=\alpha+\beta+1$. The solution of the second question requires only at present a certain finite integration of the preceding series, that is, that we make π vary from $-p$ to q, according to a method exposed preliminarily; & we find finally, by supposing, for brevity $n\alpha-q=\delta$, & $n\alpha+q=\gamma$, that the probability that the mean error falls between $\frac{-p}{n}$ & $\frac{q}{n}$, is expressed by

$$\frac{1}{1.2.3\ldots\varsigma^n}\Big(\gamma(\gamma+1)\cdots(\gamma+n-1)-(\delta+1)(\delta+3)\cdots(\delta+n)-n\big((\gamma-\varsigma)(\gamma-\varsigma+1)\cdots(\gamma-\varsigma+n-1)-(\delta-\varsigma+1)(\delta-\varsigma+2)\cdots(\delta-\varsigma+n)\big)+\frac{n(n-1)}{2}\big((\gamma-2\varsigma)(\gamma-2\varsigma+1)\cdots(\gamma-2\varsigma+n-1)-(\delta-2\varsigma+1)(\delta-2\varsigma+2)\cdots(\delta-2\varsigma+n)\big)-\text{\&c.}\Big)$$

This series must be continued to this that some one of the factors $\gamma-\varsigma$, $\gamma-2\varsigma$, &c. becomes negative; & as much to the other factors $\delta-\varsigma+1$, $\delta-2\varsigma+1$, &. If some one of among them is found negative, then it will be necessary to increase the number δ by as many units as it will be necessary in order to render it positive. Moreover, these problems, the more they become general & complicated, the more they admit corollaries; but, not

[2]*Translator's note*: On the plates accompanying the text there is no figure which corresponds to the statement.

being able to stop myself at each, I leave to the observers to simplify, according to the case that they will have to develop, the fundamental results that I indicate.

Problem VIII. Supposing that the errors that we are able to commit in each observation are $-\omega\ldots-2,-1,0,1,2\ldots\omega$, & that the number of cases which correspond to each of these errors is respectively proportional to $1,2,3,\ldots\alpha+1\ldots3,2,1$. We demand the probability that the error of the mean result of m observations is contained between the limits $\frac{-p}{m}$ & $\frac{q}{m}$?

Solution. It is found expressed by $\frac{1}{1.2.3\ldots2m\varsigma^{2m}}\big(\gamma(\gamma+1)\cdots(\gamma+2m-1)-(\delta+1)(\delta+1)(\delta+2)\cdots(\delta+2m)\big)-2m\big((\gamma-\varsigma)((\gamma+1-\varsigma)\cdots(\gamma+2m-1-\varsigma)-(\delta+1-\varsigma)(\delta+2-\varsigma)\cdots(\delta+2m-\varsigma)\big)+\frac{2m(2m-1)}{2}\big((\gamma-2\varsigma)(\gamma+1-2\varsigma)\cdots(\gamma+2m-1-2\varsigma)-(\delta+1-2\varsigma)(\delta+2-2\varsigma)\cdots(\delta+2m-2\varsigma)\big)-\text{&c.}\big)$ γ being $=m\alpha+q$ & $\delta=m\alpha-p$; & in regard to the continuation of the series, it will be necessary to follow the same rule as for the preceding.

Here are yet two other problems that Mr. de la Grange resolves in this memoir; but they demand so great preparations of calculations, that I would not be able to flatter myself by rendering them applicable by means of a few lines; I spare myself so much more easily to hold it, that the first eight problems appear to me to face all the cases: I will give however, according to Mr. de la Grange, the spirit of the solution of Problem IX, of which the last is next only a particular case.

Problem IX. We suppose that each observation is subject to all the possible errors comprehended between these two limits p & $-q$, & that the facility of each error x, that is, the number of cases where is it able to take place, divided by the total number of cases, is represented by a function any whatsoever of x designated by y: we demand the probability that the mean error of n observations will be comprehended between the limits r & $-s$.

Proceeded from the solution. We will commence first by seeking the probability that the mean error will be z, & this probability being represented by a function of z, there will be only to take of it the integral from $Z=1-r$ to $Z=-s$, this will be the sought probability. Now, in order to have the probability that the mean error of n observations will be Z, it will be necessary to consider the polynomial, which is represented by the integral of $ya^x dx$, by supposing this integral taken in a manner that it extends from $x=p$ to $x=-q$, we will raise this polynomial to the power n, & we will seek the coefficient of the power Z of a, this coefficient, which will be a function of Z, will express the probability that the mean error will be Z; all difficulty consists to find this coefficient in a direct & general manner; this is why Mr. de la Grange arrives by a new method, founded on some considerations sufficiently delicate & on an analysis completely particular.

Problem X. Supposing that each observation is subject to all the possible errors comprehended between the limits p & $-q$ (p being the arc of ninety degrees), & that the facility of each error x is proportional to $\cos x$, we demand the probability that the mean error of n observations will be contained between the limits r & $-s$. (*J. B.*)