RECHERCHES

SUR

LES SUITES RÉCURRENTES

DONT LES TERMES VARIENT DE PLUSIERS MANIERES DIFFÉRENTES, OU SUR L'INTEGRATION DES ÉQUATIONS LINÉARES AUX DIFFÉRENCES FINIES ET PARTIELLES; ET SUR L'USAGE DE CES ÉQUATIONS DANS LA THÉORIE DES HASARDS*

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I have given, in the first volume of the *Mémoires de la Societé des Sciences de Turin*, a new method for treating the theory of recurrent series, and by making it depend on the integration of linear equations in the finite differences. I myself proposed then to push these Researches further, and to apply them principally to the solution of several problems of the theory of chances; but other objects having since made me lose this from sight, M. de Laplace has preceded me in great part, in two excellent Memoirs *sur les suites récurro-récurrentes*, and *sur l'intégration des équations différentielles finies et leur usage dans la théorie des hasards*, printed in volumes VI and VII of the *Mémoires* presented to the Academy of Sciences of Paris. I believe however that we can again add something to the work of this illustrious Geometer, and to treat the same subject in a more direct, more simple and especially more general manner; this is the object of the Researches that I am going to give in this Memoir; we will find some new methods for the integration of linear equations in finite and partial differences, and the application of these methods to several interesting Problems of the Calculus of probabilities; but the question here is only of equations of which the coefficients are constants, and I reserve for another Memoir examination of those which have some variable coefficients.

ARTICLE I. — *On simple recurrent sequences, or on the integration of linear equations in finite differences between two variables.*

Although the theory of the ordinary recurrent sequences is enough known, I believe I must begin by treating it in a few words in order to serve as introduction to that of the

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récurro-récurrente sequences which form the principal object of this Memoir. Besides I will have need to employ, as much as it will be possible, only some new and more simple methods than those which one has already.

1. Let the sequence be

$$y_0, y_1, y_2, y_3, \cdots y_x, y_{x+1}, y_{x+2}, y_{x+3}, \cdots,$$

in which one has constantly this linear equation among n successive terms

$$Ay_x + By_{x+1} + Cy_{x+2} + \cdots + Ny_{x+n} = 0, \tag{A}$$

A, B, C, $\ldots N$ being any constant coefficients whatsoever; this will be a simple recurrent sequence of order n, and equation (A) will be the finite differential equation which there is question to integrate in order to have the expression of the general term y_x of the proposed sequence.

For this I suppose

$$y = a\alpha^x,$$

a and α being some undetermined constants; I will have therefore

$$y_{x+1} = a\alpha^{x+1},\ y_{x+2} = a\alpha^{x+2}, \ldots,$$

and the substitutions being made into equation (A), it will become divisible by $a\alpha^x$; and one will have after this division

$$A + B\alpha + C\alpha^2 + \ldots + N\alpha^n = 0. \tag{B}$$

One sees by this equation: 1° that, since the coefficient a is not found, this coefficient remains arbitrary: 2° that the equation being with respect to α of degree n, it will furnish, in general, n different values of α, which I will denote by α, β, γ,.... One will have therefore thus, by taking also different coefficients a, b, c, $\ldots n$ different values of y, namely $a\alpha^x$, $a\beta^x$, $a\gamma^x$,...; and, as equation (A) is linear, it is easy to see that the sum of these different values of y_x will satisfy it also. So that one will have, in general,

$$y = a\alpha^x + a\beta^x + a\gamma^x + \ldots;$$

and as this value of y_x contains n arbitrary constants a, b, c, $\ldots$, it will be the complete integral of equation (A) of the n^{th} order.

2. If one supposes that the first n terms of the proposed sequence are given, one could by their means determine the n arbitrary constants a, b, c, ...; there will be for this only to resolve the n equations

$$\begin{aligned}
y_0 =&\quad a + b + c + \ldots,\\
y_1 =&\quad a\alpha + b\beta + c\gamma + \ldots,\\
y_2 =&\quad a\alpha^2 + b\beta^2 + c\gamma^2 + \ldots,\\
&\quad \ldots\\
y_{n-1} =&\quad a\alpha^{n-1} + b\beta^{n-1} + c\gamma^{n-1} + \ldots,
\end{aligned}$$

In the case $n = 1$, one has

$$a = y_0,$$

in the case of $n = 2$, one will have

$$a = \frac{y_1 - \beta y_0}{\alpha - \beta}, \qquad b = \frac{y_1 - \alpha y_0}{\beta - \alpha};$$

in the case of $n = 3$, one will have

$$a = \frac{y_2 - (\beta+\gamma)y_1 + \beta\gamma y_0}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{y_2 - (\alpha+\gamma)y_1 + \alpha\gamma y_0}{(\beta-\alpha)(\beta-\gamma)}, \quad c = \frac{y_2 - (\alpha+\beta)y_1 + \alpha\beta y_0}{(\gamma-\alpha)(\gamma-\beta)}$$

and thus in sequence.

Thence and from the known theory of equations it is easy to conclude that if one makes, for brevity,

$$\begin{aligned}
&A + B\alpha + C\alpha^2 + D\alpha^3 + \ldots + N\alpha^n = P,\\
&B + C\alpha + D\alpha^2 + \ldots = Q,\\
&C + D\alpha + \ldots = R,\\
&D + \ldots = S,\\
&\ldots,
\end{aligned}$$

one will have, in general,

$$a = \frac{Qy_0 + ry_1 + Sy_2 + \ldots}{\frac{dP}{d\alpha}},$$

and changing in this expression of a the quantity α into β, $\gamma, \ldots$, one will have the values of the other coefficients b, c, $\ldots$.

If it happens that two or more roots are equals, there will be only to suppose their differences infinitely small, and one will find, in the case of $\beta = \alpha$, that the two terms

$$a\alpha^x + b\beta^x$$

of the expression of y_x will become of this form

$$a'\alpha^x + b'x\alpha^{x-1},$$

where one will have

$$\begin{aligned}
a' &= \frac{\frac{dQ}{d\alpha}y_0 + \frac{dR}{d\alpha}y_1 + \frac{dS}{d\alpha}y_2 + \ldots}{\frac{1}{2}\,\frac{d^2P}{d\alpha^2}},\\
b' &= \frac{Qy_0 + Ry_1 + Sy_2 + \ldots}{\frac{1}{2}\,\frac{d^2P}{d\alpha^2}};
\end{aligned}$$

and if one has $\gamma = \beta = \alpha$, then the three terms

$$a\alpha^x + b\beta^x + c\gamma^x$$

will become

$$a''\alpha^x + b''x\alpha^{x-1} + c''\frac{x(x-1)}{2}\alpha^{x-2},$$

whence one will have

$$a'' = \frac{\frac{1}{2}\,\frac{d^2Q}{d\alpha^2}y_0 + \frac{1}{2}\,\frac{d^2R}{d\alpha^2}y_1 + \frac{1}{2}\,\frac{d^2S}{d\alpha^2}y_2 + \dots}{\frac{1}{2.3}\,\frac{d^3P}{d\alpha^3}},$$

$$b'' = \frac{\frac{dQ}{d\alpha}y_0 + \frac{dR}{d\alpha}y_1 + \frac{dS}{d\alpha}y_2 + \dots}{\frac{1}{2.3}\,\frac{d^3P}{d\alpha^3}},$$

$$c'' = \frac{Qy_0 + Ry_1 + Sy_2 + \dots}{\frac{1}{2.3}\,\frac{d^3P}{d\alpha^3}};$$

and thus of the rest.

3. If in the proposed equation (A) the coefficients A, B, $C, \dots N$, instead of being constants, are some given functions of x, which we will designate by A_x, B_x, C_x, $\dots N_x$, so that one has the equation

$$A_x y_x + B_x y_{x+1} + C_x y_{x+2} + \dots + N_x y_{x+n} = 0, \tag{C}$$

one will not be able, by the preceding method nor by any other known method, to integrate it in general, unless it is only of the first order; but if one supposes that one knows *a posteriori* n particular values of y_x which we will designate by α_x, β_x, γ_x, $\dots$ it is clear that one will have, in general,

$$y_x = a\alpha_x + b\beta_x + c\gamma_x + \dots,$$

and that this expression of y_x will be complete, since it contains n arbitrary constants a, b, c, $\dots$.

4. Moreover one could in this same case find the complete integral of the equation

$$A_x y_x + B_x y_{x+1} + C_x y_{x+2} + \dots + N_x y_{x+n} = X_x, \tag{D}$$

X_x being any function of x whatsoever.

Because since, in the case of $X_x = 0$, one has

$$y_x = a\alpha_x + b\beta_x + c\gamma_x + \dots$$

for the complete integral, a, b, $c, \dots$ being some constants, we suppose now that the quantities a, b, $c, \dots$ are, in general, some functions of x which we will designate by a_x, b_x, $c_x, \dots$, so that the integral of equation (D) is

$$y_x = a_x\alpha_x + b_x\beta_x + c_x\gamma_x + \dots; \tag{E}$$

making x vary, one will have

$$y_{x+1} = a_{x+1}\alpha_{x+1} + b_{x+1}\beta_{x+1} + c_{x+1}\gamma_{x+1} + \dots,$$

or else, by designating by the characteristic Δ the finite differences, so that

$$\Delta a_x = a_{x+1} - a_x,$$

and thus of the others,

$$\begin{aligned} y_{x+1} = & a_x\alpha_{x+1} + b_x\beta_{x+1} + c_x\gamma_{x+1} + \dots \\ & + \alpha_{x+1}\Delta a_x + \beta_{x+1}\Delta b_x + \gamma_{x+1}\Delta c_x + \dots \end{aligned}$$

Therefore, if I make

$$\alpha_{x+1}\Delta a_x + \beta_{x+1}\Delta b_x + \gamma_{x+1}\Delta c_x + \dots = 0, \tag{1}$$

I will have

$$y_{x+1} = a_x\alpha_{x+1} + b_x\beta_{x+1} + c_x\gamma_{x+1} + \dots$$

as if the quantities $a_x,\ b_x,\ c_x, \dots$ had varied not at all.

Making x vary anew, I will have therefore

$$\begin{aligned} y_{x+2} &= a_{x+1}\alpha_{x+2} + b_{x+1}\beta_{x+2} + c_{x+1}\gamma_{x+2} + \dots \\ &= a_x\alpha_{x+2} + b_x\beta_{x+2} + c_x\gamma_{x+2} + \dots \\ &+ \alpha_{x+2}\Delta a_x + \beta_{x+2}\Delta b_x + \gamma_{x+2}\Delta c_x + \dots, \end{aligned}$$

and, making similarly

$$\alpha_{x+2}\Delta a_x + \beta_{x+2}\Delta b_x + \gamma_{x+2}\Delta c_x + \dots = 0, \tag{2}$$

I will have

$$y_{x+2} = a_x\alpha_{x+2} + b_x\beta_{x+2} + c_x\gamma_{x+2} + \dots$$

Likewise, by making x vary and supposing

$$\alpha_{x+3}\Delta a_x + \beta_{x+3}\Delta b_x + \gamma_{x+3}\Delta c_x + \dots = 0, \tag{3}$$

one will have

$$y_{x+3} = a_x\alpha_{x+3} + b_x\beta_{x+3} + c_x\gamma_{x+3} + \dots$$

I continue thus to make x vary and to suppose null the part of y depending on the variations of $a_x,\ b_x,\ c_x, \dots$ to the following equations inclusively,

$$\begin{aligned} &\alpha_{x+n-1}\Delta a_x + \beta_{x+n-1}\Delta b_x + \gamma_{x+n-1}\Delta c_x \cdots = 0 && (n-1) \\ &y_{x+n-1} = a_x\alpha_{x+n-1} + b_x\beta_{x+n-1} + c_x\gamma_{n+n-1} + \cdots; \end{aligned}$$

and, making again x vary in the last equation, I will have

$$\begin{aligned} y_{x+n} = & a_x\alpha_{x+n} + b_x\beta_{x+n} + c_x\gamma_{x+n} + \dots \\ & + \alpha_{x+n}\Delta a_x + \beta_{x+n}\Delta b_x + \gamma_{x+n}\Delta c_x + \dots \end{aligned}$$

Let one substitute now these values of $y_x, y_{x+1}, \dots, y_{x+n}$ into equation (D); and as all these values, the last excepted, are the same as if $a_x,\ b_x, c_x \dots$ had not varied, and as the last differs from that which it was under this hypothesis only by the terms

$$\alpha_{x+n}\Delta a_x + \beta_{x+n}\Delta b_x + \gamma_{x+n}\Delta c_x + \dots$$

which are added to it; that besides the values of $y_x, y_{x+1}, \ldots$ in the case of $\alpha_x, \beta_x, \gamma_x, \ldots$ constants, satisfy by hypothesis equation (C), whatever be the values of these constants; it follows that the first member of equation (D) will be reduced to

$$N_x(\alpha_{x+n}\Delta a_x + \beta_{x+n}\Delta b_x + \gamma_{x+n}\Delta c_x + \ldots),$$

so that one will have the equation

$$\alpha_{x+n}\Delta a_x + \beta_{x+n}\Delta b_x + \gamma_{x+n}\Delta c_x + \ldots = \frac{X_x}{N_x}. \tag{n}$$

Therefore one has thus n linear equations (1), (2), (3), ..., (n) among the quantities $\Delta a_x, \Delta b_x, \Delta c_x, \ldots$, whence one will draw the values of these quantities as functions of x, which I will designate by $P_x, Q_x, R_x, \ldots$. Therefore, passing from the differences to the sums and designating these by the characteristic Σ, one will have

$$a_x = \sum P_x, \quad b_x = \sum Q_x, \quad c_x = \sum R_x, \ldots,$$

this which being substituted into formula (E), it will become

$$y_x = \alpha_x \sum P_x + \beta_x \sum Q_x + \gamma_x \sum R_x + \ldots$$

for the complete integral of equation (D).

It follows thence that the equation

$$A_x y_x + B_x y_{x+1} + C_x y_{x+2} + \ldots + N_x y_{x+n} = X_x$$

is generally integrable all the time that one knows n particular values of y_x in the case of $X_x = 0$; the Theorem analogous to this that I have given for linear differential equations in Volume III of the *Mémoires de Turin*.[1] Mr. le Marquis de Condorcet and Mr. de Laplace had already remarked that this Theorem on the equations in infinitely small differences was also applicable to the case of the finite differences: and this last has given a general and ingenious demonstration of it, but a little complicated (*see* Tome IV of the *Mémoires de Turin* and the *Mémoires* presented to the Academy of Sciences of Paris in 1773).[2] It is this which has engaged me to treat here this matter by a new and as simple a method as one can desire it.

5. REMARK. — The principles of the preceding method can be applied also to ordinary differential equations, and are, in general, of very great usage in all the integral Calculus. Although this is not the place here to occupy ourselves with this matter, I am going nevertheless to treat it in a few words, reserving for myself to treat it elsewhere with more extent.

And first, if one has a linear equation of order n such as

$$Py + Q\frac{dy}{dx} + R\frac{d^2y}{dx^2} + \ldots + V\frac{d^ny}{dx^n} = X,$$

[1] *Oeuvres de Lagrange*, t. 1, p. 471.

[2] The paper of Laplace is "Recherches, sur l'integration des équations differentielles aux différences finies, & sur leur usage dans la théorie des hasards," *Savants étranges*, 1773 (1776) p. 37-162. See also *Oeuvres* 8, p. 69-197.

where P, Q, R, $\ldots$, V and X are some given functions of x, and if one knows the complete integral of this equation in the case of $X = 0$, which will be necessarily of the form

$$y = ap + bq + cr + \ldots,$$

$a, b,\ c, \ldots$ being some arbitrary constants in the number of n, and p, q, r, $\ldots$ some functions of x where the constants a, b, c, $\ldots$ do not enter, and which are so many particular values of y under the hypothesis of $X = 0$, one will be able to deduce easily from it the complete integral of the proposed. Because by regarding the arbitraries $a, b,\ c, \ldots$ as some indeterminate variables, and supposing null in the values of dy, d^2y, d^3y, $\ldots$, $d^{n-1}y$ the parts which depend on the variability of these quantities $a, b,\ c, \ldots$, one will have

$$\begin{array}{ll}
dy = a\,dp + b\,dq + c\,dr + \ldots, & 0 = p\,da + q\,db + r\,dc + \ldots, \\
d^2y = a\,d^2p + b\,d^2q + c\,d^2r + \ldots, & 0 = dp\,da + dq\,db + dr\,dc + \ldots, \\
d^3y = a\,d^3p + b\,d^3q + c\,d^3r + \ldots, & 0 = d^2p\,da + d^2q\,db + d^2r\,dc + \ldots, \\
\ldots & \ldots \\
d^{n-1}y = a\,d^{n-1}p + b\,d^{n-1}q + c\,d^{n-1}r + \ldots, & 0 = d^{n-1}p\,da + d^{n-1}q\,db + d^{n-1}r\,dc + \ldots,
\end{array}$$

next

$$d^ny = a\,d^np + b\,d^nq + c\,d^nr + \ldots + d^{n-1}p\,da + d^{n-1}q\,db + d^{n-1}r\,dc + \ldots$$

In this manner one sees that the expressions of y, dy, d^2y, $\ldots$, $d^{n-1}y$ have the same form as if a, b, $c, \ldots$ were constants, and that that of d^ny differs from that which it was in this case only by the terms

$$d^{n-1}p\,da + d^{n-1}q\,db + d^{n-1}r\,dc + \ldots$$

which are added; now as in the case of a, b, $c, \ldots$ constants, the values of y, dy, $d^2y, \ldots$, d^ny satisfy by the hypothesis to the proposed equation when one supposes $X = 0$, whatever be besides the values of these constants, it is easy to conclude that if one substitutes into this equation the values above of y, dy, d^2y, $\ldots$, d^ny, all the terms will be destroyed, with the exception of the terms of the value of d^ny which depends on the variation of the quantities a, b, $c, \ldots$ and on the term X, which had been supposed before null. So that one will have, in dividing by V, the equation

$$d^{n-1}p\,da + d^{n-1}q\,db + d^{n-1}r\,dc + \ldots = \frac{X}{V}dx^n;$$

and this equation being combined with the $n - 1$ equations of condition

$$\begin{aligned}
&p\,da + q\,db + r\,dc + \ldots = 0, \\
&dp\,da + dq\,db + dr\,dc + \ldots = 0, \\
&\ldots \\
&d^{n-2}p\,da + d^{n-2}q\,db + d^{n-2}r\,dc + \ldots = 0,
\end{aligned}$$

one will draw from it by the ordinary rules of the elimination of values of the n differentials da, db, $dc, \ldots$; and thence one will have by integration those of a, b, c, $\ldots$

which one will substitute into the expression of y. This which is much more simple than all that which one finds in Tomes III and IV of the *Mémoires de Turin on this matter.*

In general, if one knows the complete integral of any equation whatever of order n such as

$$\frac{d^n y}{dx^n} + P = 0,$$

P being a function of x, y, $\frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}$, one will be able to make this integral serve to find that of the equation

$$\frac{d^n y}{dx^n} + P = \Pi,$$

Π being also a given function of x, y, $\frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}$.

Because let $M = 0$ be the complete integral of which there is question, M will be a function of x, y and of n arbitrary constants a, b, c, $\ldots$; so that y will be reciprocally a function of x and of the same constants, which will satisfy consequently the equation

$$\frac{d^n y}{dx^n} + P = 0,$$

whatever be the values of these constants.

We suppose now that $M = 0$ is equally the integral of the equation

$$\frac{d^n y}{dx^n} + P = \Pi,$$

but in regarding there the quantities a, b, c, $\ldots$ as variables; under this hypothesis, the expression of y in x, a, b, c, $\ldots$ will be the same as in the case of a, b, c, $\ldots$ constants, but those of dy, $d^2y, \ldots$ will be different; however, if in the successive differentiations one supposes null the parts of the differentials dy, $d^2y, \ldots, d^{n-1}y$ which result from the variability of the quantities a, b, c, $\ldots$, one will have these $n-1$ equations of condition

$$\begin{aligned}
&\frac{dy}{da}\,da + \frac{dy}{db}\,db + \frac{dy}{dc}\,dc + \ldots = 0,\\
&\frac{d^2y}{dx\,da}\,da + \frac{d^2y}{dx\,db}\,db + \frac{d^2y}{dx\,dc}\,dc + \ldots = 0,\\
&\cdots,\\
&\frac{d^{n-1}y}{dx^{n-2}\,da}\,da + \frac{d^{n-1}y}{dx^{n-2}\,db}\,db + \frac{d^{n-1}y}{dx^{n-2}\,dc}\,dc + \ldots = 0,
\end{aligned}$$

by means of which the values of these differentials will be yet the same as if a, b, c, $\ldots$ were constants; so that by substituting these values like that of y in the quantity P, one will have again the same function of x, a, b, c, $\ldots$ as in the case where the quantities a, b, c, $\ldots$ would be constants. Now as the value of $\frac{d^{n-1}y}{dx^{n-1}}$ is the same as in the case of a, b, c, $\ldots$ constants, it is clear that that of $d\,\frac{d^{n-1}y}{dx^{n-1}}$ will be equal to that which it would be in the same case, more to the variation of $\frac{d^{n-1}y}{dx^{n-1}}$ owing to the quantities a, b, c, $\ldots$, which is

$$\frac{d^n y}{dx^{n-1}\,da}\,da + \frac{d^n y}{dx^{n-1}\,db}\,db + \frac{d^n y}{dx^{n-1}\,dc}\,dc + \ldots;$$

consequently, if one denotes by $Y\,dx$ the first part of this value, one will have for the complete value of $d\,\frac{d^{n-1}y}{dx^{n-1}}$ the quantity

$$Y\,dx + \frac{d^n y}{dx^{n-1}\,da}\,da + \frac{d^n y}{dx^{n-1}\,db}\,db + \frac{d^n y}{dx^{n-1}\,dc}\,dc + \dots,$$

where Y will be, after the substitutions, the same function of x, a, b, c, $\dots$ as in the case of a, b, c, $\dots$ constants; but in this case one has, by hypothesis,

$$Y + P = 0,$$

whatever be the values of these constants; therefore the same equations will yet hold in the case where the quantities a, b, c, $\dots$ are not constants; consequently in this last case the equation

$$\frac{d^n y}{dx^n} + P = \Pi$$

will become, being multiplied by dx,

$$\frac{d^n y}{dx^{n-1}\,da}\,da + \frac{d^n y}{dx^{n-1}\,db}\,db + \frac{d^n y}{dx^{n-1}\,dc}\,dc + \dots = \Pi\,dx.$$

This equation being combined with the $n-1$ equations of condition found above, one will have, after having substituted throughout the values of y and of its differentials in x, a, b, $c, \dots$ drawn from the finite equation $M = 0$, values which are the same as in the case of a, b, $c, \dots$ constants, one will have, I say, n differential equations of the first order among the n variables a, b, $c, \dots$ and the variable x; if therefore one integrates these equations, one will have the values of a, b, $c, \dots$ in x, which being next substituted into the equation $M = 0$ will give the integral of the proposed equation.

I swear that the integration of the equations in a, b, $c, \dots$ and x will be most often very difficult, at least as difficult as that of the proposed equation

$$\frac{d^n y}{dx^n} + P = \Pi;$$

and there is perhaps only the single case of the linear equations which we have treated above, where the integration of the equations of which there is question succeeds, in general, because the constants a, b, $c, \dots$ are also necessarily linear in the complete integral $M = 0$; but the grand use of the preceding method is in order to integrate by approximation the equations of which one knows the complete integral very nearly, that is to say by neglecting some quantities which one regards as very small.

For example, if in the equation

$$\frac{d^n y}{dx^n} + P = \Pi,$$

one supposes that the function Π is very small vis-a-vis P, and that one knows already the complete integral $M = 0$, in the case of $\Pi = 0$, by employing the preceding method, and drawing from the n differential equations in a, b, $c, \dots$ and x, the values of da, db, dc, $\dots$, one will have some equations of this form

$$da = A\,\Pi\,dx, \quad db = B\,\Pi\,dx, \quad dc = C\,\Pi\,dx, \dots,$$

A, B, C being some finite functions of x, a, b, $c, \ldots$, and Π being also a function of the same quantities, but very small by hypothesis; whence one sees that the values of $\frac{da}{dx}$, $\frac{db}{dx}$, $\frac{dc}{dx}, \ldots$ are also very small of the same order; thus, by regarding first the quantities a, b, $c, \ldots$ as constants, one will be able by the known methods to approach more and more the true values of these quantities.

It is not to fear that the functions A, B, C,... become infinities; because that supposition contains the necessary conditions in order that the complete integral $M = 0$ of the equation

$$\frac{d^n y}{dx^n} + P = 0,$$

becomes from it a particular integral; for why one can see my Memoir *sur les intégrales particulières des équations différentialles*.[3]

It is clear moreover that this method, that I only exhibited here in passing, can be applied equally to the case where one would have many differential equations among many variables of which one would know the approximate complete integral, that is to say by neglecting some quantities supposed very small. It will be consequently quite useful for calculating the movements of the planets as much as they are altered by their mutual action, since by setting aside this action the complete solution of the Problem is known; and it is good to remark that, as in this case the constants a, b, c, ... represent that which one names the *elements of the planets*, our method will give immediately the variations of these elements originating from the action that the planets exert on one another. I have already given an essay on this method in my Researches on the theory of Jupiter and of Saturn [*Mémoires de Turin*, Tome III[4]]. It is presented here in a more direct and more general manner; but I myself propose to develop it elsewhere with more extension, and to apply it to the solution of some important Problems on the System of the world.

ARTICLE II. — *On doubly recurrent sequences, or on the integration of linear equations in finite and partial differences among three variables.*

6. We suppose that one has a sequence of which the terms vary in two different ways and form a kind of Table in double entry of this form

$$\begin{array}{llllll}
y_{0,0}, & y_{1,0}, & y_{2,0}, & y_{3,0}, \ldots, & y_{x,0}, & y_{x+1,0}, \ldots, \\
y_{0,1}, & y_{1,1}, & y_{2,1}, & y_{3,1}, \ldots, & y_{x,1}, & y_{x+1,1}, \ldots, \\
y_{0,2}, & y_{1,2}, & y_{2,2}, & y_{3,2}, \ldots, & y_{x,2}, & y_{x+1,2}, \ldots, \\
y_{0,3}, & y_{1,3}, & y_{2,3}, & y_{3,3}, \ldots, & y_{x,3}, & y_{x+1,3}, \ldots, \\
\ldots, & \ldots, & \ldots, & \ldots\ldots\ldots, & \ldots, & \ldots\ldots\ldots, \\
y_{0,t}, & y_{1,t}, & y_{2,t}, & y_{3,t}, & y_{x,t}, & y_{x+1,t}, \\
y_{0,t+1}, & y_{1,t+1}, & y_{2,t+1}, & y_{3,t+1}, \ldots, & y_{x,t+1}, & y_{x+1,t+1}, \\
\ldots, & \ldots, & \ldots, & \ldots\ldots\ldots, & \ldots, & \ldots\ldots\ldots,
\end{array}$$

and that one had constantly among the terms of this sequence a linear equation of this

[3] *Oeuvres de Lagrange*, T. IV, p. 5.

[4] *Oeuvres de Lagrange*, T. 1, p. 609.

form

$$\left.\begin{aligned} Ay_{x,t} + By_{x+1,t} + Cy_{x+2,t} + \ldots + Ny_{x+n,t} \\ + B'y_{x,t+1} + C'y_{x+1,t+1} + \ldots + N'y_{x+n-1,t+1} \\ + C''y_{x,t+2} + \ldots + N''y_{x+n-2,t+2} \\ + \ldots\ldots\ldots\ldots\ldots \\ + N^{(n)}y_{x,t+n} \end{aligned}\right\} = 0,$$

in which A, B, B', C, C', C'', $\ldots$, N, N', $\ldots$ are any constant coefficients whatever; the sequence of which there is question will be a double recurrent sequence of the order n, and the preceding equation will be a linear equation in finite and partial differences among three variables, of the integration on which will depend the research of the general term $y_{x,t}$ of the sequence.

7. We suppose first that the proposed differential equation had only four terms and that it is of the form

$$Ay_{x,t} + By_{x+1,t} + B'y_{x,t+1} + C'y_{x+1,t+1} = 0. \tag{F}$$

I make

$$y_{x,t} = a\alpha^x\beta^t,$$

a, α, β being some indeterminate constants; I will have thus

$$y_{x+1,t} = a\alpha^{x+1}\beta^t, \quad y_{x.t+1} = a\alpha^x\beta^{t+1}, \quad y_{x+1,t+1} = a\alpha^{x+1}\beta^{t+1};$$

substituting these values and dividing next each equation by $a\alpha^x\beta^t$, there will come this here

$$A + B\alpha + B'\beta + C'\alpha\beta = 0,$$

by which one will be able to determine one of the two constants α, β by the other.

I draw β from this equation, I have

$$\beta = -\frac{A + B\alpha}{B' + C'\alpha};$$

therefore, substituting this value of β, I will have

$$y_{x,t} = a\alpha^x \left(-\frac{A + B\alpha}{B' + C'\alpha}\right)^t,$$

where a and α remain indeterminate.

Let one reduce now the quantity $\left(-\frac{A+B\alpha}{B'+C'\alpha}\right)^t$ into a series which proceeds according to the powers of α, but so that these powers go by diminishing, and if one supposes, in general,

$$\left(-\frac{A + B\alpha}{B' + C'\alpha}\right)^t = T\alpha^{\mu t} + T'\alpha^{\mu t-1} + T''\alpha^{\mu t-2} + T'''\alpha^{\mu t-3} + \ldots,$$

one will have

$$y_{x,t} = Ta\alpha^{x+\mu t} + T'a\alpha^{x+\mu t-1} + T''a\alpha^{x+\mu t-2} + \ldots$$

Now, as a and α are arbitraries, one will have an infinity of different values of $y_{x,t}$, and it follows from this that the differential equation (F) is linear, that one will be able equally to take for $y_{x,t}$ the sum of as many of these different values as one will wish.

Therefore, if one takes any number whatever of different constants a, b, c, ..., α, β, γ, ..., one will have, in general,

$$\begin{aligned} y_{x,t} = T &\left(a\alpha^{x+\mu t} + b\beta^{x+\mu t} + c\gamma^{x+\mu t} + \dots\right) \\ &+ T' \left(a\alpha^{x+\mu t-1} + b\beta^{x+\mu t-1} + c\gamma^{x+\mu t-1} + \dots\right) \\ &+ T''T \left(a\alpha^{x+\mu t-2} + b\beta^{x+\mu t-2} + c\gamma^{x+\mu t-2} + \dots\right) \\ &+ T'''T \left(a\alpha^{x+\mu t-3} + b\beta^{x+\mu t-3} + c\gamma^{x+\mu t-3} + \dots\right) \\ &\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

I remark now that because of the indefinite number of arbitrary constants a, b, c,..., α, β, γ, ..., the quantity

$$a\alpha^{x+\mu t} + b\beta^{x+\mu t} + c\gamma^{x+\mu t} + \dots$$

must be able to represent any function of $x+\mu t$ that I will designate by the characteristic f thus, $f(x+\mu t)$; and then it is clear that the similar quantities

$$a\alpha^{x+\mu t-1}+b\beta^{x+\mu t-1}+c\gamma^{x+\mu t-1}+\dots, \quad a\alpha^{x+\mu t-2}+b\beta^{x+\mu t-2}+c\gamma^{x+\mu t-2}+\dots, \dots$$

will become

$$f(x+\mu t-1), \quad f(x+\mu t-2), \dots;$$

therefore making these substitutions one will have, in general,

$$y_{x,t} = Tf(x+\mu t) + T'f(x+\mu t-1) + T''f(x+\mu t-2) + \dots$$

8. The determination of the form of the function $f(x+\mu t)$ depends on the values of $y_{x,t}$ when $t=0$; indeed, if one makes $t=0$, one has

$$T=1, \quad T'=0, \quad T''=0, \dots;$$

therefore

$$y_{x,0} = f(x).$$

Whence it follows that one will have, in general,

$$y_{x,t} = Ty_{x+\mu t,0} + T'y_{x+\mu t-1,0} + T''y_{x+\mu t-2,0} + \dots,$$

where one sees that the quantities $y_{x+\mu t,0}$, $y_{x+\mu t-1,0}$, ... are contained among the terms which form the first horizontal rank of the Table of n° 6, provided that one supposes that the sequence of this rank is also continued to the left in this manner

$$\dots, \quad y_{-(x+1),0}, \quad y_{-x,0}, \quad \dots, \quad y_{-3,0}, \quad y_{-2,0}, \quad y_{-1,0}, \quad y_{0,0}.$$

If therefore one regards all these terms as givens, one will have by the preceding formula the value of any term $y_{x,t}$ whatever of the Table of which there is question, in the case where it is supposed formed by one such law, that one has constantly, among four terms contiguous or arranged in a square, an equation of the form (F) of n° 7.

9. If one supposes that all the terms of the first horizontal rank, which precede $y_{0,0}$, that is to say the terms of this rank continued to rear, are nulls, this which can take place in a great number of Problems, then the expression of $y_{x,t}$ will be always composed of a finite number of terms, because it will be necessary to reject all those where $y_{s,0}$ will be found s being any negative number whatever. One will have therefore in this case

$$y_{x,t} = Ty_{x+\mu t,0} + T'y_{x+\mu t-1,0} + T''y_{x+\mu t-2,0} + \ldots + T^{(x+\mu t)}y_{0,0}.$$

In all the other cases the series will go to infinity, unless one has $B' = 0$ or $C' = 0$; since then, because t equal to a positive whole number, the sequence of quantities T, T', ... will be finite and will have only $t+1$ terms.

10. In order to show, by a known example, the application of the preceding formula, I take the one of the Table of Pascal for the combinations, in which one knows that each term is equal to the sum of the one which precedes it in the same horizontal rank and of the one which is above this last in the same vertical rank; moreover the first horizontal rank is entirely formed of units and the first vertical rank is entirely zero. Whence it follows that one has first, in general, this equation

$$y_{x+1,t+1} = y_{x,t+1} + y_{x,t},$$

and that next one has

$$y_{x,0} = 1, \text{ as much as } x = 0,\ 1,\ 2, \ldots,$$
$$y_{0,t} = 0, \text{ as much as } t = 0,\ 1,\ 2, \ldots.$$

This equation being compared to that of n° 7, one has

$$A = 1, \quad B = 0, \quad B' = 0, \quad C' = -1;$$

therefore

$$-\frac{A + B\alpha}{B' + C'\alpha} = \frac{1}{\alpha - 1};$$

this which being raised to the power t gives the series

$$\alpha^{-t} + t\alpha^{-t-1} + \frac{t(t+1)}{2}\alpha^{-t-2} + \frac{t(t+1)(t+2)}{2.3}\alpha^{-t-3} + \ldots,$$

so that one will have in the general formula of the number cited $\mu = -1$ and

$$T = 1, \quad T' = t, \quad T'' = \frac{t(t+1)}{2}, \ldots.$$

Therefore, by the formula of n° 8, one will have, in general,

$$y_{x,t} = y_{x-t,0} + ty_{x-t-1,0} + \frac{t(t+1)}{2}y_{x-t-2,0} + \ldots$$

But by making $x = 0$ one must have, by hypothesis, $y_{0,t} = 0$, by supposing $t = 1, 2, 3, \ldots$; therefore it will be necessary that one has, in general,

$$y_{-t,0} = ty_{-t-1,0} + \frac{t(t+1)}{2}y_{-t-2,0} + \ldots,$$

whatever be t, provided that this is a whole positive number; whence it is easy to conclude that one must have

$$y_{-1,0} = 0, \quad y_{-2,0} = 0, \ldots,$$

and, in general,

$$y_{s,0} = 0,$$

as much as s will be an entire negative, this which is the case of n° 9, in which we have seen that the series becomes finite.

One will have therefore, according to the formula of this number,

$$y_{x,t} = y_{x-t,0} + ty_{x-t-1,0} + \frac{t(t+1)}{2}y_{x-t-2,0} + \ldots + \frac{t(t+1)\ldots(x-1)}{1.2\ldots(x-t)}y_{0,0}.$$

Such is the general expression of any term whatsoever of the Table of Pascal, by supposing that the terms which form the first horizontal rank, and which are represented by $y_{0,0}$, $y_{1,0}$, $y_{2,0}, \ldots$, are any. But in the case of the Table of Pascal these terms are all equal to unity; substituting therefore unity in the place of these quantities in the formula above, one will have

$$y_{x,t} = 1 + t + \frac{t(t+1)}{2} + \frac{t(t+1)(t+2)}{2.3} + \ldots + \frac{t(t+1)\ldots(x-1)}{1.2\ldots(x-t)},$$

this which is reduced, as one knows, to this more simple expression

$$y_{x,t} = \frac{(t+1)(t+2)(t+3)\ldots x}{1.2.3\ldots(x-t)}.$$

11. We have remarked above that the preceding solution gives, in general, a finite expression of $y_{x,t}$, when $C' = 0$ or $B' = 0$; we examine therefore first these two cases.

1° Let $C' = 0$; then the differential equation (F) will have only three terms and will be of the first order. And if one makes, for brevity,

$$-\frac{B}{B'} = p, \quad \frac{A}{B} = q,$$

one will have

$$-\frac{A + B\alpha}{B'} = p\alpha\left(1 + \frac{q}{x}\right),$$

this which being raised to the power t and next compared to the general formula $T\alpha^{\mu t} + T'\alpha^{\mu t - 1} + \ldots$, will give

$$\mu = 1, \quad T = p^t, \quad T' = tp^tq, \quad T'' = \frac{t(t-1)}{2}p^tq^2, \ldots$$

Therefore (8)

$$y_{x,t} = p^t \left[y_{x+t,0} + tqy_{x+t-1,0} + \frac{t(t-1)}{2} q^2 y_{x+t-2,0} + \dots \right].$$

One sees here not only that the series is always finite when t is an entire positive number, but yet that it contains only some quantities of the form $y_{s,0}$, s being positive; whence it follows that in this case it suffices that the first horizontal rank of the Table of n° 6 be given, in order that one can determine the value of any term that there is of the same Table.

2° We suppose that one has $B' = 0$; the differential equation will have also only three terms, but it will be of the second order. Making in this case

$$-\frac{B}{C'} = p, \quad \frac{A}{B} = q,$$

one will have

$$-\frac{A + B\alpha}{C'\alpha} = p\left(1 + \frac{q}{\alpha}\right);$$

raising this quantity to the power t, and comparing with the general formula, one will have $\mu = 0$, and the values T, T', T'', ... will be the same as in the preceding case.

Thus one will have

$$y_{x,t} = p^t \left[y_{x,0} + tqy_{x-1,0} + \frac{t(t-1)}{2} q^2 y_{x-2,0} + \dots \right].$$

This expression is always finite as long as t is an entire positive number; but, when t is $> x$, it contains necessarily some quantities such as $y_{s,0}$, s being negative. Thus it will not suffice, in this case, that the first horizontal rank of the Table of n° 6 is given, it will be necessary again to suppose given the preceding terms $y_{-1,0}$, $y_{-2,0}$, ... If one does not know these terms, but that one knows those which form the first vertical rank of the Table, one will be able then to deduce these from those in the following manner.

I make $x = 0$ and t successively $= 1, 2, 3, \dots$; I will have

$$\begin{aligned}
&y_{0,1} = p(y_{0,0} + qy_{-1,0}),\\
&y_{0,2} = p^2(y_{0,0} + 2qy_{-1,0} + q^2 y_{-2,0}),\\
&y_{0,3} = p^3(y_{0,0} + 3qy_{-1,0} + 3q^2 y_{-2,0} + q^3 y_{-3,0}),\\
&\dots\dots\dots\dots\dots\dots,
\end{aligned}$$

whence it is easy to draw

$$\begin{aligned}
qy_{-1,0} &= \frac{1}{p} y_{0,1} - y_{0,0},\\
q^2 y_{-2,0} &= \frac{1}{p^2} y_{0,2} - \frac{2}{p} y_{0,1} + y_{0,0},\\
q^3 y_{-3,0} &= \frac{1}{p^3} y_{0,3} - \frac{3}{p^2} y_{0,2} + \frac{3}{p} y_{0,1} - y_{0,0},\\
&\dots\dots\dots\dots\dots\dots,
\end{aligned}$$

and, in general,

$$q^s y_{-s,0} = \frac{1}{p^s} y_{0,s} - \frac{s}{p^{s-1}} y_{0,s-1} + \frac{s(s-1)}{2p^{s-2}} y_{0,s-2} - \dots$$

I conclude thence that, if one considers these two sequences

$$\begin{array}{llll} y_{0,0}, & \frac{1}{q} y_{1,0}, & \frac{1}{q^2} y_{2,0}, & \frac{1}{q^3} y_{3,0}, \ \dots, \\ y_{0,0}, & \frac{1}{p} y_{0,1}, & \frac{1}{p^2} y_{0,2}, & \frac{1}{p^3} y_{0,3}, \ \dots, \end{array}$$

which are supposed givens, and if one denotes for more simplicity the terms of the first by

$$Y,\ Y_1,\ Y_2,\ Y_3, \dots,$$

and those of the second by

$$Y,\ Y',\ Y'',\ Y''',\ \dots;$$

if next one takes the successive differences of the terms of this last, which are denoted by the characteristic Δ, so that one has, as one knows,

$$\begin{aligned} \Delta Y &= Y' - Y, \\ \Delta^2 Y &= Y'' - 2Y' + Y, \\ \Delta^3 Y &= Y''' - 3Y'' + 3Y' - Y, \\ &\dots\dots\dots\dots\dots\dots; \end{aligned}$$

if one supposes finally that the first sequence is continued to the rear by the terms

$$Y_{-1},\ Y_{-2},\ Y_{-3},\ \dots,$$

which are respectively equal to

$$\Delta Y,\ \Delta^2 Y,\ \Delta^3 Y,\ \dots,$$

so that one has, in general

$$Y_{-3} = \Delta^s Y;$$

one will have the formula

$$y_{x,t} = (pq)^t \left[Y_x + tY_{x-1} + \frac{t(t-1)}{2} Y_{x-2} + \frac{t(t-1)(t-2)}{2.3} Y_{x-3} + \dots \right],$$

in which all the quantities $Y_x,\ Y_{x-1}, \dots$ are known.

12. But, if neither C' nor B' is equal to zero, then it is impossible to have, in general, a finite expression for $y_{x,t}$ by the method of n° 7; however one can arrive there by another method which we are going to expose.

I take the expression of β in α (7), which is

$$\beta = -\frac{A + B\alpha}{B' + C'\alpha};$$

I make

$$B' + C'\alpha = -\omega,$$

whence I draw

$$\alpha = -\frac{\omega + B'}{C'},$$

and substituting in the value of β, there arrives to me

$$\beta = -\frac{B}{C'} + \left(A - \frac{BB'}{C'}\right)\frac{1}{\omega}.$$

I will have therefore thus

$$\alpha = -\frac{\omega}{C'}\left(1 + \frac{B'}{\omega}\right), \qquad \beta = -\frac{B}{C'}\left[1 + \left(B' - \frac{AC'}{B}\right)\frac{1}{\omega}\right].$$

These values being substituted into the quantity $\alpha^x\beta^t$, reducing next this quantity to a series according to the powers of $\frac{1}{\omega}$, one will have an expression of the form

$$\alpha^x\beta^t = V\omega^x + V'\omega^{x-1} + V''\omega^{x-2} + V'''\omega^{x-3} + \dots,$$

which will be composed always of a finite number of terms, x and t being some entire positive numbers.

Now, since ω is an indeterminate constant, it is easy to prove, by a reasoning similar to the one which one has made in n° 7 relatively to the indeterminate α, that one will have, in general,

$$y_{x,t} = Vf(x) + V'f(x-1) + V''f(x-2) + V'''f(x-3) + \dots,$$

the characteristic f denoting any function whatever.

Such is therefore the general expression of $y_{x,t}$, and this expression has over that of the number cited the advantage to be always finite.

13. We suppose now that the values given of y are those which form the first horizontal rank, and the first vertical rank of the Table of n° 6, that is to say which corresponds to $t = 0$ and to $x = 0$; and we see how one must determine by their means the different values of the function $f(x),\ f(x-1), \dots$.

1° Let therefore $t = 0$, and making for more simplicity

$$-\frac{1}{C'} = m, \quad B' = n,$$

so that

$$\alpha = m\omega\left(1 + \frac{n}{\omega}\right) \quad \text{and} \quad \alpha^x = m^x\left[\omega^x + xn\omega^{x-1} + \frac{x(x-1)}{2}n^2\omega^{x-2} + \dots\right],$$

one will have

$$V = m^x, \quad V' = xnm^x, \quad V'' = \frac{x(x-1)}{2}n^2m^x, \dots;$$

therefore

$$y_{x,0} = m^x \left[f(x) + xnf(x-1) + \frac{x(x-1)}{2} n^2 f(x-2) + \dots \right].$$

We suppose successively

$$x = 0, 1, 2, 3, \dots;$$

one will have

$$\begin{aligned}
y_{0,0} &= f(0), \\
y_{1,0} &= m[f(1) + nf(0)], \\
y_{2,0} &= m^2[f(2) + 2nf(1) + n^2 f(0)], \\
y_{3,0} &= m^3[f(3) + 3nf(2) + 3n^2 f(1) + n^3 f(0)], \\
&\dots\dots\dots\dots\dots\dots\dots,
\end{aligned}$$

whence one draws

$$\begin{aligned}
f(0) &= y_{0,0}, \\
\frac{1}{n} f(1) &= \frac{1}{mn} y_{1,0} - y_{0,0}, \\
\frac{1}{n^2} f(2) &= \frac{1}{m^2 n^2} y_{2,0} - \frac{2}{mn} y_{1,0} + y_{0,0}, \\
\frac{1}{n^3} f(3) &= \frac{1}{m^3 n^3} y_{3,0} - \frac{3}{m^2 n^2} y_{2,0} + \frac{3}{mn} y_{1,0} - y_{0,0}, \\
&\dots\dots\dots\dots\dots\dots\dots,
\end{aligned}$$

Whence one can conclude that, if one considers the sequence of terms

$$y_{0,0}, \quad \frac{1}{mn} y_{1,0}, \quad \frac{1}{m^2 n^2} y_{2,0}, \quad \frac{1}{m^3 n^3} y_{3,0}, \dots,$$

and if one designates them by Y, Y', Y'', Y''', $\dots,$ if next one takes the successive differences of these terms and if one designates them in the ordinary manner by the characteristic Δ, one will have

$$f(0) = Y,\ f(1) = n\Delta Y,\ f(2) = n^2 \Delta^2 Y,\ f(3) = n^3 \Delta^3 Y, \dots,\ f(s) = n^s \Delta^s Y,$$

2° Let $x = 0$, and making, for brevity,

$$-\frac{B}{C'} = p, \quad B' - \frac{AC'}{B} = q,$$

so that

$$\beta = p \left(1 + \frac{q}{\omega} \right),$$

and consequently

$$\beta^t = p^t \left[1 + \frac{tq}{\omega} + \frac{t(t-1)q^2}{2\omega^2} + \frac{t(t-1)(t-2)q^3}{2.3.\omega^3} + \dots \right],$$

one will have

$$V = p^t, \quad V' = tqp^t, \quad V'' = \frac{t(t-1)}{2} q^2 p^t, \ldots;$$

therefore

$$y_{0,t} = p^t \left[f(0) + tqf(-1) + \frac{t(t-1)}{2} q^2 f(-2) + \ldots \right].$$

Making successively

$$t = 0,\ 1,\ 2,\ 3,\ \ldots,$$

one will have

$$\begin{aligned}
y_{0,0} &= f(0),\\
y_{0,1} &= p[f(0) + qf(-1)],\\
y_{0,2} &= p^2[f(0) + 2qf(-1) + q^2 f(-2)],\\
y_{0,3} &= p^3[f(0) + 3qf(-1) + 3q^2 f(-2) + q^3 f(-3)],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

whence one draws

$$\begin{aligned}
f(0) &= y_{0,0},\\
qf(-1) &= \frac{1}{p} y_{0,1} - y_{0,0,}\\
q^2 f(-2) &= \frac{1}{p^2} y_{0,2} - \frac{2}{p} y_{0,1} + y_{0,0},\\
q^3 f(-3) &= \frac{1}{p^3} y_{0,3} - \frac{3}{p^2} y_{0,2} + \frac{3}{p} y_{0,1} - y_{0,0},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots;
\end{aligned}$$

therefore if one considers the series

$$y_{0,0}, \quad \frac{1}{p} y_{0,1}, \quad \frac{1}{p^2} y_{0,2}, \quad \frac{1}{p^3} y_{0,3}, \ldots,$$

and if one designates the terms of this sequence by $Y,\ {}^{\prime}Y,\ {}^{\prime\prime}Y,\ {}^{\prime\prime\prime}Y,\ \ldots,$ if next one takes the successive differences of these terms and if one designates them by the characteristic δ, one will have

$$f(0) = Y,\ f(-1) = \frac{\delta Y}{q},\ f(-2) = \frac{\delta^2 Y}{q^2},\ f(-3) = \frac{\delta^3 Y}{q^3}, \ldots,\ f(-s) = \frac{\delta^s Y}{q^s}.$$

Thus one will know the values of $f(s)$, be it that s is positive or negative; and one will have, in general, as above,

$$y_{x,t} = Vf(x) + V'f(x-1) + V''f(x-2) + \ldots$$

In regard to the values of V, V', V'', ..., it is clear that in order to find them it will be only to multiply together the series above which gives the values of α^x and of β^t; one will have by this means

$$\begin{aligned}
V =& m^x p^t,\\
V' =& m^x p^t (xn + tq),\\
V'' =& m^x p^t \left[\frac{x(x-1)}{2} n^2 + xn.tq + \frac{t(t-1)}{2} q^2\right],\\
V''' =& m^x p^t \left[\frac{x(x-1)(x-2)}{2.3} n^3 + \frac{x(x-1)}{2} n^2.tq\right.\\
&\left. + xn.\frac{t(t-1)}{2} q^2 + \frac{t(t-1)(t-2)}{2.3} q^3\right]\\
&\dots\dots\dots\dots\dots\dots\dots,
\end{aligned}$$

And if $q = n$, this which has place when $A = 0$, one will have more simply

$$\begin{aligned}
&V = m^x p^t,\\
&V' = m^x p^t (x + t) n,\\
&V'' = m^x p^t \frac{(x+t)(x+t-1)}{2} n^2,\\
&V''' = m^x p^t \frac{(x+t)(x+t-1)(x+t-2)}{2.3} n^3,\\
&\dots\dots\dots\dots\dots\dots\dots,
\end{aligned}$$

The Problem is therefore resolved with all the simplicity and the generality that one can desire.

14. In the Example of n° 10 one has

$$A = 1, \quad B = 0, \quad B' = 1, \quad C' = -1;$$

therefore

$$m = 1, \quad n = 1, \quad p = 0, \quad q = \infty \quad \text{and} \quad pq = 1.$$

Therefore one will find (because of $p = 0$, $q = \infty$ and $p = 1$)

$$\begin{array}{lllll}
V = 0, & V' = 0, & V'' = 0,\ \dots, & V^{(t-1)} = 0, & \\
V^{(t)} = m^x, & V^{(t+1)} = m^x xn, & V^{(t+2)} = m^x \frac{x(x-1)}{2} n^2, & V^{(t+2)} = m^x \frac{x(x-1)(x-2)}{2.3} n^3, & \dots
\end{array}$$

Next the sequence Y, Y', Y'', ... will become $y_{0,0}$, $y_{1,0}$, $y_{2,0}$, ..., so that one will have, in general,

$$f(s) = \Delta^s y_{0,0},$$

s being a positive number. Finally, because $p = 0$, $q = \infty$ and $pq = 1$, one will find

$$f(0) = y_{0,0},\ f(-1) = -y_{0,1},\ f(-2) = y_{0,2}, \dots,\ f(-s) = \pm y_{0,s};$$

the superior sign being for the case of s even, and the inferior for the one of s odd.

Substituting therefore these values into the general expression of $y_{x,t}$, one will have

$$y_{x,t} = m^x \left\{ \begin{array}{l} \Delta^{x-t} y_{0,0} + x\Delta^{x-t-1} y_{0,0} + \dfrac{x(x-1)}{2} \Delta^{x-t-2} y_{0,0} + \ldots + \dfrac{x(x-1)(x-2)\ldots(t+1)}{1.2.3\ldots(x-t)} y_{0,0} \\ - \dfrac{x(x-1)(x-2)\ldots t}{1.2.3\ldots(x-t+1)} y_{0,0} + \dfrac{x(x-t)\ldots(t-1)}{1.2\ldots(x-t+2)} y_{0,0} - \dfrac{x(x-1)\ldots(t-2)}{1.2\ldots(x-t+3)} y_{0,0} + \ldots \end{array} \right\},$$

where the differences $\Delta y_{0,0},\ \Delta^2 y_{0,0}, \ldots$ return uniquely to the terms of the first horizontal rank $y_{0,0}, y_{1,0}, y_{2,0}, \ldots$, so that

$$\Delta y_{0,0} = y_{1,0} - y_{0,0}, \quad \Delta^2 y_{0,0} = y_{2,0} - 2y_{1,0} + y_{0,0}, \ \ldots$$

By means of this formula one can therefore have the value of any term whatsoever of the Table of Pascal, by supposing that in this Table the first horizontal rank and the first vertical rank are anything.

In the same Table of Pascal, the first horizontal rank is entirely formed of units, and the first vertical rank is entirely zero with the exception of the first term, so that one has

$$\begin{array}{l} y_{0,0} = 1, \quad y_{1,0} = 1, \quad y_{2,0} = 1, \ldots, \\ y_{0,1} = 0, \quad y_{0,2} = 0, \ldots; \end{array}$$

therefore

$$\Delta y_{0,0} = 0, \quad \Delta^2 y_{0,0} = 0, \ldots,$$

Thus the preceding formula will become in this case

$$y_{x,t} = \frac{x(x-1)(x-2)\ldots(t+1)}{1.2.3\ldots(x-t)};$$

this which accords with that which one has found at the end of n° 10.

15. Let be proposed now the general equation of the second order

$$\left. \begin{array}{l} Ay_{x,t} + By_{x+1,t} + Cy_{x+2,t} \\ + B'y_{x,t+1} + C'y_{x+1,t+1} \\ + C''y_{x,t+2} \end{array} \right\} = 0. \tag{G}$$

I make, as above,

$$y_{x,t} = a\alpha^x \beta^t;$$

substituting and dividing next all the terms by $a\alpha^x\beta^t$, there comes to me this equation in α and β

$$A + B\alpha + B'\beta + C\alpha^2 + C'\alpha\beta + C''\beta^2 = 0, \tag{H}$$

by which one will be able to determine β in α.

I seek therefore by the known method of Newton the value of β in α expressed by a descending series, that is to say in which the exponents of α go by diminishing. I raise next this series to the power t by means of the formulas known for this object; I obtain thence a value of β^t in α of the following form

$$\beta^t = T\alpha^{\mu t} + T'\alpha^{\mu t - \mu'} + T''\alpha^{\mu t - \mu''} + T'''\alpha^{\mu t - \mu'''} + \ldots,$$

where the numbers μ', μ'', μ''', ... will be necessarily all positives and increasing.

Therefore, substituting this value of β^t, one will have this particular expression of $y_{x,t}$, namely

$$y_{x,t} = Ta\alpha^{x+\mu t} + T'a\alpha^{x+\mu t-\mu'} + T''a\alpha^{x+\mu t-\mu''} + \dots,$$

in which a and α will be some indeterminate constants.

Thence, by a reasoning similar to the one of n° 7, one will draw immediately the general expression

$$y_{x,t} = Tf(x+\mu t) + T'f(x+\mu t-\mu') + T''f(x+\mu t-\mu'') + \dots,$$

the characteristic f denoting any undetermined function whatsoever.

Now, as long as C'' will not be null, the equation in β will rise to the second degree and will have consequently two roots; one will have therefore for β, and consequently also for β^t, two different series; therefore, if the other value of β^t is represented by the series

$$\beta^t = U\alpha^{\nu t} + U'\alpha^{\nu t-\nu'} + U''\alpha^{\nu t-\nu''} + U'''\alpha^{\nu t-\nu'''} + \dots,$$

then numbers ν', ν'', ν''', ... being also positives and increasing, one will draw from it similarly a value of $y_{x,t}$, which will be

$$y_{x,t} = U\phi(x+\nu t) + U'\phi(x+\nu t-\nu') + U''\phi(x+\nu t-\nu'') + \dots,$$

the characteristic ϕ designating also any indeterminate function whatsoever.

Reuniting now the two values of $y_{x,t}$, one will have, in general,

$$\begin{aligned} y_{x,t} =& Tf(x+\mu t) + T'f(x+\mu t-\mu') + T''f(x+\mu t-\mu'') + \dots \\ &+ U\phi(x+\nu t) + U'\phi(x+\nu t-\nu') + U''\phi(x+\nu t-\nu'') + \dots, \end{aligned}$$

an expression which is necessarily the complete integral of the proposed, since it contains two indeterminate functions.

16. It is clear that this expression of $y_{x,t}$ will always be composed of an infinite number of terms, unless the two values of β in α are not finite; this which takes place only when equation (H) can be decomposed into two equations of the first degree. In this case one will have for $y_{x,t}$ a finite expression, and consequently one will have the finite integral of the proposed differential equation. But it can happen in this same case that the two values of β in α are equals; this which will give

$$\begin{aligned} &U = T, \quad U' = T', \dots, \\ &\nu = \mu, \quad \nu' = \mu', \dots, \end{aligned}$$

so that the two arbitrary functions will merge into one alone; this which will render the value of $y_{x,t}$ incomplete.

In order to remedy this inconvenience one will suppose, according to the method used in these sorts of cases, that the two values of β differ between them by a very small quantity, that is to say that one will take for the second value of β, $\beta + d\beta$; this which will give for the second value of β^t, $\beta^t + t\beta^{t-1}d\beta$, where it is necessary to remark that the differential $d\beta$ remains undetermined, because by differentiating equation (H) it will happen necessarily that the quantities by which the two differentials $d\alpha$ and $d\beta$ will be found multiplied, will be null all at once. Thence it is easy to conclude that if one denotes by ${}_{,}T$, ${}_{,}T'$, ${}_{,}T''$, ${}_{,}T'''$, … the values of T, T', T'', T''', … which correspond to $t-1$, that is to say which result from the substitution of $t-1$ in the place of t, one will have for $y_{x,t}$ this other expression

$$\begin{aligned} y_{x,t} =& Tf(x+\mu t) + T'f(x+\mu t-\mu') + T''f(x+\mu t-\mu'') + \dots \\ &+ t\,{}_{,}TF[x+\mu(t-1)] + t\,{}_{,}T'F[x+\mu(t-1)-\mu'] + t\,{}_{,}T''F[x+\mu(t-1)-\mu''] + \dots, \end{aligned}$$

in which the characteristics f and F denote some functions any whatsoever.

17. In order to determine now the arbitrary functions, one supposes that the first two horizontal ranks of the Table of n° 6 are given, that is to say one knows all the values of $y_{x,0}$ and $y_{x,1}$; one will make therefore 1° $t = 0$, and, as in this case one has

$$T = 1, \quad T' = 0, \quad T'' = 0, \dots$$

and likewise

$$U = 1, \quad U' = 0, \quad U'' = 0, \dots$$

the formula of n° 15 will give

$$y_{x,0} = f(x) + \phi(x);$$

one will make 2° $t = 1$, and, denoting by θ, θ', θ'', …, υ, υ', υ'', …the values of T, T', T'', …, U, U', U'', … which correspond to $t = 1$, the same formula will give

$$\begin{aligned} y_{x,t} =& \theta f(x+\mu) + \theta' f(x+\mu-\mu') + \theta'' f(x+\mu-\mu'') + \dots \\ &+ \upsilon\phi(x+\nu) + \upsilon\phi'(x+\nu-\nu') + \upsilon\phi''(x+\nu-\nu'') + \dots; \end{aligned}$$

thus one will have two equations, by the aid of which, by giving successively to x all the values $0, 1, 2, 3, \dots$, one will be able to determine those of the functions $f(x)$ and $\phi(x)$; but it is clear that this determination will be very difficult, in general, unless the expression of $y_{x,t}$ is not finite, this which will happen only when the value of β in α is finite.

If the two values of β are equal, the determination of the functions $f(x)$ and $F(x)$ of the formula of n° 6 will be very easy; because by making $t = 0$ one will have first

$$y_{x,0} = f(x);$$

and making next $t = 1$, one will have

$${}_{,}T = 1, \quad {}_{,}T' = 0, \quad {}_{,}T'' = 0, \dots,$$

therefore

$$y_{x,1} = \theta f(x+\mu-\mu') + \theta' f(x+\mu-\mu'') + \dots + F(x);$$

so that one will know immediately thence the general values of the two functions.

18. Moreover, although the expression of $y_{x,t}$ found by the preceding method is, in general, composed of an infinite number of terms, it is however a very extended case, and which takes place in the greater part of the questions which lead to these sorts of differential equations, in which the expression becomes finite; so that the determination of the arbitrary functions are no longer of difficulty. This case is the one where one supposes that if one continues to the rear the first two horizontal ranks of the Table of n° 6, all the terms which would form these ranks so continued are nulls; that is to say when one will have, in general,

$$y_{x,0} = 0, \quad y_{x,1} = 0$$

as long as x will be negative.

Indeed, it is easy to see that one will have then

$$f(-s) = 0 \quad \text{and} \quad \phi(-s) = 0,$$

as long as s will be greater than μ and ν; so that as the numbers which are after the characteristics f and ϕ in the general expression of $y_{x,t}$ go continually by diminishing, the functions of these numbers will become finally null, this which will render the expression of which there is the question is finite.

It is easy now to apply to the differential equations of all orders, included under the general formula of n° 6, the method that we just exposed for the equations of the second order, and to draw some similar conclusions from it; thus we ourselves will not expand further on this method.

19. In the case of the equations of the second order in three terms we have found means to remedy the inconvenience of the general method, and to obtain a finite expression for $y_{x,t}$ (12); by considering the artifice that one has employed in the place cited, and that consists in expressing the two quantities α and β by a third indeterminate ω, in a finite manner, one will be convinced easily that it can also serve for all the equations of second order, as one is going to see.

I take again therefore equation (H) of n° 15, and I make first the terms vanish where the indeterminates are in the first dimension, by supposing

$$\alpha = m + \epsilon, \quad \beta = n + \theta,$$

and taking m and n such that one has

$$B + 2Cm + C'n = 0, \quad B' + 2C''n + C'm = 0,$$

this which gives

$$m = \frac{2BC'' - B'C'}{C'^2 - 4CC''}, \quad n = \frac{2B'C - BC'}{C'^2 - 4CC''};$$

for what if one makes, for brevity,

$$K = A + Bm + B'n + Cm^2 + C'mn + C''n^2,$$

one has this transformed by ϵ and θ

$$C\epsilon^2 + C'\epsilon\theta + C''\theta^2 + K = 0,$$

which being multiplied by C can be put under this form

$$(C\epsilon + h\theta)(C\epsilon + l\theta) + CK = 0,$$

by supposing

$$h = \frac{C'}{2} + \sqrt{\frac{C'^2}{4} - CC''}, \quad l = \frac{C'}{2} - \sqrt{\frac{C'^2}{4} - CC''}.$$

I make now

$$C\epsilon + h\theta = \omega,$$

I will have

$$C\epsilon + l\theta = -\frac{CK}{\omega},$$ [5]

where I draw immediately

$$\epsilon = \frac{l\omega + \frac{hCK}{\omega}}{Cl - h}, \quad \theta = \frac{\omega + \frac{CK}{\omega}}{h - l};$$

therefore finally

$$\alpha = m + \frac{l\omega + \frac{hCK}{\omega}}{C(l - h)}, \quad \beta = n + \frac{\omega + \frac{CK}{\omega}}{h - l}.$$

Thus the two indeterminates α and β are expressed by a third indeterminate ω in a finite manner and without complex fraction, so that the value of $\alpha^x\beta^t$ will be always finite as long as x and t will be integral positive.

And one remarks with regard to the preceding expressions that the ambiguity of the radical which enters into the values of h and of l influence not at all on the form of these expressions; because by changing the sign of this radical one does only to change h into l and *vice versa*; now by making this change and setting at the same time $-\frac{CK}{\eta}$ in the place of ω, and consequently η in the place of $-\frac{CK}{\omega}$, one will see that the new expressions of α and β in η will be the same as the first in ω.

This set, if one makes, in order to abridge further,

$$p = \frac{l}{C(l - h)}, \quad q = \frac{hK}{l - h}, \quad r = \frac{1}{h - l}, \quad s = \frac{CK}{h - l},$$

one will have

$$\alpha = m + p\omega + \frac{q}{\omega}, \quad \beta = n + r\omega + \frac{s}{\omega},$$

[5] *Editor's note*: In the original text the second member of this formula has the $+$ sign, which has the effect to change K into $-K$ in the expressions of α and of β. We have believed we must restore the exactness of the formulas.

consequently

$$\alpha^x\beta^t = \left(m + p\omega + \frac{q}{\omega}\right)^x \left(n + r\omega + \frac{s}{\omega}\right)^t.$$

This expression of $\alpha^x\beta^t$, being developed and ordered according to the powers of ω, will be reduced to a finite series of the form

$$\begin{aligned} V+&V'\omega + V''\omega^2 + V'''\omega^3 + \ldots + V^{(x+t)}\omega^{x+t} \\ &+\frac{{}'V}{\omega} + \frac{{}''V}{\omega^2} + \frac{{}'''V}{\omega^3} + \ldots + \frac{{}^{(x+t)}V}{\omega^{x+t}}, \end{aligned}$$

where the coefficients V, V', V'', $\ldots$, ${}'V$, ${}''V$, ${}'''V$, $\ldots$ will be some functions of x and t, which one can determine by different means after known methods.

Therefore as ω is an absolutely arbitrary quantity, one will be able to conclude from it immediately by some reasonings analogous to those that we have made above (7) the general expression of $y_{x,t}$, which will be

$$\begin{aligned} y_{x,t} =&Vf(0) + V'f(1) + V''f(2) + V'''f(3) + \ldots + V^{(x+t)}f(x+t) \\ &+ {}'Vf(-1) + {}''Vf(-2) + {}'''Vf(-3) + \ldots + {}^{(x+t)}Vf(-x-t), \end{aligned}$$

the characteristic f denoting an arbitrary function.

20. In order to determine this function, or at least its different particular values which enter into the preceding expression, we will suppose that in the Table of n° 6 the first horizontal rank and the first vertical rank are given, so that one knows all the values of $y_{x,0}$ and of $y_{0,t}$. One will suppose therefore first $t = 0$ and x successively $0, 1, 2, 3, \ldots$; next $x = 0$ and t successively $0, 1, 2, 3, \ldots$; one will have by this means the equations necessary in order to determine the values of $f(0)$, $f(1)$, $f(-1)$, $\ldots$ But as by taking thus one falls into some rather complicated formulas, I am going to give another way to arrive more easily to the end.

21. For this I remark first that as

$$p\omega + \frac{q}{\omega} = \epsilon, \quad r\omega + \frac{s}{\omega} = \theta,$$

one will have, by known formulas,

$$p\omega + \frac{q}{\omega} = \epsilon, \quad p^2\omega^2 + \frac{q^2}{\omega^2} = \epsilon^2 - 2pq, \quad p^3\omega^3 + \frac{q^3}{\omega^3} = \epsilon^3 - 3pq\epsilon, \ldots,$$

and, in general,

$$p^\lambda\omega^\lambda + \frac{q^\lambda}{\omega^\lambda} = \epsilon^\lambda - \lambda pq\epsilon^{\lambda-2} + \frac{\lambda(\lambda-3)}{2}p^2q^2\epsilon^{\lambda-4} - \frac{\lambda(\lambda-4)(\lambda-5)}{2.3}p^3q^3\epsilon^{\lambda-6} + \ldots,$$

and likewise one will have

$$r^\lambda\omega^\lambda + \frac{s^\lambda}{\omega^\lambda} = \theta^\lambda - \lambda rs\theta^{\lambda-2} + \frac{\lambda(\lambda-3)}{2}r^2s^2\theta^{\lambda-4} - \frac{\lambda(\lambda-4)(\lambda-5)}{2.3}r^3s^3\theta^{\lambda-6} + \ldots,$$

whence one draws

$$\omega^\lambda = \frac{(s^\lambda\epsilon^\lambda - q^\lambda\theta^\lambda) - \lambda(pqs^\lambda\epsilon^{\lambda-2} - rsq^\lambda\theta^{\lambda-2}) + \dots}{p^\lambda s^\lambda - q^\lambda r^\lambda},$$
$$\frac{1}{\omega^\lambda} = \frac{(r^\lambda\epsilon^\lambda - p^\lambda\theta^\lambda) - \lambda(pqr^\lambda\epsilon^{\lambda-2} - rsp^\lambda\theta^{\lambda-2}) + \dots}{q^\lambda r^\lambda - p^\lambda s^\lambda}.$$

If one substitutes these values into the series

$$V + V'\omega + \frac{{}'V}{\omega} + V''\omega^2 + \frac{{}''V}{\omega^2} + \dots$$

it is clear that one will have a conversion to this form

$$\begin{aligned} Z+&Z'\epsilon + Z''\epsilon^2 + Z'''\epsilon^3 + \dots + Z^{(x+t)}\epsilon^{x+t} \\ &+{}'Z\theta + {}''Z\theta^2 + {}'''Z\theta^3 + \dots + {}^{(x+t)}Z\theta^{x+t}, \end{aligned}$$

which will be consequently equal and identical to the quantity

$$\alpha^x\beta^t = (m+\epsilon)^x(n+\theta)^t,$$

by supposing that there is between ϵ and θ (19) the equation

$$C\epsilon^2 + C'\epsilon\theta + C''\theta^2 + K = 0.$$

Now as ϵ and θ are two different functions of the indeterminate ω, one can conclude from it immediately, by a reasoning analogous to the one of n° 7, this general expression of $y_{x,t}$, namely

$$\begin{aligned} y_{x,t} =& ZF(0) + Z'F(1) + Z''F(2) + \dots + Z^{(x+t)}F(x+t) \\ &+ {}'Z\phi(1) + {}''Z\phi(2) + {}'''Z\theta^3 + \dots + {}^{(x+t)}Z\phi(x+t), \end{aligned}$$

where the characteristics F and ϕ denote any functions whatsoever.

22. Let one suppose now, in order to determine these functions, $t = 0$ and next $x = 0$, one will have:

1° When $t = 0$,

$$(m+s)^x(n+\theta)^t = (m+\epsilon)^x = m^x + xm^{x-1}\epsilon + \frac{x(x-1)}{2}m^{x-2}\epsilon^2 + \dots;$$

therefore

$$Z = m^x, \quad Z' = xm^{x-1}, \quad Z'' = \frac{x(x-1)}{2}m^{x-2}, \dots,$$
$${}'Z = 0, \quad {}''Z = 0, \dots,$$

therefore

$$y_{x,0} = m^x\left[F(0) + x\frac{F(1)}{m} + \frac{x(x-1)}{2}\,\frac{F(2)}{m^2} + \dots\right],$$

whence by making successively $x = 0, 1, 2, \ldots$ one will draw easily the values of $F(0),\ F(1),\ F(2), \ldots$. And by the method of n° 13 one will find that if one designates the sequence of quantities

$$y_{0,0}, \quad \frac{1}{m}y_{1,0}, \quad \frac{1}{m^2}y_{2,0}, \quad \frac{1}{m^3}y_{3,0}, \ldots$$

by Y, Y', Y'', ..., and if one denotes by Δ, Δ^2, ... the successive differences of the terms of this sequence, one will have, in general,

$$F(\mu) = m^\mu \Delta^\mu Y.$$

2° When $x = 0$,

$$(m+\epsilon)^x(n+\theta)^t = (n+\theta)^t = n^t + tn^{t-1}\theta + \frac{t(t-1)}{2}n^{t-2}\theta^2 + \ldots;$$

therefore

$$Z = n^t, \quad {}'Z = tn^{t-1}, \quad {}''Z = tn^{t-2}, \ldots,$$
$$Z' = 0, \quad Z'' = 0, \ldots,$$

therefore

$$y_{0,t} = n^t\left[\phi(0) + t\frac{\phi(1)}{n} + \frac{t(t-1)}{2}\frac{\phi(2)}{n^2} + \ldots\right],$$

by supposing $\phi(0) = F(0)$.

Thence one will find, as before, that if one considers the sequence

$$y_{0,0}, \quad \frac{1}{n}y_{0,1}, \quad \frac{1}{n^2}y_{0,2}, \quad \frac{1}{n^3}y_{0,3}, \ldots,$$

and if one designates the terms of it by $Y,\ {}'Y,\ {}''Y,\ {}'''Y, \ldots,$ if one denotes next by δ, δ^2, ... the successive differences of these terms, one will find, I say, in general,

$$\phi(\nu) = n^\nu \delta^\nu Y.$$

Now by making $\mu = 0, \nu = 0$, one has

$$F(0) = Y = \phi(0),$$

as this must be by the hypothesis.

Therefore if one substitutes these values into the expression of $y_{x,t}$ of the preceding number, one will have

$$\begin{aligned} y_{x,t} = ZY &+ mZ'\Delta Y + m^2 Z''\Delta^2 Y + \ldots + m^{x+t}Z^{(x+t)}\Delta^{x+t}Y \\ &+ n\,{}'Z\delta Y + n^2\,{}''Z\delta^2 Y + \ldots + n^{x+t}\,{}^{(x+t)}Z\delta^{x+t}Y, \end{aligned}$$

a formula by which one will be able to know any term whatsoever of the Table of n° 6, as soon as one will know those of the first two ranks, the one horizontal, the other vertical.

23. If now one compares together the two expressions of $y_{x,t}$ in n$^{\text{os}}$ 19 and 21, it will be easy to conclude from it the values of the function f through those of the functions F and ϕ; and it is not difficult to see that one will have, in general, among

$$f(\lambda),\, f(-\lambda),\, F(\lambda),\, F(\lambda-2), \ldots \phi(\lambda),\, \phi(\lambda-2), \ldots$$

the same relations as among

$$\omega^\lambda,\, \frac{1}{\omega^\lambda},\, \epsilon^\lambda,\, \epsilon^{\lambda-2},\, \ldots, \theta^\lambda,\, \theta^{\lambda-2}, \ldots$$

So that if one substitutes the values of the functions F and ϕ found above, and if one makes, for brevity,

$$p^\lambda s^\lambda - q^\lambda r^\lambda = \frac{1}{\Lambda},$$

one will have, λ being positive,

$$\begin{aligned}
f(\lambda) =& \Lambda s^\lambda m^\lambda \left[\Delta^\lambda Y - \lambda \frac{pq}{m^2}\Delta^{\lambda-2}Y + \frac{\lambda(\lambda-3)}{2}\,\frac{p^2q^2}{m^4}\Delta^{\lambda-4}Y - \ldots\right] \\
&+ \Lambda q^\lambda n^\lambda \left[\delta^\lambda Y - \lambda \frac{rs}{n^2}\delta^{\lambda-2}Y + \frac{\lambda(\lambda-3)}{2}\,\frac{r^2s^2}{n^4}\delta^{\lambda-4}Y - \ldots\right] \\
f(-\lambda) =& \Lambda p^\lambda n^\lambda \left[\delta^\lambda Y - \lambda \frac{rs}{n^2}\delta^{\lambda-2}Y + \frac{\lambda(\lambda-3)}{2}\,\frac{r^2s^2}{n^4}\delta^{\lambda-4}Y - \ldots\right] \\
&- \Lambda r^\lambda m^\lambda \left[\Delta^\lambda Y - \lambda \frac{pq}{m^2}\Delta^{\lambda-2}Y + \frac{\lambda(\lambda-3)}{2}\,\frac{p^2q^2}{m^4}\Delta^{\lambda-4}Y - \ldots\right];
\end{aligned}$$

these are the values of the function f which would result from the equations of n° 20, as it is easy to convince ourselves of it by the calculus; thus there will be only to substitute these values into the formula of n° 19.

24. The method by which we just integrated in a finite and complete manner all the differential equations of the second order among three variables could be extended also to the equations of the superior orders if, in any equation in two indeterminates, it was always possible to express each of these indeterminates by a finite rational function and without a complex fraction of a third indeterminate; but as this takes place, for the equations which pass the second degree, only in some particular cases, one must regard the preceding method as limited to the differential equations of the first and of the second order.

In order to take the place of this defect, we are going to give in the following Article another method which will extend to the equations of all orders, and which will join in the advantage to give always finite integrals, the one to render the determination of the arbitrary functions very easy in all the cases.

ARTICLE III. — *Where one gives a general method for the integration of linear equations in finite differences among three variables.*

25. We consider the differential equation of the n^{th} degree of n° 6, and we make, in general,

$$y_{x,t} = a\alpha^x\beta^t;$$

it is easy to see that after the substitutions and division by $a\alpha^x\beta^t$, there will come this equation of the n^{th} degree in α and β

$$\left.\begin{cases} A + B\alpha + C\alpha^2 + \ldots + N\alpha^n \\ + B'\beta + C'\alpha\beta + \ldots + N'\alpha^{n-1}\beta \\ + C''\beta^2 + \ldots + N''\alpha^{n-2}\beta \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ + N^{(n)}\beta^n \end{cases}\right\} = 0, \tag{I}$$

by which it will be necessary to determine β in α or *vice versa*.

I remark now that one can express, in general, β in powers of α only by an infinite series, this which will give, as one has seen it in Article II, an expression of $y_{x,t}$ in infinite series; but as one has no need of the value of β, but only of that of β^t, where t is counted greater than n, I observe that one can reduce this value to a rational and finite series of terms ordered according to the powers of α, provided that one admits also the powers of β inferior to β^n; because it is clear that if one takes the value of β^n given by the preceding equation, and if one substitutes it as much as is possible into the value of β^t, if next in the terms resulting from this first substitution, one substitutes anew as much as it is possible the same value of β^n, and thus in sequence until one has lowered the powers of β to below β^n; it is clear, I say, that one will arrive to a formula of this form

$$\left.\begin{cases} \beta^t = T + T'\alpha + T''\alpha^2 + T'''\alpha^3 + \ldots + T^{(t)}\alpha^t \\ \qquad + {}_{,}T\beta + {}_{,}T'\alpha\beta + {}_{,}T''\alpha^2\beta + \ldots + {}_{,}T^{(t-1)}\alpha^{t-1}\beta \\ \qquad + {}_{,,}T\beta^2 + {}_{,,}T'\alpha\beta^2 + \ldots + {}_{,,}T^{(t-2)}\alpha^{t-2}\beta^2 \\ \qquad \ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad + {}_{(n-1)}T\beta^{n-1} + \ldots + {}_{(n-1)}T^{(t-n+1)}\alpha^{t-n+1}\beta^{n-1}, \end{cases}\right\} \tag{K}$$

where the coefficients $T,\ T',\ T'', \ldots,\ {}_{,}T,\ {}_{,}T', \ldots$ will be some given rational functions of t and of the coefficients $A,\ B,\ B', \ldots$ of the equation in α and β.

26. Multiplying therefore this expression of β^t by $a\alpha^x$, one will have a particular value of $y_{x,t}$ in which the two constants a and α will be at will; and as the proposed differential equation is linear and contains only terms without y, it is clear that one will be able also to take for $y_{x,t}$ the sum of as many similar particular values as one will wish, by supposing that the quantities a and α are different in each of these values.

Thence and from this that the quantities $\beta,\ \beta^2,\ \beta^3, \ldots$ to β^{n-1} are necessarily some irrational functions of α, irreducibles among them, it is easy to conclude by a reasoning analogous to the one which one has employed in n° 7 that one will have, in general,

$$\begin{aligned} y_{x,t} = {} & Tf(x) + T'f(x+1) + T''f(x+2) + T'''f(x+3) + \ldots + T^{(t)}f(x+t) \\ & + {}_{,}T\ {}^{1}f(x) + {}_{,}T'\ {}^{1}f(x+1) + {}_{,}T''\ {}^{1}f(x+2) + \ldots + {}_{,}T^{(t-1)}\ {}^{1}f(x+t-1) \\ & + {}_{,,}T\ {}^{2}f(x) + {}_{,,}T'\ {}^{2}f(x+1) + \ldots + {}_{,,}T^{(t-2)}\ {}^{2}f(x+t-2) \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & + {}_{(n-1)}T\ {}^{n-1}f(x) + \ldots + {}_{(n-1)}T^{(t-n+1)\ n-1}f(x+t-n+1), \end{aligned}$$

where the characteristics $f,\ {}^1f,\ {}^2f,\ldots,\ {}^{n-1}f$ denote some arbitrary functions any whatsoever independent among themselves; so that as the number of these different functions is n, and consequently equal to the exponent of the order of the proposed differential equation, one must regard the preceding expression as the complete integral of this same equation.

27. In order to determine now the values of these different functions, I suppose that the first n horizontal ranks of the Table of n° are given, so that one knows all the different values of $y_{x,0},\ y_{x,1},\ y_{x,2},\ \ldots,\ y_{x,n-1}$, that is to say all the values of $y_{x,t}$ which correspond to $t = 0,\ 1,\ 2,\ \ldots,\ n-1$.

Now making $t = 0$ one has $\beta^t = 1$; therefore in formula (K) of n° 25 one will have

$$T = 0,\ T' = 0,\ \ldots,\quad {}_{,}T = 0,\ \ldots,\ldots;$$

making $t = 1$ one will have $\beta_t = \beta$; therefore

$${}_{,}T = 1,\quad \text{and all the other coefficients nulls;}$$

making $t = 2$ one has $\beta_t = \beta^2$; therefore

$${}_{,,}T = 2,\quad \text{and all the other coefficients nulls;}$$

and thus in sequence.

Therefore: if one makes $t = 0$, one will have in the formula of n° 26

$$y_{x,0} = f(x);$$

if one makes $t = 1$, one will have

$$y_{x,1} = {}^1f(x);$$

if one makes $t = 2$, one will have

$$y_{x,2} = {}^2f(x);$$

and thus in sequence until

$$y_{x,n-1} = {}^{n-1}f(x).$$

One knows therefore by this means all the arbitrary functions; and substituting their values into the general formula, one will have

$$\begin{aligned} y_{x,t} =& Ty_{x,0} + T'y_{x+1,0} + T''y_{x+2,0} + T'''y_{x+3,0} + \ldots + T^{(t)}y_{x+t,0} \\ &+ {}_{,}Ty_{x,1} + {}_{,}T'y_{x+1,1} + {}_{,}T''y_{x+2,1} + \ldots + {}_{,}T^{(t-1)}y_{x+t-1,1} \\ &+ {}_{,,}Ty_{x,2} + {}_{,,}T'y_{x+1,2} + \ldots + {}_{,,}T^{(t-2)}y_{x+t-2,2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ {}_{(n-1)}Ty_{x,n-1} + \ldots + {}_{(n-1)}T^{(t-n+1)}y_{x+t-n+1,n-1}. \end{aligned}$$

28. In order to determine the coefficients $T,\ T',\ T'',\ \ldots,\ {}_{,}T,\ {}_{,}T',\ {}_{,}T'',\ \ldots,\ \ldots$ of formula (K) of n° 25, one can employ different methods.

And first it is clear that if one draws from equation (I) the value of β in α, if one substitutes it next into equation (K), and if after having ordered the terms according to the powers of α, one makes each term equal to zero, one will have a sequence of equations by which one will be able to determine the sought coefficients.

This method can be rendered simpler by the consideration of the different roots of equation (I). Indeed, if one represents equation (K) thus

$$\beta^t = A + {}_{\prime}A\beta + {}_{\prime\prime}A\beta^2 + \ldots + {}_{(n-1)}A\beta^{n-1},$$

A being a polynomial in α of degree t, ${}_{\prime}A$ another polynomial in α of degree $t-1$ and thus in sequence; and if on the other hand one designates by β', β'', ... the n roots of equation (I) ordered with respect to β, one will have these n different equations

$$\begin{aligned}\beta'^t &= A + {}_{\prime}A\beta' + {}_{\prime\prime}A\beta'^2 + \ldots + {}_{(n-1)}A\beta'^{n-1},\\ \beta''^t &= A + {}_{\prime}A\beta'' + {}_{\prime\prime}A\beta''^2 + \ldots + {}_{(n-1)}A\beta''^{n-1},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

by means of which one will determine separately the n quantities A, ${}_{\prime}A$, ${}_{\prime\prime}A, \ldots$ in β', β'', ... Then there will be no more than to substitute in the place of β', β'', ... their values in α reduced to ascendant series, and advanced only to the t^{th} power for the quantity A, to the $(t-1)^{\text{st}}$ power for the quantity ${}_{\prime}A$, and thus in sequence.

29. But as soon as one will have determined by this method or by any other whatsoever the first terms of the polynomials A, ${}_{\prime}A$, ${}_{\prime\prime}A, \ldots$, one will be able to find all the following in a more simple manner while searching by aid of the differential Calculus the law which must rule among them. For this one will differentiate logarithmically the equation

$$\beta^t = A + {}_{\prime}A\beta + {}_{\prime\prime}A\beta^2 + \ldots + {}_{(n-1)}A\beta^{n-1},$$

by making vary altogether the quantities α and β, this which will give

$$\frac{td\beta}{\beta} = \frac{dA + \beta\, d{}_{\prime}A + \beta^2\, d{}_{\prime\prime}A + \ldots + (A + 2{}_{\prime\prime}A\beta + \ldots)\, d\beta}{A + {}_{\prime}A\beta + {}_{\prime\prime}A\beta^2 + \ldots};$$

one will substitute in the place of $d\beta$ its value in α and $d\alpha$ drawn from equation (I) by differentiation, and making the fractions vanish one will order all the terms with respect to the powers of β; it is easy to understand that in this new equation the highest power of β will be able to be only β^{2n-1}; thus there will be only to lower the $n-1$ powers β^n, β^{n+1}, $\ldots, \beta^{2n-1}$ below the n^{th} degree by means of equation (I); after what one will order the equation with respect to the n remaining powers of β and one will make separately equal to zero all the quantities multiplied by each of these different powers of β; one will have n differential equations of the first order between α and the n quantities A, ${}_{\prime}A$, ${}_{\prime\prime}A, \ldots$. One will substitute now into each of these equations the expressions of A, ${}_{\prime}A$, ... in α, and by the comparison of the terms one will obtain some equations among the coefficients T, T', T'', $\ldots$, ${}_{\prime}T$, ${}_{\prime}A', \ldots, \ldots$ by which one will be able to determine the coefficients.

30. If instead of supposing givens the n first horizontal ranks of the Table of n° 6, in the same way as one has done in the preceding solution, one would wish to regard as givens the n first vertical ranks of the same Table, that is to say the values of

$y_{0,t},\ y_{1,t}, y_{2,t}, \dots y_{n-1,t}$; it is clear that one could resolve this case by the same method by changing only t into x, that is to say β into α, or, that which reverts to the same, by operating in regard to β and to α as one has done in regard to α and β; there will be for this no new difficulty.

It will not be the same if the given ranks were in part horizontal and in part vertical; however, as this case can take place in many of the questions, we are going to give the method to resolve it.

31. We suppose therefore that one knows the first m horizontal ranks of the Table of n° 6 and the first $n-m$ vertical ranks of the same Table, that is to say that one knows the values of $y_{x,0},\ y_{x,1},\ y_{x,2}, \dots, y_{x,m}$ at the same time as $y_{0,t},\ y_{1,t},\ y_{2,t}, \dots, y_{n-m,t}$ and that one demands the value of any term whatever $y_{x,t}$.

Having made $y_{x,t} = a\alpha^x\beta^t$, one will have (25) equation (I) between α and β; I consider in this equation the term $N^{(m)}\alpha^{n-m}\beta^m$, which is given by all the other terms of the same equation, and I observe that by substituting the value of $\alpha^{n-m}\beta^m$ which comes from this term in the quantity $\alpha^x\beta^t$, and next in the terms arising from this substitution, as much as it will be possible, one will arrive necessarily to an expression of $\alpha^x\beta^t$ by the powers of α and of β, in which the highest of these powers will be the $(x+t)^{\text{th}}$, and where the two powers α^{n-m} and β^m will never be found together, since one supposes that one has made them vanish by the substitution of the value of $\alpha^{n-m}\beta^m$.

This equation of $\alpha^x\beta^t$ will be therefore of the following form

$$\begin{aligned}
\alpha^x\beta^t = {} & V + V'\alpha + V''\alpha^2 + V'''\alpha^3 + \dots + V^{(x+t)}\alpha^{x+t} \\
& + {}_{,}V\beta + {}_{,}V'\alpha\beta + {}_{,}V''\alpha^2\beta + {}_{,}V'''\alpha^3\beta + \dots + {}_{,}V^{(x+t-1)}\alpha^{x+t-1}\beta \\
& + {}_{,,}V\beta^2 + {}_{,,}V'\alpha\beta^2 + {}_{,,}V''\alpha^2\beta^2 + {}_{,,}V'''\alpha^3\beta^2 + \dots + {}_{,,}V^{(x+t-2)}\alpha^{x+t-2}\beta^2 \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& + {}_{(m)}V\beta^m + {}_{(m)}V'\alpha\beta^m + {}_{(m)}V''\alpha^2\beta^m + \dots + {}_{(m)}V^{(n-m\text{-}1)}\alpha^{n-m-1}\beta^m \\
& + {}_{(m+1)}V\beta^{m+1} + {}_{(m+1)}V'\alpha\beta^{m+1} + {}_{(m+1)}V''\alpha^2\beta^{m+1} + \dots + {}_{(m+1)}V^{(n-m-1)}\alpha^{n-m-1}\beta^{m+1} \\
& + {}_{(m+2)}V\beta^{m+2} + {}_{(m+2)}V'\alpha\beta^{m+2} + {}_{(m+2)}V''\alpha^2\beta^{m+2} + \dots + {}_{(m+2)}V^{(n-m-1)}\alpha^{n-m-1}\beta^{m+2} \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& + {}_{(x+t)}V\beta^{x+t} + {}_{(x+t)}V'\alpha\beta^{x+t} + {}_{(x+t)}V''\alpha^2\beta^{x+t} + \dots + {}_{(x+t)}V^{(n-m-1)}\alpha^{n-m-1}\beta^{x+t},
\end{aligned}$$

where the coefficients $V,\ V', \dots,\ {}_{,}V,\ {}_{,}V', \dots$ will be some known functions of x and t, and some coefficients of equation (I).

32. I remark now the values of the powers and of the products of α and of β which comprise the preceding expression of $\alpha^x\beta^t$ are necessarily different and irreducibles among them, since equation (I), whence the relation between α and β depends, contains moreover the product $\alpha^{n-m}\beta^m$, which is not found at all in this expression. From this consideration and the principles posed higher, it is easy to conclude immediately the general expression of $y_{x,t}$ by doing in it only to substitute into that of $\alpha^x\beta^t$, in the place of each product such as $\alpha^r\beta^s$, any function whatever of r and s, that one will be

able to designate by $f(r,s)$; one will have thus

$$
\begin{aligned}
y_{x,t} = {} & Vf(0,0) + V'f(1,0) + V''f(2,0) + V'''f(3,0) + \ldots + V^{(x+t)}f(x+t,0) \\
& + {}_{,}Vf(0,1) + {}_{,}V'f(1,1) + {}_{,}V''f(2,1) + {}_{,}V'''f(3,1) + \ldots + {}_{,}V^{(x+t-1)}f(x+t-1,1) \\
& + {}_{,,}Vf(0,2) + {}_{,,}V'f(1,2) + {}_{,,}V''f(2,2) + {}_{,,}V'''f(3,2) + \ldots + {}_{,,}V^{(x+t-2)}f(x+t-2,2) \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + {}_{(m)}Vf(0,m) + {}_{(m)}V'f(1,m) + {}_{(m)}V''f(2,m) + \ldots + {}_{(m)}V^{(n-m\text{-}1)}f(n-m-1,m) \\
& + {}_{(m+1)}Vf(0,m+1) + {}_{(m+1)}V'f(1,m+1) + {}_{(m+1)}V''f(2,m+1) + \ldots + {}_{(m+1)}V^{(n-m-1)}f(n-m-1,m+1) \\
& + {}_{(m+2)}Vf(0,m+2) + {}_{(m+2)}V'f(1,m+2) + {}_{(m+2)}V''f(2,m+2) + \ldots + {}_{(m+2)}V^{(n-m-1)}f(n-m-1,m+2) \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + {}_{(x+t)}Vf(0,x+t) + {}_{(x+t)}V'f(1,x+t) + {}_{(x+t)}V''f(2,x+t) + \ldots + {}_{(x+t)}V^{(n-m-1)}f(n-m-1,x+t).
\end{aligned}
$$

33. In order to determine the values of the function f, one will suppose successively $t = 0, 2, \ldots, m-1$, and next $x = 0, 1, 2, \ldots, n-m-1$, since by the hypothesis the corresponding values of $y_{x,t}$ are givens.

Now, by making $t = 0$, the quantity $\alpha^x\beta^t$ becomes α^x; therefore in the formula of n° 31, one will have then

$$V^{(x)} = 1, \quad \text{and the other coefficients nulls;}$$

by making $t = 1$, one has $\alpha^x\beta$; therefore

$${}_{,}V^{(x)} = 1, \quad \text{and the other coefficients nulls;}$$

by making $t = 2$, one has $\alpha^x\beta^2$; therefore

$${}_{,,}V^{(x)} = 1, \quad \text{and the other coefficients nulls;}$$

and thus in sequence until

$${}_{(m-1)}V^{(x)} = 1,$$

when $t = m-1$.

Likewise, by making $x = 0$, $\alpha^x\beta^t$ becomes β^t; therefore one will have, in the same formula,

$${}_{(t)}V = 1, \quad \text{and the other coefficients nulls;}$$

by making $x = 1$, one will have $\alpha\beta^t$; therefore

$${}_{(t)}V' = 1, \quad \text{and the other coefficients nulls;}$$

one will have similarly, when $x = 2$,

$${}_{(t)}V'' = 1, \quad \text{and the other coefficients nulls;}$$

and thus in sequence until

$${}_{(t)}V^{n-m-1} = 1,$$

when $x = n - m - 1$.

If one makes therefore in the expression of $y_{x,t}$ of the preceding number successively $t = 0,\ 1,\ 2,\ \ldots,\ m-1$, one will have

$$y_{x,0} = f(x,0),\ y_{x,1} = f(x,1),\ y_{x,2} = f(x,2),\ \ldots,\ y_{x,m-1} = f(x,m-1),$$

whatever be x. And if one makes successively $x = 0,\ 1,\ 2,\ \ldots,\ n-,-1$, one will have

$$y_{0,t} = f(0,t),\ y_{1,t} = f(1,t),\ y_{2,t} = f(2,t),\ \ldots,\ y_{n-m-1,t} = f(n-m-1,t),$$

whatever be t. One will know therefore in this manner the values of the functions which enter into the expression of which there is question, and substituting these values, one will have the following formula, which contains only some known quantities.

$$\begin{aligned}
&y_{x,t} = Vy_{0,0} + V'y_{1,0} + V_{2,0} + V'''y_{3,0} + \ldots + V^{(x+1)}y_{x+t,0}\\
&+ {}_{,}Vy_{0,1} + {}_{,}V'y_{1,1} + {}_{,}V''y_{2,1} + {}_{,}V'''y_{3,1} + \ldots + {}_{,}V^{(x+t-1)}y_{x+t,1}\\
&+ {}_{,,}Vy_{0,2} + {}_{,,}V'y_{1,2} + {}_{,,}V''y_{2,2} + {}_{,,}V'''y_{3,2} + \ldots + {}_{,,}V^{(x+t-2)}y_{x+t-2,2}\\
&+ {}_{,,,}Vy_{0,3} + {}_{,,,}V'y_{1,3} + {}_{,,,}V''y_{2,3} + {}_{,,,}V'''y_{3,3,} + \ldots + {}_{,,,}V^{(x+t-3)}y_{x+t-3,3}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+ {}_{(m)}Vy_{0,m} + {}_{(m)}V'y_{1,m} + {}_{(m)}V''y_{2,m} + {}_{(m)}V'''y_{3,m} + \ldots + {}_{(m)}V^{(n-m-1)}y_{n-m-1,m}\\
&+ {}_{(m+1)}Vy_{0,m+1} + {}_{(m+1)}V'y_{1,m+1} + {}_{(m+1)}V''y_{2,m+1} + {}_{(m+1)}V'''y_{3,m+1} + \ldots + {}_{(m+1)}V^{(n-m-1)}y_{n-m-1,m+1}\\
&+ {}_{(m+2)}Vy_{0,m+2} + {}_{(m+2)}V'y_{1,m+2} + {}_{(m+2)}V''y_{2,m+2} + {}_{(m+2)}V'''y_{3,m+2} + \ldots + {}_{(m+2)}V^{(n-m-1)}y_{n-m-1,m+2}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+ {}_{(x+t)}Vy_{0,x+t} + {}_{(x+t)}V'y_{1,x+t} + {}_{(x+t)}V''y_{2,x+t} + {}_{(x+t)}V'''y_{3,x+t} + \ldots + {}_{(x+t)}V^{(n-m-1)}y_{n-m-1,x+t}
\end{aligned}$$

34. As for the manner to determine the coefficients $V,\ V',\ V'',\ldots,\ {}_{,}V,\ {}_{,}V',\ {}_{,}V'',\ \ldots,$ one will be able to employ some methods analogous to those that we have proposed above (28).

Indeed, if one seeks the value of β in α or of α in β by equation (I), and if one substitutes it into the formula of n° 31, one will have, by the comparison of the terms affected with the same powers of α and of β, a sequence of equations by which one will be able to determine the coefficients of which there is question. One will be able also to employ the differential Calculus in order to find the law of these coefficients: because by differentiating logarithmically the equation

$$\alpha^x\beta^t = V + \ldots$$

of n° 31, substituting next in the place of $\frac{d\beta}{d\alpha}$ its value drawn from equation (I), and making vanish, by means of this equation, the terms where will be found $\alpha^{n-m}\beta^m$, in the same way one has taught it in n° 29, one will have an equation of which each term must next be supposed equal to zero; this which will give a sequence of equations which will contain the relation which must rule among the coefficients of which there is question.

Further, as all that is no more than an affair of analysis, we will not occupy ourselves with it further, contenting ourselves for the present to have reduced the integration of the linear equations in the finite and partial differences to a known theory, which demands of other help only those which the ordinary methods can furnish.

REMARK I.

35. I am going to terminate this Article with some important Remarks. The first, that one will be able always to find as many different expressions of $y_{x,t}$ as there will be terms in the last column of equation (I), which correspond to the last column, or to the highest rank of the proposed differential equation of n° 6. Indeed, to each of the terms to which of which there is question such as $N^{(m)}\alpha^{n-m}\beta^m$, which comes from the term $N^{(m)}y_{x+n-m,t+m}$ of the differential equation, will correspond, as one has seen, an expression for $y_{x,t}$ in which the terms given from the Table of n° 6 will be those which form the first m horizontal ranks, and the first $n-m$ vertical ranks; and it is easy to be convinced, with a little reflection, that one would know how to find one such expression only by means of a similar term; so that, if the term of this form lacked in the differential equation, it would be then impossible to be able to express, in general, the value of $y_{x,t}$ by means of the first m horizontal ranks and of the first $n-m$ vertical ranks of the Table of n° 6. For example, in the case of the differential equation (F) of n° 7, where one has only a single term of the highest order, so that n being equal to 2, m has only a single value equal to 1, the general expression of $y_{x,t}$ demands necessarily that one knows the first horizontal rank and the first vertical rank of the Table cited, and it is also that which we have supposed in the solution of n° 13.

REMARK II.

36. The second Remark concerns the case where equation (I) has some rational factors, so that it can be decomposed into as many particular equations. In this case, one can simplify the general method by considering in part each of these equations and seeking the expression of $y_{x,t}$ which results from each of them; because the sum of these different expressions of $y_{x,t}$ will be the complete expression of $y_{x,t}$ which agrees with the differential equation proposed. Indeed, we suppose that equation (I) of degree n can be decomposed into two rational equations of degrees p and q, so that $p+q=n$; it is easy to prove that, if one makes

$$y_{x,t} = y'_{x,t} + y''_{x,t},$$

the differential equation in $y_{x,t}$ of order n will be able also to be decomposed into two equations, the one in $y'_{x,t}$ of order p, the other in $y''_{x,t}$ of order q; and these two equations will be such that, if one sets into the first $\alpha^x\beta^t$ in the place of $y'_{x,t}$ and in the second place $\alpha^x\beta^t$ in the place of $y''_{x,t}$, there will result from it the two equations of degrees p and q which are the factors of equation (I) resulting from the substitution of $\alpha^x\beta^t$ in the place of $y_{x,t}$ in the proposed differential equation. And this conclusion will take place for all the factors of the same equation (I).

In regard to the arbitrary functions, it is clear that the expression of $y'_{x,t}$ will contain in it a number p, and that the expression of $y''_{x,t}$ will contain in it a number q; so that the expression of $y_{x,t}$ will contain in it a number equal to $p+q$, that is to say equal to n; consequently this expression will be complete.

In order to determine now these functions according to the values given by $y_{x,0}$, $y_{x,1}, \ldots, y_{0,t}, y_{1,t}, \ldots$ (33), one will suppose first that the given quantities are

$$y'_{x,0},\ y'_{x,1}, \ldots,\ y'_{0,t},\ y'_{1,t}, \ldots, \quad \text{at the same time as } y''_{x,0},\ y''_{x,1}, \ldots,\ y''_{0,t},\ y''_{1,t}, \ldots;$$

one will determine, by aid of the first ones, the arbitrary functions of the expression of $y'_{x,t}$, and, by aid of the second ones, the arbitrary functions of the expression of $y''_{x,t}$, by the general method of the number cited; next there will be no more than to substitute in the place of these quantities their values in $y_{x,0},\, y_{x,1},\, \ldots,\, y_{0,t},\, y_{1,t}, \ldots$.

For this one will remark that, since one has a differential equation in $y'_{x,t}$, and one in $y''_{x,t}$, and since moreover

$$y_{x,t} = y'_{x,t} + y''_{x,t},$$

one can always, by elimination, find the value of $y'_{x,t}$, at the same time as that of $y''_{x,t}$, in $y_{x,t}$ and its differences; thus one will know thence the values of the quantities of which there is question through those of $y_{x,0},\, y_{x,1},\, \ldots$

If equation (I) had many rational factors, one would make relatively to all these factors some reasonings analogous to the preceding, and one will draw from it some similar conclusions.

REMARK III.

37. The third Remark has for object the case where equation (I) has some equal factors; in this case one knows by the theory of equations that these factors will be necessarily rationals; so that, according to the method of the previous number, one will be able to consider these factors equal apart and independently of the others; thus the difficulty is reduced to the case where the equation in α and β will be any power whatsoever of another equation. We designate this last equation by

$$\Pi = 0,$$

and let the proposed equation in α and β be,

$$\Pi^m = 0;$$

I say that if one seeks the general expression of $y_{x,t}$ according to the equation $\Pi = 0$ by the methods explicated above, and if one names this value $y'_{x,t}$, if next one designates by $y''_{x,t}$, $y'''_{x,t}$, ... some other similar expressions, in which the arbitrary functions are different, one will have for the general expression of $y_{x,t}$ resulting from the equation $\Pi^m = 0$,

$$y_{x,t} = y'_{x,t} + y''_{x-1,t}, \quad \text{or} \quad y_{x,t} = y'_{x,t} + y''_{x,t-1},$$

if $m = 2$;

$$y_{x,t} = y'_{x,t} + xy''_{x-1,t} + x(x-1)y''_{x-2,t},$$

or

$$y_{x,t} = y'_{x,t} + xy''_{x-1,t} + xy'''_{x-1,t-1},$$

or

$$y_{x,t} = y'_{x,t} + ty''_{x,t-1} + txy'''_{x-1,t-1},$$

or finally

$$y_{x,t} = y'_{x,t} + ty''_{x,t-1} + t(t-1)y'''_{x,t-2},$$

if $m = 3$; and thus in sequence; these different expressions of $y_{x,t}$ revert always to the same.

Indeed, if one seeks the value of $\alpha^x\beta^t$ according to the equation $\Pi = 0$, one will have for the equation $\Pi^2 = 0$ the same value of $\alpha^x\beta^t$ and moreover this here

$$x\alpha^{x-1}\beta^t d\alpha \quad \text{or} \quad t\alpha^x\beta^{t-1}\, d\beta,$$

$d\alpha$ and $d\beta$ being some indeterminate quantities; and for the equation $\Pi^3 = 0$, one will have, beyond the value of $\alpha^x\beta^t$ which corresponds to $\Pi = 0$, these two others here

$$x\alpha^{x-1}\beta^t d\alpha, \quad x(x-1)\alpha^{x-2}\, d\alpha^2,$$

or else these two here

$$x\alpha^{x-1}\beta^t d\alpha, \quad xt\alpha^{x-1}\beta^{t-1}\, d\alpha d\beta,$$

or

$$t\alpha^x\beta^{t-1} d\beta, \quad tx\alpha^{x-1}\beta^{t-1}\, d\alpha d\beta,$$

or else again

$$t\alpha^x\beta^{t-1} d\beta, \quad t(t-1)\alpha^x\beta^{t-2}\, d\beta^2,$$

and thus in sequence; being indifferent to make α or β vary at each new differentiation. Thence and from this which we have already said in n [os] 16 and 36 it is easy to deduce the preceding formulas for the general expression of $y_{x,t}$ and to continue them farther for all the exponents m.

As for the determination of the arbitrary functions, it has no difficulty; because there will be only to determine first those which enter into the expressions of $y'_{x,t}$, of $y''_{x,t}, \ldots$ by $y'_{x,0},\, y'_{x,1},\, \ldots,\, y'_{0,t},\, y'_{1,t},\, \ldots$, by the values of $y''_{x,0},\, y''_{x,1},\, \ldots,\, y''_{0,t},\, y''_{1,t},\, \ldots$; next one will determine these last by those of $y_{x,0},\, y_{x,1},\, \ldots,\, y_{0,t},\, y_{1,t},\, \ldots$ according to the formulas

$$y_{x,t} = y'_{x,t} + xy''_{x-1,t} + \ldots$$

given above, combined with the differential equation which corresponds to the equation $\Pi = 0$, and which is the same for all the quantities $y'_{x,t},\, y''_{x,t}, \ldots$ since they differ among themselves only by the arbitrary functions.

REMARK IV.

38. The fourth Remark will take in some transformations that one can use in order to facilitate the integration of the equations in the finite and partial differences. If in the equations in α and β resulting from the substitution of $\alpha^x\beta^t$ in the place of $y_{x,t}$ in the proposed differential equation, one makes

$$\alpha = a\epsilon^m\gamma^n + a'\epsilon^{m'}\gamma^{n'} + a''\epsilon^{m''}\gamma^{n''} + \ldots,$$
$$\beta = b\epsilon^p\gamma^q + b'\epsilon^{p'}\gamma^{q'} + b''\epsilon^{p''}\gamma^{q''} + \ldots,$$

$a,\, a',\, a'',\, \ldots,\, b,\, b',\, b'',\, \ldots,\, m,\, m',\, m'',\, \ldots,\, n,\, n',\, n'',\, \ldots,\, p,\, p',\, p'',\ldots,\, q,\, q',\, q'',\, \ldots$ being some given constants any whatsoever, and ϵ, γ two new indeterminates, one will have one transformed from ϵ, γ which will be able in many cases to be simpler and more tractable than the first equation in α, β.

Now I say that if one regards this equation in ϵ, γ as resulting immediately from an equation in finite and partial differences among the three variables x, t and $z_{x,t}$, by the substitution of $\epsilon^x\gamma^t$ in the place of $z_{x,t}$, and if one deduces from it by the methods above the general expression of $z_{x,t}$, it will be easy to conclude from it the general expression of $y_{x,t}$ in the following manner. One will substitute for that the same values of α and β in the quantity $\alpha^x\beta^t$, and developing the terms one will have an expression of this form

$$\alpha^x\beta^t = A\epsilon^{mx+pt}\gamma^{nx+qt} + B\epsilon^{mx+pt+\mu}\gamma^{nx+qt+\nu} + C\epsilon^{mx+pt+\pi}\gamma^{nx+qt+\rho} + \dots$$

Now $\epsilon^x\gamma^t$ is a particular value of $z_{x,t}$, likewise as $\alpha^x\beta^t$ is a particular value of $y_{x,t}$; thus passing from the particular values to the general expressions, one will have immediately

$$y_{x,t} = Az_{mx+pt,\,nx+qt} + Bz_{mx+pt+\mu,nx+qt+\nu} + Cz_{mx+pt+\pi,nx+qt+\rho} + \dots$$

One would be able to transform anew the equation in ϵ and γ, and one would find in the same manner the corresponding value of $z_{x,t}$.

We suppose, for example,

$$\alpha = a + pt, \quad \beta = b + q\gamma,$$

one will have

$$\alpha^x\beta^t = A + B\epsilon + C\gamma + D\epsilon^2 + E\epsilon\gamma + F\gamma^2 + \dots$$

by making

$$\begin{aligned} &A = a^x b^t, \\ &B = xa^{t-1}p \times b^t, && C = tb^{t-1}q \times a^x, \\ &D = \frac{x(x-1)}{2}a^{x-2}p^2 \times b^t, && E = xa^{x-1}p \times tb^{t-1}q, \dots, \\ &\dots\dots\dots\dots\dots\dots\dots\dots, \end{aligned}$$

and thence

$$y_{x,t} = Az_{0,0} + Bz_{1,0} + Cz_{0,1} + Dz_{2,0} + Ez_{1,1} + \dots$$

If one would wish to make successively the two substitutions, one would have first

$$y_{x,t} = a^x y'_{0,t} + xa^{x-1}py'_{1,t} + \frac{x(x-1)}{2}a^{x-1}p^2y'_{2,t} + \dots,$$

and next

$$y'_{x,t} = b^t z_{x,0} + tb^{t-1}qz_{x,1} + \frac{t(t-1)}{2}b^{t-2}q^2z_{x,2} + \dots$$

Reciprocally one will be able to determine the values of $z_{x,t}$ by those of $y_{x,t}$ by substituting into $\epsilon^x\gamma^t$ the values of ϵ and γ in α and β and changing next each product from α and β such as $\alpha^r\beta^s$ into $y_{r,s}$.

REMARK V.

39. The fifth Remark is that it can happen in the solution of the Problems that the terms given in the Table of n° 6 are not those which form the first horizontal or vertical rank of this Table, as we have always supposed until here, but of some others whatever. Then among the different forms that one can give to the general expression of $y_{x,t}$ it will be necessary to choose that which will render the determination of the arbitrary functions by the given terms, the easiest; but one would not know how to give the general rules for this, and it is necessary to abandon this research to the sagacity of the Analyst.

In general, it will be necessary always that there be as many indefinite lines of terms given in the Table of n° 6, as there are units in the exponent of the order of the differential equation; but it is not necessary that these lines be horizontal or vertical; they can equally be inclined in any manner whatever, and even they can be curved or rather composed of an assembly of straight lines differently inclined. We will see some Examples of it in Article V.

REMARK VI.

40. My last Remark concerns the case where one has to integrate many linear equations which contain as many different unknowns such as $y_{x,t}$, $z_{x,t}$, $u_{x,t}, \ldots$; it is easy to convince oneself that one can always by elimination arrive to a single final equation which contains only a single unknown $y_{x,t}$; but it will be often very painful to start in this way, and one will arrive to the end in a much simpler manner by applying immediately our methods to the proposed equations. For this one will make first

$$y_{x,t} = a\alpha^x\beta^t, \quad z_{z,t} = b\alpha^x\beta^t, \quad u_{x,t} = c\alpha^x\beta^t, \ldots,$$

this which will give, after having divided each equation by $\alpha^x\beta^t$, as many equations in α, β and in a, b, c,... as there are of these last quantities, and where these quantities will be all linear; so that if one eliminates the quantities $\frac{b}{a}$, $\frac{c}{a}$, ..., one will arrive to a final equation in α and β which will contain the relation which it must have between these two indeterminates, and which will be the same as one had found by the substitution of $a\alpha^x\beta^t$ in the place of $y_{x,t}$ in the equation in $y_{x,t}$ resulting from the elimination of the other unknowns $z_{x,t}$, $u_{x,t}, \ldots$ One will be able therefore to find according to this equation and by means of the methods exposed until here, the general expression of $y_{x,t}$; we will denote this expression by $y'_{x,t}$. Now as the equations in a, b, $c, \ldots$ are purely linear, one will be able to find the values of $\frac{b}{a} \cdot \frac{c}{a}, \ldots$ through some rational functions of α and β; let be therefore

$$\frac{b}{a} = \frac{A}{B}, \quad \frac{c}{a} = \frac{C}{A}, \ldots,$$

A, B, $C, \ldots$ being some rational entire functions of α and β; as a is a constant which remains arbitrary, one will be able to set everywhere aA in the place of a; by this means the quantities $a\alpha^x\beta^t, b\alpha^x\beta^t, c\alpha^x\beta^t, \ldots$ which are the particular values of $y_{x,t}$, $z_{x,t}$,

$u_{x,t}, \ldots$ will become $aA\alpha^x\beta^t$, $bB\alpha^x\beta^t$, $cC\alpha^x\beta^t, \ldots$ Now the quantities A, B, $C \ldots$ are necessarily of this form

$$\begin{aligned}
A &= P\alpha^m\beta^n + P'\alpha^{m'}\beta^{n'} + \ldots,\\
B &= Q\alpha^p\beta^q + Q'\alpha^{p'}\beta^{q'} + \ldots,\\
C &= R\alpha^r\beta^s + R'\alpha^{r'}\beta^{s'} + \ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;
\end{aligned}$$

P, Q, R, $P', \ldots, m, n, m', \ldots$ being some given constants; therefore the particular values of which the is question will become of the form

$$\begin{aligned}
&Pa\alpha^{x+m}\beta^{t+n} + P'a\alpha^{x+m'}\beta^{t+n'} + \ldots,\\
&Qa\alpha^{x+p}\beta^{t+q} + Q'a\alpha^{x+p'}\beta^{t+q'} + \ldots,\\
&Ra\alpha^{x+r}\beta^{t+s} + R'a\alpha^{x+r'}\beta^{t+s'} + \ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;
\end{aligned}$$

but, by hypothesis, $a\alpha^x\beta^t$ is the particular value of $y'_{x,t}$; therefore passing from the particular values to the general expressions, one will have also, in general

$$\begin{aligned}
y_{x,t} &= Py'_{x+m,t+n} + P'y'_{x+m',t+n'} + \ldots,\\
z_{x,t} &= Qy'_{x+p,t+q} + Q'y'_{x+p',t+q'} + \ldots,\\
u_{x,t} &= Ry'_{x+r,t+s} + R'y'_{x+r',t+s'} + \ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;
\end{aligned}$$

and there will be no more than to substitute in the place of $y'_{x,t}$ its general expression found previously.

ARTICLE IV. — *On the triply recurrent sequences, or on the integration of the linear equation in finite and partial differences among four variables.*

41. If one imagines a sequence of which the terms vary in three different ways, and if one supposes that there is always among a certain number of successive terms of this sequence one same linear equation, of which the coefficients are constants, this will be thence a triply recurrent sequence; and the equation of which there is question will be a linear equation in the finite and partial differences among four variables, of which the integration will be the object of this Article.

In imitation of that which we have practiced with regard to the doubly recurrent sequences, we will designate any term whatsoever of a triply recurrent sequence by $y_{x,t,u}$, so that by making successively

$$x = 0, 1, 2, 3, \ldots, \quad t = 0, 1, 2, 3, \ldots, \quad u = 0, 1, 2, 3, \ldots,$$

one will have all the terms which will be able to enter into this sequence; whence one sees that these terms will be able to form a Table in triply entry in the form of a parallelepiped, just as the terms $y_{x,t}$ of the doubly recurrent sequences form a Table in double entry in the form of a rectangle (6).

42. This put, let the equation of the third order be

$$\left\{\begin{array}{l} Ay_{x,t,u}+By_{x+1,t,u}+Cy_{x+1,t+1,u}+Dy_{x+1,t+1,u+1} \\ \qquad +B'y_{x,t+1,u}+C'y_{x+1,t,u+1} \\ \qquad +B''y_{x,t,u+1}+C''y_{x,t+1,u+1} \end{array}\right\} = 0, \tag{L}$$

which is, as one sees, of a form similar to that of equation (F) of n° 7.

In order to integrate this equation I suppose

$$y_{x,t,u} = a\alpha^x\beta^t\gamma^u,$$

a, α, β, γ being some constants; substituting this value and dividing next the whole equation by $a\alpha^x\beta^t\gamma^u$, I have this here

$$A + B\alpha + B'\beta + B''\gamma + C\alpha\beta + C'\alpha\gamma + C''\beta\gamma + D\alpha\beta\gamma = 0. \tag{M}$$

whence I draw the value of γ in α and β, namely

$$\gamma = -\frac{A + B\alpha + B'\beta + C\alpha\beta}{B'' + C'\alpha + C''\beta + D\alpha\beta},$$

or else, by dividing the top and the bottom of this fraction by $\alpha\beta$,

$$\gamma = -\frac{C + \frac{B'}{\alpha} + \frac{B}{\beta} + \frac{A}{\alpha\beta}}{D + \frac{C''}{\alpha} + \frac{C'}{\beta} + \frac{B''}{\alpha\beta}}.$$

I raise now this quantity to the power u, and developing the terms according to the different powers of $\frac{1}{\alpha}$ and of $\frac{1}{\beta}$, I will have

$$\begin{aligned}\gamma^u = V &+ V'\frac{1}{\alpha} + V''\frac{1}{\alpha^2} + V'''\frac{1}{\alpha^3} + \dots \\ &+ {}_{\prime}V\frac{1}{\beta} + + {}_{\prime}V'\frac{1}{\alpha\beta} + {}_{\prime}V''\frac{1}{\alpha^2\beta} + \dots \\ &+ {}_{\prime\prime}V\frac{1}{\beta^2} + {}_{\prime\prime}V'\frac{1}{\alpha\beta^2} + \dots \\ &+ {}_{\prime\prime\prime}V\frac{1}{\beta^3} + \dots,\end{aligned}$$

where the coefficients V, V', $\dots$, ${}_{\prime}V$, ${}_{\prime}V'$, ... will be some known functions of u and of the constants A, B, B', $\dots$.

Multiplying therefore this expression of γ^u in series by $a\alpha^x\beta^t$, one will have a particular value of $y_{x,t,u}$; and because a, α and β are indeterminates and because the equation is linear, one will be able to take also for $y_{x,t,u}$ the sum of as many similar expressions as one will wish by changing at will the values of a, α and β. Thence it is easy to conclude, by a reasoning analogous to the one of n° 7, that one will have the general expression of $y_{x,t,u}$ by setting in that of $\alpha^x\beta^t\gamma^u$ in the place of each product of α and β such as $\alpha^r\beta^s$ any function whatsoever of r and s, that one will be able to designate by $f(r,s)$. Thus therefore, one will have immediately

$$\begin{aligned} y_{x,t,u} =& Vf(x,t) + V'f(x-1,t) + V''f(x-2,t) + V'''f(x-3,t) + \dots \\ &+ {}_{,}Vf(x,t-1) + {}_{,}V'f(x-1,t-1) + {}_{,}V''f(x-2,t-1) + \dots \\ &+ {}_{,,}Vf(x,t-2) + {}_{,,}V'f(x-1,t-2) + \dots \\ &+ {}_{,,,}Vf(x,t-3) + \dots \\ &\dots\dots\dots\dots\dots\dots \end{aligned}$$

43. In order to determine now the values of the function $f(x,t)$, I suppose that one knows all the values of $y_{x,t,u}$ when $u=0$; now making $u=0$, it is clear that one has $\gamma^u = 1$; therefore $V=1$, and all the other coefficients are null; therefore the preceding formula will give, when $u=0$,

$$y_{x,t,u} = f(x,t).$$

Therefore, if one makes this substitution, one will have

$$\begin{aligned} y_{x,t,u} =& Vy_{x,t,0} + V'y_{x-1,t,0} + V''y_{x-2,t,0} + V'''y_{x-3,t,0} + \dots \\ &+ {}_{,}Vy_{x,t-1,0} + {}_{,}V'y_{x-1,t-1,0} + {}_{,}V''y_{x-2,t-1,0} + \dots \\ &+ {}_{,,}Vy_{x,t-2,0} + {}_{,,}V'y_{x-1,t-2,0} + \dots \\ &+ {}_{,,,}Vy_{x,t-3,0} + \dots \\ &\dots\dots\dots\dots\dots\dots \end{aligned}$$

This solution is, as one sees, completely analogous to that of n° 8; also it is subject to the same inconvenience, which is to give for $y_{x,t,u}$ an expression composed of an infinite number of terms, unless three of the four quantities B'', C', C'', D vanish at once, in which case the value of γ^u will be finite, u being (hypothesis) a integral positive number.

However the preceding solution will be able to be used in each case where the given terms $y_{x,t,0}$ are null for all the negative values of x and of t; because then it is clear that the expression above of $y_{x,t,u}$ will always be terminated; and this is that which can take place in a great number of questions.

44. But one can, by means similar to the one of n° 12, obtain a finite expression of $y_{x,t,u}$ in all cases. Indeed, if in the value of γ of n° 42 one makes

$$B'' + C'\alpha + C''\beta + D\alpha\beta = \omega,$$

this which gives

$$\beta = \frac{\omega - B'' - C'\alpha}{C'' + D\alpha},$$

and if next one makes in this value of β

$$C'' + D\alpha = \eta, \quad \text{whence} \quad \alpha = \frac{\eta - C''}{D},$$

one will have, by substituting successively these values

$$\alpha = \frac{\eta - C''}{D},$$
$$\beta = \frac{C'C'' - DB'' - C'\eta + D\omega}{D\eta},$$
$$\gamma = \frac{D(BC' - DA - B\eta) + (C'C'' - DB' - C\eta)(C'C'' - DB'' - C'\eta + D\omega)}{D^2\eta\omega},$$

expressions which have the advantage of being under a finite form and of not containing at all a complex fraction; so that if one multiplied together these quantities raised respectively to the powers $x,\ t,\ u$, and if one developed the terms according to the powers and the products of η and of ω, one will have for $\alpha^x\beta^t\gamma^u$ a finite expression, as long as $x,\ t,\ u$ will be entire positives.

We suppose therefore

$$\begin{aligned}\left(\frac{\eta - C''}{D}\right)^x &\times \left(\frac{C'C'' - DB'' - C'\eta + D\omega}{D}\right)^t\\ &\times \left(\frac{BC' - DA - B\eta}{D} + \frac{(C'C'' - DB' - C\eta)(C'C'' - DB'' - C'\eta + D\omega)}{D^2}\right)^u\\ &= Z + Z'\eta + Z''\eta^2 + Z'''\eta^3 + \ldots + Z^{(x+t+2u)}\eta^{x+t+2u}\\ &+ {}_{,}Z\omega + {}_{,}Z'\eta\omega + {}_{,}Z''\eta^2\omega + \ldots\\ &+ {}_{,,}Z\omega^2 + {}_{,,}Z'\eta\omega^2 + \ldots\\ &+ {}_{,,,}Z\omega^3 + \ldots\\ &\ldots\ldots\ldots\ldots\\ &+ {}_{(t+u)}Z\omega^{t+u} + \ldots + {}_{(t+u)}Z^{(x+u)}\eta^{x+u}\omega^{t+u}\end{aligned}$$

where the coefficients $Z,\ Z',\ \ldots,\ {}_{,}Z,\ \ldots$ are some known functions of $x,\ t,\ u$ and of the constants $A,\ B,\ B',\ \ldots$; there will be no more than to multiply this series by $\eta^{-t-u}\omega^{-u}$ in order to have the value of $\alpha^x\beta^t\gamma^u$ in η and ω; and as η and ω are indeterminates, one will be able to draw from it immediately the complete value of $y_{x,t,u}$ by doing only to change each product such as $\eta^{-r}\omega^{-s}$ into $f(r,s)$; in such manner, one will have therefore

$$\begin{aligned}y_{x,t,u} =& Zf(t+u,u) + Z'f(t+u-1,u) + Z''f(t+u-2,u) + Z'''f(t+u-3,u) + \ldots\\ &+ {}_{,}Zf(t+u,u-1) + {}_{,}Z'f(t+u-1,u-1) + {}_{,}Z''f(t+u-2,u-1) + \ldots\\ &+ {}_{,,}Zf(t+u,u-2) + {}_{,,}Z'f(t+u-1,u-2) + \ldots\\ &+ {}_{,,,}Zf(t+u,u-3) + \ldots\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

45. In order to determine now the values of the function $f(r,s)$, one will do as above $u = 0$, and one will suppose next successively $x = 0,\ 1,\ 2, 3,\ \ldots,\ t = 0,\ 1,\ 2,\ 3,\ \ldots$; one will have, by this means, a sequence of equations, whence one will draw the different values of the function of which there is question in $y_{0,0,0},\ y_{1,0,0},\ y_{0,1,0}, \ldots$; but it

will be difficult to arrive by this means to some formulas simple enough, such as those which we have found for the case of three variables alone (13).

46. One can also apply to the equations which are the object of this Article the general method of the preceding Article, and to draw from it some similar conclusions.

Indeed, it is first evident that if $\alpha^l\beta^m\gamma^n$ is one of the terms of the highest dimension of the equation in α, β, γ resulting from the substitution of $\alpha^x\beta^t\gamma^u$ in the place of $y_{x,t,u}$ in the proposed differential equation, it is evident, I say, that by substituting successively, as much as it is possible, the value of this term into the quantity $\alpha^x\beta^t\gamma^u$, one will be able to reduce it to a finite sequence of powers of α, β, γ, among which there will never be found the product $\alpha^l\beta^m\gamma^n$. Next one will be able to prove by the principles of n° 32 that there will be only to put into this expression of $\alpha^x\beta^t\gamma^u$ in the place of any product whatsoever such as $\alpha^r\beta^s\gamma^q$ any function whatsoever of the three numbers r, s, q that one will be able to designate by $f(r,s,q)$, in order to have immediately the general and complete expression of $y_{x,t,u}$. Finally one will demonstrate as in n° 33, that these functions will be respectively equal to the first terms of the proposed recurrent sequence, so that one will have, in general,

$$f(r,s,q) = y_{x,t,u};$$

it will be necessary therefore to suppose given all the terms of the form $y_{r,s,q}$ in which one will not have all at once

$$r = \text{or } > l, \quad s = \text{or} > m, \quad q = \text{or} > n;$$

and then one will have, by means of these terms, the general expression of $y_{x,t,u}$.

47. For example in the case of n° 42, the equation in α, β, γ, containing in the rank of the highest dimension the term $D\alpha\beta\gamma$, one will be able to reduce the quantity $\alpha^x\beta^t\gamma^u$ to a finite series of the form

$$\begin{array}{lllll}
P & +Q\alpha & +R\alpha^2 & +S\alpha^3 & +T\alpha^4\ldots \\
 & +Q'\beta & +R'\alpha\beta & +S'\alpha^2\beta & +T'\alpha^3\beta\ldots \\
 & +Q''\gamma & +R''\alpha\gamma & +S''\alpha^2\gamma & +T''\alpha^3\gamma\ldots \\
 & & +R'''\beta^2 & +S'''\alpha\beta^2 & \ldots\ldots\ldots \\
 & & +R^{\text{iv}}\beta\gamma & +S^{\text{iv}}\alpha\gamma^2 & \ldots\ldots\ldots \\
 & & +R^{\text{v}}\gamma^2 & +S^{\text{v}}\beta^2 & \ldots\ldots\ldots \\
 & & & +S^{\text{vi}}\beta^2\gamma & \ldots\ldots\ldots \\
 & & & +S^{\text{vii}}\beta\gamma^2 & \ldots\ldots\ldots \\
 & & & +S^{\text{viii}}\gamma^3 & \ldots\ldots\ldots \\
 & & & & \ldots\ldots\ldots
\end{array}$$

in which is found all the powers of α, β, γ, either alone, or combined among themselves two by two, but never the three quantities together. And as the term $D\alpha\beta\gamma$ is the only one of the highest dimension in the equation of which there is question, it follows that one will be able to find only this single finite expression of $\alpha^x\beta^t\gamma^u$. Consequently one will be able to have only a single complete expression of $y_{x,t,u}$, which will result from the substitution of $f(r, s, q)$ or else of $y_{r,s,q}$ in the place of $\alpha^r \beta^s \gamma^q$ in the preceding formula.

In this case therefore it will be necessary to suppose known all the terms such as $y_{r,s,q}$, one of the three numbers r, s, q being null; that is to say all the terms $y_{r,s,0}$, $y_{r,0,q}$, $y_{0,\,s,q}$, which form the three faces of the parallelepiped of the Table in triple entry of which we have spoken in n° 41.

In regard to the coefficients P, Q, Q', Q'', $R, \dots$ one will be able to employ some methods analogous to those that one has proposed in Article III; but as in the case of equation (M) one can represent the three quantities α, β, γ by some finite and rational functions of two other indeterminates, as one has found in n° 44, it will be simpler to substitute these values of α, β, γ into η and ω into the expression $\alpha^x \beta^t \gamma^u$, and next in the series

$$P + Q\alpha + Q'\beta + \dots,$$

and to determine next by the comparison of the homologous terms the values of the coefficients P, Q, Q'.

48. I will not push further these Researches on the integration of the linear equations in the partial and finite differences of which the coefficients are constants; it is easy to see through what means one will be able to apply to the equations of all orders the methods which we just exposed; I am going to show now the usage of these methods in a small number of chosen Problems concerning the theory of probabilities, this which will serve not only to cast more publicly on these methods, but yet to give to the Analysis of hazards a new degree of perfection.

ARTICLE V. — *Application of the preceding methods to the solution of different Problems in the Analysis of chances.*

PROBLEM I.

49. *A player wagers to bring forth a given event, b times at least, in a number a of trials, the probability of bringing it forth at each trial being p; we demand the lot of this player.*

We designate by $y_{x,t}$ his lot when there are more than x trials to play, and when he has yet to bring forth the event in question t times; it is clear that the sought lot will be $y_{a,b}$. Now by supposing that we play a trial, it is easy to form by the known principles of the Analysis of chances the following equation

$$y_{x,t} = py_{x-1,t-1} + (1-p)y_{x-1,t};$$

which is, as we see, linear of the second order in finite and partial differences among three variables.

Moreover, we see by the conditions of the Problem that the player wins when $t = 0$, x being any whatsoever; and that he loses when x being equal to zero, t is greater than zero; thus we will have $y_{x,0} = 1$, x being any whatsoever, and $y_{0,t} = 0$, t being > 0; so that, in this case, the given terms of the Table of no. 6 will be those which form the first horizontal rank and the first vertical.

Such are therefore the conditions of the Problem; in order to resolve it, there is no concern but to integrate conveniently the differential equation found according to the methods exposed in Article II.

For this, I put this equation under the following form, by increasing by one unit the numbers x and t,

$$py_{x,t} + (1-p)y_{x,t+1} - y_{x+1,t+1} = 0,$$

and I note that it is contained in formula (F) of no. 7 by making

$$A = p, \quad B = 0, \quad B' = 1-p, \quad C' = -1.$$

Employing therefore the solution of the same section, we will have

$$\beta = -\frac{p}{1-p-\alpha} = \frac{p}{\alpha} \times \frac{1}{1-\frac{1-p}{\alpha}};$$

therefore

$$\beta^t = p^t \left[\alpha^{-t} + t(1-p)\alpha^{-t-1} + \frac{t(t+1)}{2}(1-p)^2\alpha^{-t-2} + \cdots\right];$$

whence we deduce (8) the general expression

$$y_{x,t} = p^x \left[y_{x-t,0} + t(1-p)y_{x-t-1,0} + \frac{t(t+1)}{2}(1-p)^2 y_{x-t-2,0} + \cdots\right].$$

This expression goes to infinity; but as it is necessary, by the conditions of the Problem, that we have $y_{0,t} = 0$ when t is > 0, it is clear that it will be necessary that we have separately

$$y_{-t,0} = 0, \quad y_{-t-1,0} = 0, \quad y_{-t-2,0} = 0, \ldots,$$

whatever be t, provided that it be > 0; whence it follows that the quantities $y_{s,0}$ must always be null when s will be a negative number, which is the case of no. 9, where we have seen that the series must be finite. Next the conditions of the Problem give also $y_{x,0} = 1$, whatever be x; therefore, substituting these values into the preceding expression, we will have

$$y_{x,t} = p^t \left[1 + t(1-p) + \frac{t(t+1)}{2}(1-p)^2 + \cdots\right],$$

where it will be necessary to take only as many terms as there are units in $x - t - 1$.

Therefore finally, changing x to a and t to b, we will have for the sought lot

$$y_{a,b} = p^b \left[1 + b(1-p) + \frac{b(b+1)}{2}(1-p)^2 + \cdots + \frac{b(b+1)\cdots(a-1)}{1.2\ldots(a-b)}(1-p)^{a-b}\right],$$

This Problem is resolved in the *Doctrine of Chances* of Moivre (page 13, edition of 1756), by induction, and our solution agrees perfectly.

COROLLARY.

50. If the question were to *bring forth the given event b times neither more nor less, in a trials*, conserving the same denominations as above, we would find first the same differential equation, and consequently the same general expression of $y_{x,t}$; next we will prove also that $y_{0,t}$ must be zero when $t > 0$; which will render null all the quantities $y_{s,0}$, where s will be negative; but, in regard to $y_{x,0}$, it will be necessary to consider that this quantity expresses the lot of the player when he must again play x trials, and that he must no longer bring forth the given event; now, as the probability of not bringing forth this event at each trial is $1 - p$, that of not bringing it forth in x successive trials will be $(1 - p)^x$; thus we will have

$$y_{x,0} = (1 - p)^x,$$

and, in general,

$$y_{s,0} = (1 - p)^s,$$

s being any positive number or zero. By these substitutions, the expression $y_{x,t}$ will become

$$y_{x,t} = p^t(1 - p)^{x-t}\left[1 + t + \frac{t(t+1)}{2}(1 - p)^2 + \cdots + \frac{t(t+1)\cdots(x-1)}{1.2\ldots(x-t)}\right],$$

which can be reduced to this simpler form

$$y_{x,t} = \frac{(t+1)(t+2)\cdots x}{1.2\ldots(x-t)}p^t(1-p)^{x-t}.$$

Therefore, changing x to a, t to b, we will have for the sought lot

$$y_{a,b} = \frac{(b+1)(b+2)\cdots a}{1.2\ldots(a-b)}p^b(1-p)^{a-b}.$$

We would be able, besides, to deduce immediately the solution of this last Problem from that of the preceding section; because it is easy to understand that, if from the probability of bringing forth a given event at least b times in a trials we take off that of the bringing it forth at least $b + 1$ times in a similar number of trials, there must remain the probability of bringing forth the same event only b times in a trials; whence it follows that, if we designate by $Y_{a,b}$ the value of $y_{a,b}$ of no. 49, we will have for the case of the present Corollary

$$y_{a,b} = Y_{a,b} - Y_{a,b+1}.$$

It is in this manner that the Problem in question is resolved in the Work cited of de Moivre, page 15; but what we have just given for it is not only simpler, but it has the advantage of being deduced from immediate principles.

PROBLEM II.

51. *We suppose that at each trial there can happen two events of which the respective probabilities are p and q; and we demand the lot of a player who would wager to bring forth the first of these events at least b times and the second at least c times, in a number a of trials.*

Let, in general, $y_{x,t,u}$ be the lot of the player when he has yet x trials to play, and when he must yet bring forth the two events, one t times and the other u times; it is clear that the sought lot will be $y_{a,b,c}$.

Now if we suppose that we play one trial, and if we consider the different cases which can occur, we will form easily, according to the known principles of the theory of chances, the equation

$$y_{x,t,u} = py_{x-1,t-1,u} + qy_{x-1,t,u-1} + (1-p-q)y_{x-1,t,u};$$

which is, as we see, in finite and partial differences among four variables.

Now it is clear: 1° that the player loses when x being null, t and u have still any positive value; whence it follows that we must have, in general, $y_{0,t,u} = 0$ when t or $u > 0$; 2° that if we make $u = 0$, we have the case of the Problem preceding, so that the value of $y_{x,t,0}$ must be the same as that of $y_{x,t}$ of no. 49 above; 3° that if we make $t = 0$, we will have also the case of the same Problem by changing only p to q and t to u; consequently, the value of $y_{x,0,u}$ will be also the same as that of $y_{x,t}$ of no. 49, but by changing t to u, p to q.

This put, I set the differential equation under the following form

$$py_{x,t,u+1} + qy_{x,t+1,u} + (1-p-q)y_{x,t+1,u+1} - y_{x+1,t+1,u+1} = 0$$

and comparing it to formula (L) of no. 42, I will have, by making, for brevity, $n = 1-p-q$,

$$\gamma = \frac{p}{q} \times \frac{1}{1 - \frac{n}{\alpha} - \frac{p}{\alpha\beta}};$$

whence

$$\gamma^u = \frac{q^u}{\alpha^u}\left[1 + \frac{u}{\alpha}\left(n + \frac{p}{\beta}\right) + \frac{u(u+1)}{2\alpha^2}\left(n^2 + \frac{2np}{\beta} + \frac{p^2}{\beta^2}\right)\right]$$
$$+ \frac{u(u+1)(u+2)}{2.3.\alpha^3}\left(n^3 + \frac{3n^2p}{\beta} + \frac{3np^2}{\beta^2} + \frac{p^3}{\beta^3}\right) + \cdots,$$

and from there I will have immediately, by the formula of no. 43, this general expression

$$\begin{aligned} y_{x,t,u} = q^u[&y_{x-u,t,0} + u(ny_{x-u-1,t,0} + py_{x-u-1,t-1,0}) \\ &+ \frac{u(u+1)}{2}(n^2y_{x-u-2,t,0} + 2npy_{x-u-2,t-1,0} + p^2y_{x-u-2,t-2,0}) \\ &+ \frac{u(u+1)(u+2)}{2.3}(n^3y_{x-u-3,t,0} + 3n^2py_{x-u-3,t-1,0} + 3np^2y_{x-u-3,t-2,0} + p^3y_{x-u-3,t-3,0}) \\ &+ \cdots]. \end{aligned}$$

This formula goes to infinity; but as it is necessary when $x=0$ we have $y_{0,t,u}=0$, whatever be t and u, provided that they are not null at once, it is easy to see that all the terms of the form $y_{r,s,0}$ in which r will be negative, must necessarily be null; so that the formula will become finite, and that it should be advanced only to the terms, inclusively, which will be affected with the coefficient

$$\frac{u(u+1)u+2)\cdots(x-1)}{1.2.3\ldots(x-u)}.$$

Thus therefore we will have no more than some terms of the form $y_{r,s,0}$ where r will always be positive, but where s can become negative. In order to know the values of $y_{r,s,0}$ when s is negative, I make in the general formula above $t=0$ in which case the value of $y_{x,0,u}$ must be equal to that of $y_{x,t}$ of no. 49 by changing t to u and p to q; and as this equality must take place whatever be x and u, I deduce easily from it, by the comparison of the terms affected with the same coefficients u, $\frac{u(u+1)}{2}, \ldots,$ these equalities

$$\begin{aligned}
&y_{x-u,0,0}=1,\\
&ny_{x-u-1,0,0}+py_{x-u-1,-1,0}=1-q,\\
&n^2y_{x-u-2,0,0}+2npy_{x-u-2,-1,0}+p^2y_{x-u-2,-2,0}=(1-q)^2,
\end{aligned}$$

and thus in sequence; whence we deduce successively, because $n=1-p-q$,

$$y_{x-u,0,0}=1,\quad y_{x-u-1,-1,0}=1,\quad y_{x-u-2,-2,0}=1,\ldots,$$

so that we will have, in general,

$$y_{r,s,0}=1$$

when s will be zero or negative, r being positive or zero.

We can moreover convince ourselves *a priori* that $y_{r,s,0}$ must be equal to 1 when s is negative; because by supposing s positive, this quantity expresses the lot of the player, when there remain to him still r trials to play, and when he must still bring forth one of the events s times; now, if s becomes negative, it is clear that we will have the lot of the player when he has already brought forth the event in question more than s times needed; in which case, by the conditions of the game, he is counted to have already won; consequently his lot must then be always equal to unity.

Therefore, in general, in order to have the value of the terms $y_{r,s,0}$ which can enter into the expression above of $y_{x,t,u}$, we will note: 1° that these terms are all null for all the negative values of r; 2° that these terms are all equal to unity for all the negative or null values of s, r being zero or positive; 3° that r and s being positives or zero, we will have for the preceding Problem

$$y_{r,s}=p^t\left[1+s(1-p)+\frac{s(s+1)}{2}(1-p)^2+\cdots+\frac{s(s+1)\cdots(r-1)}{1.2\ldots(r-s)}(1-p)^{r-s}\right].$$

Thus the problem is resolved.

We see from this how it would be necessary to take it if the number of events were anything; there will be difficulty only in the length of the calculation.

PROBLEM III.

52. *The same things being supposed as in the Problem II, we demand the lot of the player who would wager to bring forth, in an undetermined number of trials, the second of two events b times before the first had happened a times.*

I designate by $y_{x,t}$ the lot of the player when he must still bring forth the second event t times before the first happens x times; it is clear that $y_{a,b}$ will be the sought lot.

We imagine now that we play a trial, and as the probability of the first event is p and that of the second is q at each trial by hypothesis, we will form easily the equation

$$y_{x,t} = py_{x-1,t} + qy_{x,t-1},$$

and we will note that the player wins when $t = 0$ and x any positive, and that he loses when $x = 0$, and t any positive; so that we will have $y_{x,0} = 1$, x being > 0, and $y_{0,t} = 0$, t being > 0.

This put, if we set the differential equation under the form

$$py_{x,t+1} + qy_{x+1,t} - y_{x+1,t+1} = 0,$$

and if we compare it to formula (F) of no. 7, we will have

$$\beta = -\frac{q\alpha}{p-\alpha} = \frac{q}{1-\frac{p}{\alpha}};$$

therefore

$$\beta^t = q^t\left[1 + t\frac{p}{\alpha} + \frac{t(t+1)}{2}\frac{p^2}{\alpha^2} + \frac{t(t+1)(t+2)}{2.3}\frac{p^3}{\alpha^3} + \cdots\right],$$

and consequently (8),

$$y_{x,t} = q^t\left[y_{x,0} + tpy_{x-1,0} + \frac{t(t+1)}{2}p^2y_{x-2,0} + \cdots\right].$$

Now since $y_{0,t} = 0$, it will be necessary that we have

$$y_{0,0} = 0, \quad y_{0,-1} = 0, \quad y_{0,-2} = 0, \ldots,$$

so that we will have the case of no. 9, where the series becomes finite, and as besides we must have $y_{x,0} = 1$, there will result from it this expression

$$y_{x,t} = q^t\left[1 + tp + \frac{t(t+1)}{2}p^2 + \frac{t(t+1)(t+2)}{2.3}p^3 + \cdots + \frac{t(t+1)\cdots(t+x-2)}{1.2\ldots(x-1)}p^{x-1}\right].$$

whence there will be no more than to change x to a and t to b.

ANOTHER SOLUTION OF PROBLEM III.

53. We can also find another solution of the preceding Problem by means of the formulas of no. 13, which give in all cases a finite expression for $y_{x,t}$.

By applying these formulas to the present case, we will have

$$m = 1, \quad n = p;$$

so that the quantities $Y, Y', Y'', \dots$ will be $y_{0,0}$, $\frac{1}{p}y_{1,0}$, $\frac{1}{p^2}y_{2,0}, \dots$, and because the conditions of the Problem demand that

$$y_{0,0} = 0, \quad y_{1,0} = 1, \quad y_{2,0} = 1, \dots,$$

this series will become

$$0, \frac{1}{p}, \frac{1}{p^2}, \dots;$$

whence, by taking the successive differences, we will have

$$Y = 0, \, \Delta Y = \frac{1}{p}, \, \Delta^2 Y = \frac{1}{p^2} - \frac{2}{p}, \, \Delta^3 Y = \frac{1}{p^3} - \frac{3}{p^2} + \frac{3}{p}, \dots;$$

therefore

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 1 - 2p, \quad f(3) = 1 - 3p + 3p^2,$$
$$f(4) = 1 - 4p + 6p - 4p^3, \, \dots,$$

whence it is easy to conclude that we will have, in general,

$$f(x) = (1-p)^x - (-p)^x,$$

as long as x is 0 or > 0.

Next, as the conditions of the Problem require also that

$$y_{0,0} = 0, \quad y_{0,1} = 0, \quad y_{0,2} = 0, \dots,$$

it follows that the quantities Y, $'Y,'' \, Y, \dots$ will all be null; consequently their differences δY, $\delta^2 Y, \dots$ will be null, which will give

$$f(0) = 0, \quad f(-1) = 0, \quad f(-2) = 0, \dots \quad \text{and} \quad f(-x) = 0.$$

Finally, as $A = 0$, and as that which we just named p in the place cited is $= -\frac{B}{C'} = q$, we will have

$$V = q^t, \quad V' = (x+t)q^t p, \quad V'' = \frac{(x+t)(x+t-1)}{2} q^t p^2, \dots$$

Therefore substituting these values into the formula

$$y_{x,t} = Vf(x) + V'f(x-1) + V''f(x-2) + \cdots,$$

we will have

$$y_{x,t} = q^t \Big[(1-p)^x - (-p)^x + (x+t)[(1-p)^{x-1} - (-p)^{x-1}]p \\ + \frac{(x+t)(x+t-1)}{2}[(1-p)^{x-2} - (-p)^{x-2}]p^2 + \cdots + \frac{(x+t)(x+t-1)\cdots(t+2)}{1.2\ldots(x-1)}p^{x-1}\Big]$$

We can again simplify this expressing by noting that

$$(1-p)^x - (-p)^x = (1-p)^{x-1} - (1-p)^{x-2}p + (1-p)^{x-3}p^2 - \cdots \pm p^{x-1},$$
$$(1-p)^{x-1} - (-p)^{x-1} = (1-p)^{x-2} - (1-p)^{x-3}p + \cdots \mp p^{x-2},$$

and thus in sequence; so that by substituting these values and ordering with respect to the powers of $1-p$, we will have

$$y_{x,t} = q^t \Big[(1-p)^{x-1} + (x+t-1)(1-p)^{x-2}p \\ + \left(\frac{(x+t)(x+t-1)}{2} - \frac{x+t}{1} - 1\right)(1-p)^{x-3}p^2 + \cdots + p^{x-1}\Big]$$

or else by reducing

$$y_{x,t} = q^t \Big[(1-p)^{x-1} + (x+t-1)(1-p)^{x-2}p + \frac{(x+t-1)(x+t-2)}{2}(1-p)^{x-3}p^2 + \cdots \\ + \frac{(x+t-1)(x+t-2)\cdots(t+1)}{1.2\ldots(x-1)}p^{x-1}\Big]$$

This expression of $y_{x,t}$, although under a different form from that which we have found in the preceding number, reverts however at base to that one, as we can convince ourselves easily be developing the powers of $1-p$, and ordering next the terms according to those of p; that which can serve to confirm the exactitude of our methods.

Besides, we see that in the Problem in question the method of the first solution is preferable to that of which we just made usage, not only because the process is easier, but mainly because the result is much simpler.

PROBLEM IV.

54. *We suppose that at each trial there can happen three different events, which I will designate for more clarity by P, Q, R, and that the probabilities of these events are respectively equal to p, q, r; we demand the lot of a player who would wager to bring forth the event R c times before the event Q happens b times, and the event P happens a times.*

Let $y_{x,t,u}$ be the lot of the player when he has yet to bring forth the event R u times before the event Q happens t times and the event P happens x times; we will have $y_{a,b,c}$ for the sought lot. Now, by supposing that we play one trial, we will attain the equation

$$y_{x,t,u} = py_{x-1,t,u} + qy_{x,t-1,u} + ry_{x,t,u-1};$$

and since the player is counted to have won when $u=0$ and x, t greater than zero; but on the contrary he is counted to have lost when $t=0$, and x, u greater than zero, and when $x=0$ and t, u greater than zero, it follows that we will have

$$y_{x,t,0} = 1, \quad y_{x,0,u} = 0, \quad y_{0,t,u} = 0,$$

x, t, u being any whole positive numbers.

The differential equation above being set under the form

$$py_{x,t+1,u+1} + qy_{x+1,t,u+1} + ry_{x+1,t+1,u} - y_{x+1,t+1,u+1} = 0,$$

is found contained in the formula (L) of no. 42, and we will have

$$\gamma = -\frac{r\alpha\beta}{q\alpha + p\beta - \alpha\beta} = \frac{r}{1 - \frac{p}{\alpha} - \frac{q}{\beta}};$$

whence

$$\gamma^u = r^u \left[1 + u\left(\frac{p}{\alpha} + \frac{q}{\beta}\right) + \frac{u(u+1)}{2}\left(\frac{p^2}{\alpha^2} + \frac{2pq}{\alpha\beta} + \frac{q^2}{\beta^2}\right)\right.$$
$$\left. + + \frac{u(u+1)(u+2)}{2.3}\left(\frac{p^3}{\alpha^3} + \frac{3p^2q}{\alpha^2\beta} + \frac{3pq^2}{\alpha\beta^2} + \frac{q^3}{\beta^3}\right) + \cdots\right].$$

And thence we can deduce immediately the value of $y_{x,t,u}$ by changing in the expression of γ^u each product such as $\frac{1}{\alpha^\rho\beta^\sigma}$ to $y_{x-\rho,t-\sigma,0}$, in the same way we see it by the comparison of the general formulas of no. 42 and 43. Thus we will have

$$\begin{aligned} y_{x,t,u} = r^u \big[&y_{x,t,0} + u(py_{x-1,t,0} + qy_{x,t-1,0}) \\ &+ \frac{u(u+1)}{2}(p^2y_{x-2,t,0} + 2pqy_{x-1,t-1,0} + q^2y_{x,t-2,0}) \\ &+ \frac{u(u+1)(u+2)}{2.3}(p^3y_{x-3,t,0} + 3p^2qy_{x-2,t-1,0} + 3pq^2y_{x-1,t-2,0} + q^3y_{x,t-3,0}) \\ &\cdots\big]. \end{aligned}$$

Now by the conditions of the Problem it is necessary that $y_{x,t,u}$ become equal to zero when $x = 0$, or $t = 0$ whatever be u; and it is clear, by the preceding expression, that this condition obtains the one that each quantity such as $y_{x,t,0}$ is null when $x = 0$ or negative, or when $t = 0$ or negative. Moreover, it is necessary also by the conditions of the Problem that $y_{x,t,0}$ be $= 1$ when x and t are greater than zero. Whence it follows that the general expression of $y_{x,t,u}$ will become finite, and will be represented in the following manner

$$y_{x,t,u} = r^u\left[1 - u(p+q) + \frac{u(u+1)}{2}(p^2 + 2pq + q^2)\right.$$
$$\left. - \frac{u(u+1)(u+2)}{2.3}(p^3 + 3p^2q + 3pq^2 + q^3) + \cdots\right]$$

by continuing this series only as long as the powers of p will be less than x, and those of q less than t.

So that if we designate, for more simplicity, the coefficients u, $\frac{u(u+1)}{2}$, $\frac{u(u+1)(u+2)}{2.3}$, ...

by u', u'', u''', ..., we can give to the expression in question this form

$$\begin{aligned} y_{x,t,u} = r^u \Big[& 1 + u'p + u''p^2 + \cdots + u^{(x-1)}p^x \\ & + u'q + 2u''pq + 3u'''p^2q + \cdots + xu^{(x)}p^{x-1}q \\ & + u'q^2 + 3u''pq^2 + 6u^{iv}p^3q^2 + \cdots + \frac{x(x+1)}{2}u^{(x-1)}p^{x-1}q^2 \\ & \ldots \\ & + u^{(t-1)}q^{t-1} + tu^{(t)}pq^{t-1} + \frac{t(t+1)}{2}u^{(t+1)}p^2q^{t-1} + \cdots + u^{(x+t-2)}p^{x-1}q^{t-1} \Big], \end{aligned}$$

where the coefficient of the last term $u^{(x+t-2)}p^{x-1}q^{t-1}$ will be equally

$$\frac{x(x+1)(x+2)\cdots(x+t-2)}{2.3\ldots(t-1)} \quad \text{or} \quad \frac{t(t+1)(t+2)\cdots(t+x-2)}{2.3\ldots(x-1)},$$

these two quantities being equal to each other, as we can convince ourselves by multiplying the one by the denominator of the other, and *vice versa*.

COROLLARY I.

55. If we supposed that at each trial there could happen four different events P, Q, R, S of which the respective probabilities are p, q, r, s and if we sought the lot of a player who would wager to bring forth the event S z times before the events R, Q, P could happen respectively u, t, x times, the Problem will always be resolved by the same method, and we will find, for the sought lot, the expression

$$s^c\left[1 + z(p+q+r) + \frac{z(z+1)}{2}(p+q+r)^2 + \frac{z(z+1)(z+2)}{2.3}(p+q+r)^3 + \cdots\right]$$

in which, after having developed the powers of $p+q+r$, it will be necessary to retain only the terms where p will be raised to a power less than x, q to a power less than t, and r to a power less than u; so that all the terms which must enter into the expression in question will form a rectangular parallelepiped, in which the three sides which depart from the same angle where there is the term s^c will be formed by these three series

$$\begin{aligned} & s^c\left[1 + zp + \frac{z(z+1)}{2}p^2 + \cdots + \frac{z(z+1)(z+2)\cdots(z+x-2)}{1.2\ldots(x-1)}p^{x-1}\right], \\ & s^c\left[1 + zq + \frac{z(z+1)}{2}q^2 + \cdots + \frac{z(z+1)(z+2)\cdots(z+t-2)}{1.2\ldots(t-1)}q^{t-1}\right], \\ & s^c\left[1 + zr + \frac{z(z+1)}{2}r^2 + \cdots + \frac{z(z+1)(z+2)\cdots(z+u-2)}{1.2\ldots(u-1)}r^{u-1}\right], \end{aligned}$$

and the number of all the terms will be equal to $(x-1)(t-1)(u-1)$.

COROLLARY II.

56. In general, if the events which can happen at each trial are A, B, C, D,. . ., and their respective probabilities a, b, c, d, . . . , and if we demand the lot of a player who would wager to bring forth the event A α times before B happens β times, C γ times, D δ times, . . ., we will find this expression

$$a^\alpha\left[1+\alpha(b+c+d+\cdots)+\frac{\alpha(\alpha+1)}{2}(b+c+d+\cdots)^2\right.$$
$$\left.+\ \frac{\alpha(\alpha+1)(\alpha+2)}{2.3}(b+c+d+\cdots)^3+\cdots\right],$$

in which, after having developed the powers of $b+c+d+\cdots$, it will be necessary to retain only the terms where the powers of b will be less than β, those of c less than γ, those of d less than δ, etc.; so that the number of all the terms which must enter into this expression will be

$$(\beta-1)(\gamma-1)(\delta-1)\cdots;$$

and it is easy to prove by the known Theorem on the form of the coefficients of the powers of multinomials, that each of these terms will be of the following form

$$\frac{\alpha(\alpha+1)\cdots(\alpha+l+m+n+\cdots-1)}{1.2\ldots(l+m+n+\cdots)}\times\frac{(l+1)(l+2)\cdots(l+m+n+\cdots)}{1.2.3\ldots m\times 1.2.3\ldots n\times 1\ldots}a^\alpha b^l c^m d^n\cdots$$

by giving successively to l, m, n all the integer values from zero to $\beta-1$, $\gamma-1$, $\delta-1$, . . . respectively.

REMARK.

57. The problem of which we just gave a very general and very simple solution contains in a general manner the one that we call commonly in the Analysis of chances the *Problem of points*, and which has been resolved completely only for the case of two players. (See the *Analyse* of Monmort, *Propositions XL and XLI*, second edition: *The Doctrine of Chances* of Moivre, *Problem VI*, second edition; the *Mémoire* of M. de Laplace printed among the Memoirs presented to the Académie des Sciences in 1773, *Problems XIV and XV*.)

If two players A and B playing together to several games have the respective probabilities p and q to win each game in particular, and if there is lacking to player A x games or points, and to player B t games or points to win, we will evidently have the case of Problem III (52), and $y_{x,t}$ will be the lot or expectation of player B; and our two solutions accord with those which we find in the Work cited of Monmort. Nos. 191, 192.

If there are three players A, B, C of whom the respective probabilities to win each game are p, q, r, and if there is lacking to A x games, to B t games, to C u games, we will have the case of Problem IV (54); and $y_{x,t,u}$ will be the lot or expectation of player C, and thus in sequence.

In general, if there are as many players as we wish, A, B, C, D,. . . of whom the respective probabilities to win each game are a, b, c, d, . . . , and if there is lacking to them respectively α, β, γ, δ, ... games, we will have by Corollary II above the general expression of the lot of player A and consequently also that of the lot of each of the other players by changing among them the quantities $a,\ b,\ c, \ldots$ and $\alpha,\ \beta,\ \gamma, \ldots$.

PROBLEM V.

58. *The probability to bring forth a given event at each trial being p, a player wagers that in a trials at least he will bring forth this event a number of times which will surpass by b the number of times where he will not bring it forth.*

Let $y_{x,t}$ be the lot of the player when he has no more than x trials to play, and when he must still bring forth the given event a number of times which surpasses by t the number of times where he will not bring forth this event; it is clear that the sought lot will be $y_{a,b}$.

If we imagine now that we play one trial, it is easy to from the following equation

$$y_{x,t} = py_{x-1,t-1} + (1-p)y_{x-1,\,t+1};$$

and as the player wins when $t = 0$ and x whatever, and to the contrary he loses when, x being null, t is still positive, it follows that we will have $y_{x,0} = 1$, x being whatever, and $y_{0,t} = 0$, t being > 0.

I make for greater simplicity $1 - p = q$, and I put the equation above under the following form

$$py_{x,t} + qy_{x,t+2} - y_{x+1,t+1} = 0,$$

which is, as we see, contained in the general formula (G) of no. 15; and there will come from it, according to formula (H), this equation in α and β

$$p - \alpha\beta + \alpha\beta^2 = 0;$$

whence it will be necessary to deduce the value of β and next that of β^t in α by a descending series. For this it is necessary to employ the method that I have given in my *Mémoire sur la résolution des équations littérales*, printed in the volume of this academy for the year 1768. In no. 26 of that Memoir we find two formulas which give the two values of x^m in the equation

$$a - bx + cx^2 = 0,$$

and which can be applied in the present case by making

$$x = \beta, \quad m = t, \quad a = p, \quad b = \alpha, \quad c = q;$$

we will have thus

$$\beta^t = \frac{p^t}{\alpha^t} + \frac{tp^{t+1}q}{\alpha^{t+2}} + \frac{t(t+3)}{2}\frac{p^{t+2}q^2}{\alpha^{t+4}} + \frac{t(t+4)(t+5)}{2.3}\frac{p^{t+3}q^3}{\alpha^{t+6}} + \cdots$$

or

$$\beta^t = \frac{\alpha^t}{q^t} - \frac{tp\alpha^{t-2}}{q^{t-1}} + \frac{t(t-3)}{2}\frac{p^2\alpha^{t-4}}{q^{t-2}} - \frac{t(t-4)(t-5)}{2.3}\frac{p^3\alpha^{t-6}}{\alpha^{t-3}} + \cdots$$

These two values of β^t being compared in the general expressions of β^t of no. 15, we will have

$$1^\circ \qquad \begin{cases} & \mu = -1,\ \mu' = 2,\ \mu'' = 4,\ \dots \\ T = p^t, & T' = tp^{t+1}q,\ T'' = \dfrac{t(t+3)}{2}p^{t+2}q^2,\ \dots \end{cases}$$

$$2^\circ \qquad \begin{cases} & \nu = 1,\ \nu' = 2,\ \nu'' = 4,\ \dots \\ U = \dfrac{1}{q^t}, & U' = \dfrac{tq}{q^{t-1}}.\, U'' = \dfrac{t(t-3)}{2}\dfrac{p^2}{q^{t-2}}, \dots \end{cases}$$

Therefore we will have (section cited)

$$\begin{aligned} y_{x,t} = p^t \bigg[& f(x-1) + tpqf(x-t-2) + \frac{t(t+3)}{2}p^2q^2f(x-t-4) \\ & + \frac{t(t+4)(t+5)}{2.3}p^3q^3f(x-t-6) + \cdots \bigg] \\ & + \frac{1}{q^t}\bigg[\phi(x+t) - tpq\phi(x+t-2) + \frac{t(t-3)}{2}p^2q^2\phi(x+t-4) \\ & - \frac{t(t-4)(t-5)}{2.3}p^3q^3\phi(x+t-6) + \cdots \bigg], \end{aligned}$$

the characteristics f and ϕ denoting two arbitrary functions, which we will determine in the manner following according to the conditions given in the Problem.

The first condition demands that when $x = 0$, we have $y_{0,t} = 0$, t being any whole positive number; it is easy to convince ourselves that we can satisfy this condition only by supposing that the function designated by the characteristic ϕ is always zero, and that the one designated by the characteristic f also becomes null when the number of which it is a function becomes negative. In this matter the expression of $y_{x,t}$ will become finite and will be of the form

$$\begin{aligned} y_{x,t} = p^t \bigg[& f(x-t) + tpqf(x-t-2) + \frac{t(t+3)}{2}p^2q^2f(x-t-4) \\ & + \frac{t(t+4)(t+5)}{2.3}p^3q^3f(x-t-6) + \cdots \bigg], \end{aligned}$$

by taking only as many terms as there are units in $\frac{x-t}{2} + 1$ or $\frac{x-t-1}{2} + 1$.

The other condition of the Problem demands next that when $t = 0$ and x whatever, we have $y_{x,t} = 1$; but in this case we will have, by the preceding formula, $y_{x,0} = f(x)$; therefore $f(x)$ must always be equal to 1, as long as x is not negative. Therefore the values of $f(x-t),\ f(x-t-2), \dots$ in the expression above will always be equal to 1. Thus we will have

$$y_{x,t} = p^t \left[1 + tpq + \frac{t(t+3)}{2}p^2q^2 + \frac{t(t+4)(t+5)}{2.3}p^3q^3 + \cdots \right],$$

and taking only as many terms as there will be units in $\frac{x-t}{2} + 1$ or in $\frac{x-t-1}{2} + 1$.

If we wished that t be negative, then by changing t to $-t$, in the general expression of $y_{x,t}$, we would only change p to q and f to ϕ and *vice versa*, and, making the same reasoning as before, we would find

$$y_{x,-t} = q^t \left[1 + tpq + \frac{t(t+3)}{2} p^2 q^2 + \cdots \right].$$

It is this which is moreover evident in itself ; because the case of t negative is the same as if t remaining positive, we exchanged between them the events P and Q, which produces no other difference in the solution than to substitute q in place of p and *vice versa*.

This Problem corresponds to Problem LXV of Moivre, and the preceding solution agrees with the second solution of that Author (page 210, third edition.)

ANOTHER SOLUTION TO PROBLEM V.

59. In the preceding solution we have need to resolve an equation of second degree in order to have the value of β in α through the equation

$$p - \alpha\beta + \alpha\beta^2 = 0;$$

but if instead of determining β in α we wished on the contrary to determine α in β, we would have then only one linear equation to resolve, and this value of α in β would have the advantage of being finite and of giving directly an expression for $y_{x,t}$ in finite terms.

Indeed, we will have

$$\alpha = \frac{p}{\beta} + q\beta;$$

I raise this binomial to the power x, and I reunite, for greater simplicity, the extreme terms and those which are equally extended from the extremes; I will have thus

$$\begin{aligned} \alpha^x =& (p^x \beta^{-x} + q^x \beta^x) + xpq(p^{x-2}\beta^2 + q^{x-2}\beta^{x-2}) \\ &+ \frac{x(x-1)}{2} p^2 q^2 (p^{x-4}\beta^{4-x} + q^{x-1}\beta^{x-4}) + \cdots, \end{aligned}$$

a formula which will need to grow only to the terms which will have for coefficient

$$\frac{x(x-1)(x-2)\cdots\left(\frac{x+1}{2}\right)}{1.2.3\ldots}$$

if x is odd, or else

$$\frac{x(x-1)(x-2)\cdots\left(\frac{x}{2}+1\right)}{1.2.3\ldots}$$

if x is even, by taking care, in this last case, to take only the half of this coefficient.

Multiplying this value of α^x by β^t I will have this expression for $\alpha^x\beta^t$:

$$\begin{aligned} \alpha^x\beta^t =& (p^x \beta^{t-x} + q^x \beta^{t+x}) + xpq(p^{x-2}\beta^{t+2-x} + q^{x-2}\beta^{t-2+x}) \\ &+ \frac{x(x-1)}{2.} p^2 q^2 (p^{x-4}\beta^{t-4-x} + q^{x-4}\beta^{t-4+x}) + \cdots \end{aligned}$$

whence, by the principles established in Article II above, we will deduce immediately this general expression of $y_{x,t}$ namely

$$\begin{aligned} y_{x,t} =& [p^x f(t-x) + q^x f(t+x)] \\ &+ xpq[p^{x-2} f(t+2-x) + q^{x-2} f(t-2+x)] \\ &+ \frac{x(x-1)}{2} p^2 q^2 [p^{x-4} f(t+4-x) + q^{x-4} f(t-4+x)] \\ &\ldots \end{aligned}$$

the characteristic f designating an arbitrary function, which must be determined by the conditions of the Problem.

For this result, it is necessary to recall that when $x = 0$ we must have $\nu_{0,t} = 0$, t being > 0, and that when $t = 0$ we must have $y_{x,0} = 1$, x being = or > 0; therefore: 1° we will have $f(t) = 0$, t being any positive number; 2° we will have

$$\begin{aligned} 1 =& [p^x f(-x) + q^x f(x)] + xpq[p^{x-2} f(2-x) + q^{x-2} f(x-2)] \\ &+ \frac{x(x-1)}{2} p^2 q^2 [p^{x-4} f(4-x) + q^{x-4} f(x-4)] + \cdots \end{aligned}$$

x being any number positive or zero. If we make successively $x = 0, 1, 2, 3, \ldots$ we will be able to deduce from this equation the values of

$$f(0),\ pf(-1) + qf(1),\ p^2 f(-2) + q^2 f(-2),\ \ldots,$$

and we will find, in general, by the formulas already known,

$$p^i f(-s) + q^i f(s) = 1 - spq + \frac{s(s-3)}{2} p^2 q^2 - \frac{s(s-4)(s-5)}{2.3} p^3 q^3 + \frac{s(s-5)(s-6)(s-7)}{2.3.4} p^4 q^4 + \cdots,$$

by taking in this series only as many terms as there are units in $\frac{s+1}{2}$ or in $\frac{s}{2} + 1$.

Therefore, since we must have, in general, $f(s) = 0$ as long as $s > 0$, we will have for the Problem in question

$$y_{x,t} = p^x \left[f(t-x) + x\frac{p}{q} f(t+2-x) + \frac{x(x-1)}{2} \frac{q^2}{p^2} f(t+4-x) + \cdots \right],$$

by taking only as many terms as there are units in $\frac{x-t+1}{2}$ or in $\frac{x-t}{2} + 1$; and there will be no more than to substitute into this formula, in the place of each function such as $f(-s)$, the quantity

$$f(-s) = \frac{1}{p^s} - \frac{sq}{p^{s-1}} + \frac{s(s-3)}{2} \frac{q^2}{p^{s-2}} - \cdots,$$

where the number of terms must be $\frac{s+1}{2}$ or $\frac{s}{2} + 1$.

60. As the equation which determines β in α is of the second degree, and as the given terms of the Table of no. 6 are those which form the first horizontal rank and the first vertical rank, we will have the simplest and most direct solution all at once by the method of Article III (31), by converting the quantity $\alpha^x\beta^t$ into a finite series of the following form

$$\begin{aligned}\alpha^x\beta^t =&V + V'\alpha + V''\alpha^2 + V'''\alpha^3 + \cdots\\ &+ {}_{\prime}V\beta + {}_{\prime\prime}V\beta^2 + {}_{\prime\prime\prime}V\beta^3 + \cdots;\end{aligned}$$

because then we will have at once (33)

$$\begin{aligned}y_{x,t} =&Vy_{0,0} + V'y_{1,0} + V''y_{2,0} + V'''y_{3,0} + \cdots\\ &+ {}_{\prime}Vy_{0,1} + {}_{\prime\prime}Vy_{0,2} + {}_{\prime\prime\prime}Vy_{0,3} + \cdots\end{aligned}$$

And as the conditions of the Problem demand that $y_{x,0} = 1$, x being $= 0,\ 1,\ 2,\ \ldots,$ and as $y_{0,t} = 0$, t being $= 1,\ 2,\ 3,\ \ldots,$ we will have in the case of the proposed Problem

$$y_{x,t} = V + V' + V'' + V''' + \cdots$$

Thus the difficulty is resolved by finding the sum of the coefficients V, V', V'', … of the first part of the expression of $\alpha^x\beta^t$.

For this I substitute in place of α its value in β in the quantity $\alpha^x\beta^t$; I have, as in no. 59,

$$\begin{aligned}\alpha^x\beta^t =&(p^x\beta^{t-x} + q^x\beta^{t+x}) + xpq(p^{x-2}\beta^{t-x} + q^{x-2}\beta^{t-2+x})\\ &+ \frac{x(x-1)}{2.}p^2q^2(p^{x-4}\beta^{t-4-x} + q^{x-4}\beta^{t-4+x}) + \cdots\end{aligned}$$

When $x < t$, this formula will contain only some positive powers of t, and will form consequently the second part of the sought expression of $\alpha^x\beta^t$, the first becoming then all null; which gives consequently $y_{x,t} = 0$, as it must be when the number of remaining trials is less than the number t. But, in the case where $x > t$, the preceding formula contains necessarily some negative powers of β, which it will be necessary to eliminate in the following manner.

If we raise successively to the square, the cube, etc., the equation

$$\alpha = \frac{p}{\beta} + q\beta,$$

we will be able to deduce from it the values of

$$\frac{p}{\beta} + q\beta,\ \frac{p^2}{\beta^2} + q^2\beta^2,\ \frac{p^3}{\beta^3} + q^3\beta^3,\ \ldots$$

in α, and we will have, in general, by the formulas already known,

$$\frac{p^s}{\beta^s} + q^s\beta^s = \alpha^s - spq\alpha^{s-2} + \frac{s(s-3)}{2}p^3q^3\alpha^{s-4} + \frac{s(s-4)(s-5)}{2.3}p^4q^4\alpha^{s-4} - \cdots,$$

by continuing this series only as long as the powers of α will be positives.

We designate, for greater simplicity, this series in α by $A^{(s)}$; we will have therefore

$$\frac{p^s}{\beta^s} + q^s\beta^s = A^{(s)}; \quad \text{therefore} \quad \beta^{-s} = \frac{A^{(s)}}{p^s} - \left(\frac{q}{p}\right)^s \beta^s.$$

Therefore, if by means of this formula we make vanish in the expression above of $\alpha^x\beta^t$ all the negative powers of β, it will be reduced to two series, one composed of positive powers of α, and the other composed of positive powers of β; thus it will have the form demanded.

As for our object it suffices to know the first series, we will consider uniquely the negative powers of β which enter into the expression of $\alpha^x\beta^t$, and making, for greater simplicity,

$$t = x - u,$$

we will have this formula

$$p^x\beta^{-u} + xpq \times p^{x-2}\beta^{2-u} + \frac{x(x-1)}{2}p^2q^2 \times p^{x-4}q^{4-x} + \cdots,$$

by taking in it only as many terms as there are units in $\frac{u+1}{2}$ or in $\frac{u}{2} + 1$.

Next we will put in the place of each negative power β^{-s} its value in α, $\frac{A^{(s)}}{p^s}$, by neglecting the positive powers of β; in this manner we will have, for the first part of the expression demanded of $\alpha^x\beta^t$, the formula

$$p^t\left[A^{(s)} + xpqA^{(s-2)} + \frac{x(x-1)}{2}p^2q^2A^{(s-4)} + \cdots\right],$$

in which there will be no more but to suppose $\alpha = 1$ in order to have the sought value of $y_{x,t}$.

Now, since in our case $q = 1 - p$ (58), it is clear that $\beta = 1$ gives $\alpha = 1$ in the equation

$$\alpha = \frac{p}{\beta} + q\beta,$$

therefore also in the equation which is derived from it

$$\frac{p^s}{\beta^s} + q^s\beta^s = A^{(s)},$$

by making $\beta = 1$, the quantity α contained in $A^{(s)}$ will become $= 1$; therefore we will have, when $\alpha = 1$,

$$A^{(s)} = p^s + q^s;$$

therefore, making this substitution and putting back $x - t$ in the place of u, we will have

$$\begin{aligned} y_{x,t} =& p^t\left[(p^{x-t} + q^{x-t}) + xpq(p^{x-t-2} + q^{x-t-2}\right. \\ &\left.+ \frac{x(x-1)}{2}p^2q^2(p^{x-t-4} + q^{x-t-4}) + \cdots\right] \end{aligned}$$

by continuing this series only as long as the exponent of the quantities

$$p^{x-t-\cdots} + q^{x-t-\cdots}$$

will be positive or zero, and by taking care, in this last case, to take 1 in the place of $p^0 + q^0$, because $A^{(0)} = 1$.

This solution is the same as the first solution of Moivre (page 209).

PROBLEM VI.

61. *Supposing, as in the preceding Problem, that the probability of bringing forth a given event at each trial is* p*; a player wagers that in* a *trials or less he will bring forth this event a number of times such that this number will be either greater by* b *than the number of times where he will not bring forth the same event, or else less by* c *than this last number.*

Let $y_{x,t}$ be the lot of the player when he has no more than x trials to win, and when the difference between the number of times where the given event has already happened and the number of times where this event has not happened is expressed by $t-c$; it is clear that at the beginning where $x = a$ we will have $t-c = 0$, consequently $t = c$; so that the sought lot will be $y_{a,c}$.

If we suppose now that we play one trial, we will find the equation

$$y_{x,t} = py_{x-1,t+1} + (1-p)y_{x-t,\,t-1},$$

which is, as we see, similar to that of the preceding Problem, with this sole difference that p is here in the place of $1-p$; what comes from this here the number t does not express the same thing as in the preceding Problem.

Now, according to the conditions of the Problem, it is easy to see that the player must win when $t-c = b$ and when $t-c = -c$, whatever be x, which gives $t = b+c$ or $= 0$, and consequently $y_{x,o} = 1$, $y_{x,b+c} = 1$, x being positive or zero.

Next we see that the player will lose when x being null, $t-c$ will be contained between the limits b and $-c$, that is to say that t will be between the limits 0 and $b+c$; therefore we will have $y_{0,t} = 0$, t being $1,\ 2,\ 3,\ \ldots,\ b+c-1$.

Thus the given terms of the Table of no. 6 are, in this case, those which form the first horizontal rank, next those which form the first vertical rank to the $(b+c+1)^{\text{st}}$ term alone, and finally those which form the $(b+c+1)^{\text{st}}$ horizontal rank; so that the first line of the given terms is a straight line and horizontal; and that the second is composed of two straight lines, one vertical and finite, the other horizontal and indefinite; which can serve as example of what we have observed in no. 39.

Since therefore as the differential equation is of the same form as that of the preceding Problem, we will be able to use the same means to integrate it; but I note first that the first solution leading to a general expression of $y_{x,t}$ composed of an infinite number of terms, would not know how to be applied conveniently to the present case. We will take therefore first the second solution, and we will have by changing only p

to q and q to p (59)

$$\begin{aligned} y_{x,t} =& [q^x f(t-x) + p^x f(t+x)] \\ & + xpq[q^{x-2} f(t+2-x) + p^{x-2} f(t-2+x)] \\ & + \frac{x(x-1)}{2} p^2 q^2 [q^{x-4} f(t+4-x) + p^{x-4} f(t-4+x) + \cdots , \end{aligned}$$

this formula must be continued only to the terms which will have for coefficient

$$\frac{x(x-1)\cdots\left(\frac{x+1}{2}\right)}{1.2\ldots} \quad \text{or} \quad \frac{x(x-1)\cdots\left(\frac{x}{2}+1\right)}{1.2\ldots},$$

and taking care to take only the half of this coefficient in the case of x even.

The question therefore is no more than to determine properly, according to the conditions of the Problem, the functions indicated by the characteristic f. For this result, I will make first $x = 0$, which will give $y_{0,t} = f(t)$; therefore, since we must have $y_{0,t} = 0$ as long as $t = 1, 2, 3, \ldots, b + c - 1$, it follows that we will have, in general, $f(s) = 0$, s being $= 1, 2, 3, \ldots, b + c - 1$.

Next we will make $t = 0$, in which case we must have, as in Problem V, $y_{x,0} = 1$, whatever be x; making therefore successively $x = 0, 1, 2, \ldots$, I will have, in general, as in the solution of no. 59, by changing only p into q and *vice versa*,

$$q^s f(-s) + p^s f(s) = 1 - spq + \frac{s(s-3)}{2} p^2 q^2 - \frac{s(s-4)(s-5)}{2.3} p^3 q^3 + \cdots ,$$

by taking in this series only $\frac{s+1}{2}$ or $\frac{s}{2} + 1$ terms.

Finally I will make $t = b+c = n$ and as we must have then also $y_{x,n} = 1$, whatever be x, I will deduce from it in the same manner the general formula

$$q^s f(n-s) + p^s f(n+s) = 1 - spq + \frac{s(s-3)}{2} p^2 q^2 - \frac{s(s-4)(s-5)}{2.3} p^3 q^3 + \cdots ,$$

or else

$$q^s f(n-s) + p^s f(n+s) = q^s f(-s) + p^s f(s).$$

By means of these formulas, by making successively $x = 0, 1, 2, \ldots,$ we will be able to find all the values of the unknown function which enters into the general expression above of $y_{x,t}$.

But we can much simplify this solution by the substitution of $1 - z_{x,t}$ in place of $y_{x,t}$. Because we will have first the differential equation

$$z_{x,t} = pz_{x-1,t+1} + (1-p)z_{x-1,t-1},$$

which is the same form as the equation in $y_{x,t}$; consequently we will have likewise, in general, by employing the characteristic ϕ, in order to designate an arbitrary function,

$$\begin{aligned} z_{x,t} =& [q^x \phi(t-x) + p^x \phi(t+x)] - xpq[q^{x-2}\phi(t+2-x) + p^{x-2}\phi(t-2+x)] \\ & + \frac{x(x-1)}{2} p^2 q^2 [q^{x-4}\phi(t+4-x) + p^{x-4}\phi(t-4+x)] + \cdots \end{aligned}$$

Now, as in making $x = 0$ we must have $y_{0,t} = 0$, as long as t is between the limits of 0 and $b+c$, we will have therefore $z_{0,t} = 1$, t being $= 1, 2, 3, \ldots, b+c-1$; and as in making $t = 0$ and $t = b+c$, we must have $y_{x,0} = 1$ and $y_{x,b+c} = 1$, x being positive or zero, it follows that we will have $z_{x,0} = 0$ and $z_{x,b+c} = 0$, x being $0, 1, 2, \ldots$

Therefore: 1° we will have, by making $x = 0$, $\phi(t) = 1$; therefore, in general, $\phi(s) = 1$ for all the values of s, namely $s = 1, 2, 3, \ldots, b+c-1$: 2° by making $t = 0$, and x successively $0, 1, 2, \ldots$, we will find, in general,

$$q^s\phi(-s) + p\phi(s) = 0,$$

s being likewise $= 0, 1, 2, 3, \ldots$: 3° by making $t = b+c$, and $x = 0, 1, 2, \ldots$, we will find similarly

$$q^s\phi(b+c-s) + p^s\phi(b+c+s) = 0,$$

s being also $= 0, 1, 2, 3 \ldots$.

Therefore finally, if for greater simplicity we put the expression for $z_{x,t}$ under the form

$$z_{x,t} = p^x\phi(x+t) + xp^{x-1}q\phi(x+t-2) + \frac{x(x-1)}{2}p^{x-2}q^2\phi(x+t-4) + \cdots,$$

if next we make $x = a$, $t = c$, and if we put back $1-p$ in place of q, we will find, for the sought value of $y_{a,c}$, that is to say for the lot of the player, the following formula

$$1 - p^x\phi(a+c) - ap^{x-1}\phi(a+c-2) - \frac{a(a-1)}{2}p^{x-2}q^2\phi(a+c-4) - \cdots,$$

and we will determine the values of the arbitrary function by these conditions

$$\phi(s) = 1, \quad s \text{ being} 1, 2, 3, \ldots, b+c-1,$$

and

$$\left.\begin{aligned} &p^s\phi(s) + (1-p)^s\phi(-s) = 0 \\ &p^s\phi(b+c+s) + (1-p)^s\phi(b+c-s) = 0 \end{aligned}\right\} s \text{ being } 0, 1, 2, 3, \ldots \text{ to infinity.}$$

Let for example,

$$a = 7, \quad b = 2, \quad c = 3,$$

we will have the formula

$$\begin{aligned} 1 - &p^7\phi(10) - 7p^6(1-p)\phi(8) + 21p^5(1-p)^2\phi(6) - 35p^5(1-p)^3\phi(4) \\ &-35p^3(1-p)^4\phi(2) - 21p^2(1-p)^5\phi(0) - 7p(1-p)^6\phi(-2) - (1-p)^7\phi(-4); \end{aligned}$$

now the condition

$$\phi(s) = 1$$

gives first

$$\phi(2) = 1, \quad \phi(4) = 1,$$

next the condition

$$p^s\phi(s) + (1-p)\phi(-s) = 0$$

will give

$$\phi(0) = 0, \quad \phi(-2) = -\frac{p^3}{(1-p)^3}, \quad \phi(-4) = -\frac{p^4}{(1-p)^4}$$

finally the condition

$$p^s\phi(5+s) + (1-p)\phi(5-s) = 0$$

will give

$$\phi(6) = \frac{1-p}{p}, \quad \phi(8) = -\frac{(1-p)^3}{p^3}, \quad \phi(10) = 0.$$

Therefore substituting these values we will have after the reductions

$$1 - 21p^3(1-p)^4 - 13p^4(1-p)^3$$

for the sought lot.

ANOTHER SOLUTION OF PROBLEM VI.

62. I come now to resolve the same Problem by the method of Article III; but instead of taking it as we have done in the third solution of the preceding Problem (60), where we have regarded as given the terms of the first horizontal row and those of the first vertical row of the Table no. 6, it will be more convenient here to suppose given the terms of the first two horizontal rows; which requires only to reduce the value of β^t to an expression of the following form (25)

$$\begin{aligned}\beta^t =& T + T'\alpha + T''\alpha^2 + T'''\alpha^3 + \cdots + T^{(t)}\alpha^t \\ &+ [{}_{,}T + {}_{,}T'\alpha + {}_{,}T''\alpha^2 + \cdots + {}_{,}T^{(t-1)}\alpha^{t-1}]\beta;\end{aligned}$$

because then we will have at once (27)

$$\begin{aligned}y_{x,t} =& Ty_{x,0} + T'y_{x+1,0} + T''y_{x+2,0} + T'''y_{x+3,0} + \cdots + T^{(t)}y_{x+t,0} \\ &+ {}_{,}Ty_{x,1} + {}_{,}T'y_{x+1,1} + {}_{,}T''y_{x+2,1} + \cdots + {}_{,}T^{(t-1)}y_{x+t-1,1}.\end{aligned}$$

Now as the quantity β must be determined (58, 61) by the equation

$$q - \alpha\beta + p\beta^2 = 0,$$

of which the two roots are

$$\beta = \frac{\alpha \pm \sqrt{\alpha^2 - 4pq}}{2p}$$

if we designate these two roots by β' and β'', and if we make, for brevity

$$\begin{aligned}A =& T + T'\alpha + T''\alpha^2 + T'''\alpha^3 + \cdots + T^{(t)}\alpha^t, \\ {}_{,}A =& {}_{,}T + {}_{,}T'\alpha + {}_{,}T''\alpha^2 + \cdots + {}_{,}T^{(t-1)}\alpha^{t-1},\end{aligned}$$

we will have (28)

$$\beta'^t = A + {}_{,}A\beta^t, \quad \beta''^t = A + {}_{,}A\beta'',$$

whence we will deduce

$$A = \frac{\beta'\beta''(\beta'^{t-1} - \beta''^{t-1})}{\beta'' - \beta'}, \quad {}_{,}A = \frac{\beta'^t - \beta''^t}{\beta' - \beta''};$$

namely

$$A = -q\frac{\left(\alpha + \sqrt{\alpha^2 - 4pq}\right)^{t-1} - \left(\alpha - \sqrt{\alpha^2 - 4pq}\right)^{t-1}}{(2p)^{t-1}\sqrt{\alpha^2 - 4pq}}$$

$${}_{,}A = \frac{\left(\alpha + \sqrt{\alpha^2 - 4pq}\right)^{t} - \left(\alpha - \sqrt{\alpha^2 - 4pq}\right)^{t}}{(2p)^{t}\sqrt{\alpha^2 - 4pq}}$$

Thus there will be only to develop these t^{th} and $(t-1)^{\text{st}}$ powers and to order next the terms with respect to α, we will have the values of the coefficients T, T', T'', ... at the same time those of ${}_{,}T$, ${}_{,}T'$, ${}_{,}T''$, ..., in p, q, and t; but we will not have the same need to know these values, as we are going to see.

In fact, as the conditions of the Problem demand that $y_{x,0} = 1$, x being any positive or zero (61), if we make $y_{x,t} = 1 - u_x$, it is clear that the expression of $y_{x,t}$ will become

$$y_{x,t} = A + {}_{,}A - {}_{,}Tu_x - {}_{,}T'u_{x+1} - {}_{,}T''u_{x+2} - \cdots - {}_{,}T^{(t-1)}u_{x+t-1},$$

by supposing that in the quantities A and ${}_{,}A$ we have made $\alpha = 1$; now

$$\beta^t = A + {}_{,}A\beta,$$

and, because $q = 1 - p$, if we make $\beta = 1$, we have $\alpha = 1$ according to the equation

$$q - \alpha\beta + p\beta^2 = 0;$$

therefore

$$1 = A + {}_{,}A$$

when $\alpha = 1$; therefore

$$y_{x,t} = 1 - {}_{,}Tu_x - {}_{,}T'u_{x+1} - {}_{,}T''u_{x+2} - \cdots - {}_{,}T^{(t-1)}u_{x+t-1}.$$

Next it is necessary also, by the conditions of the Problem, that $y_{x,b+c} = 1$, x being any positive or zero; therefore if we denote by B, B', B'', ... the values of ${}_{,}T$, ${}_{,}T'$, ${}_{,}T''$, ..., when $t = b + c$, we will have for the determination of the quantities u_x the equation

$$Bu_x + B'u_{x+1} + B''u_{x+2} + \cdots + B^{(t-1)}u_{x,b+c-1} = 0,$$

whence we see that these quantities form a simple recurrent series of order $b + c - 1$; so that if we make the equation

$$B + B'\alpha + B''\alpha^2 + B'''\alpha^3 + \cdots + B^{(b+c-1)}\alpha^{b+c-1} = 0, \qquad (a)$$

and if we denote by α', α'', α''', ... the different roots of this equation, we will have, in general (Article I),

$$u_x = M\alpha'^x + N\alpha''^x + P\alpha'''^x + \cdots,$$

M, N, P being some undetermined constants.

We will make therefore this substitution into the expression above of $y_{x,t}$ and as we have, in general

$$_{\prime}A = {}_{\prime}T + {}_{\prime}T'\alpha + {}_{\prime}T''\alpha^2 + \cdots$$

if we denote by $_{\prime}A$, $_{\prime}A'$, $_{\prime}A''$, ... the values of $_{\prime}A$ which correspond to $\alpha = \alpha'$, α'', α''', ..., we will have

$$y_{x,t} = 1 - M{}_{\prime}A'\alpha'^x - N{}_{\prime}A''\alpha''^x - P{}_{\prime}A'''\alpha'''^x - \cdots, \qquad (b)$$

and there will remain no more than to determine $b+c-1$ constants by means of the last condition of the Problem which is $y_{0,t} = 0$, t being $1, 2, 3, \ldots, b+c-1$; so that it will be necessary that these constants are such, that we have (x being $= 0$)

$$M{}_{\prime}A' + N{}_{\prime}A'' + P{}_{\prime}A''' + \cdots = 1, \qquad (c)$$

by supposing successively $t = 1, 2, 3, \ldots, b+c-1$ in the quantities $_{\prime}A$, $_{\prime}A'$, $_{\prime}A''$, ...

Now it is clear that equation (a) above is no other than that this one here $_{\prime}A = 0$ by making $t = b + c$; moreover, if we make

$$\alpha = 2\sqrt{bc}\cos\theta,$$

it is clear that the expression for $_{\prime}A$ found above will become

$$_{\prime}A = \left(\sqrt{\frac{q}{p}}\right)^t \frac{\sin t\theta}{\sqrt{pq}\sin\theta};$$

therefore, making $t = b + c$, the equation in question will become

$$\frac{\sin(b+c)\theta}{\sin\theta} = 0,$$

whence we deduce

$$\theta = \frac{\lambda\pi}{b+c},$$

π being the angle of 180 degrees and λ any number of the sequence $1, 2, 3, \ldots, b+c-1$. We know thence the $b+c-1$ roots α, α', α'', ..., in the same way the corresponding quantities $_{\prime}A$, $_{\prime}A'$, $_{\prime}A''$, ...; and we will have, in general,

$$\alpha^{(\lambda)} = 2\sqrt{pq}\cos\frac{\lambda\pi}{b+c}, \quad {}_{\prime}A^{(\lambda)} = \left(\sqrt{\frac{q}{p}}\right)^t \frac{\sin\frac{\lambda t\pi}{b+c}}{\sqrt{pq}\sin\frac{\lambda\pi}{b+c}}.$$

Substituting therefore these values into formula (b) and making for greater simplicity

$$b + c = n$$

and

$$\frac{M}{\sqrt{pq}\sin\frac{\pi}{n}} = (1), \quad \frac{N}{\sqrt{pq}\sin\frac{\pi}{n}} = (2), \; \ldots,$$

we will have

$$\begin{aligned} y_{x,t} =& 1 - (2\sqrt{pq})^x \left(\sqrt{\frac{q}{p}}\right)^t \\ &\times \left[(1)\left(\cos\frac{\pi}{n}\right)^x \sin\frac{t\pi}{n} + (2)\left(\cos\frac{2\pi}{n}\right)^x \sin\frac{2t\pi}{n} \right. \\ &\left. + (3)\left(\cos\frac{\pi}{n}\right)^x \sin\frac{3t\pi}{n} + \cdots + (n-1)\left(\cos\frac{(n-1)\pi}{n}\right)^x \sin\frac{(n-1)t\pi}{n}\right]; \end{aligned}$$

and equation (c) by which it will be necessary to determine the constants (1), (2), (3), $\ldots$,$(n-1)$ will be

$$(1)\sin\frac{t\pi}{n} + (2)\sin\frac{2t\pi}{n} + \cdots + (n-1)\sin\frac{(n-1)t\pi}{n} = \left(\sqrt{\frac{q}{p}}\right)^t,$$

which must take place by making successively $t = 1,\, 2,\, 3,\, \ldots,\, n-1$.

In order to deduce thence the value of each of these constants, it will be necessary only to multiply the whole equation by the sine which has for coefficient the constant which we wish to determine, and to add next together the $n-1$ particular equations which correspond to $t = 1,\, 2,\, 3,\, \ldots,\, n-1$; in this manner all the other constants will disappear, and the sought constant will be found multiplied by $\frac{2}{n}$; this is what we can be assured by the known formulas for the summation of series formed of sines or of cosines.

Thus in order to have, in general, the value of (μ) we will multiply the equation by $\sin\frac{\mu t\pi}{n}$, and, operating as we just said, there will come

$$\frac{n}{2}(\mu) = \sqrt{\frac{q}{p}}\sin\frac{\mu\pi}{n} + \frac{p}{q}\sin\frac{2\mu\pi}{n} + \cdots + \left(\sqrt{\frac{q}{p}}\right)^{n-1}\sin\frac{(n-1)\mu\pi}{n}.$$

Now the second member of this equation is reduced by known formulas to

$$\frac{\sqrt{\frac{q}{p}}\left[1 \pm \left(\sqrt{\frac{q}{p}}\right)^n\right]\sin\frac{\mu\pi}{n}}{1 - 2\sqrt{\frac{q}{p}}\cos\frac{\mu\pi}{n} + \frac{p}{q}},$$

the upper sign being for the case of μ odd, and the lower sign for the one of μ even.

We will have therefore, in general,

$$(\mu) = \frac{\frac{2}{n}\sqrt{\frac{q}{p}}\left[1 \pm \left(\sqrt{\frac{q}{p}}\right)^n\right]\sin\frac{\mu\pi}{n}}{1 - 2\sqrt{\frac{q}{p}}\cos\frac{\mu\pi}{n} + \frac{p}{q}},$$

whence, by making successively $\mu = 1,\, 2,\, 3,\, \ldots,\, n-1$, we will deduce the values of the constants (1), (2), (3), $\ldots$, which we will substitute into the expression above for $y_{x,t}$; next we will have no more than to make $x = a$ and $t = c$ in order to have the value of the demanded lot.

REMARK.

63. The preceding Problem returns to the one which concerns the duration of the games which we play by reducing, and of which Messers de Monmort, Bernoulli and Moivre have occupied themselves. (*See* the Work of Monmort, page 268, second edition; the one of Moivre, page 191, third edition.)

We propose ordinarily this Problem thus: *Two players each having a certain number of tokens play together with this condition that the one who loses a game will give a token to the other; we demand how much the odds are that the game, which can endure to infinity, will be finite in a certain number of games at most, so that one of the two players will have won all the tokens of the other.* It is easy to understand that if we denote by b and c the numbers of tokens of the two players, by p and $1 - p$ or q the respective probabilities that these players have in order to win each game, and by a the number of games in which we wager that the game will end, it is easy, I say, to understand that we will have exactly the case of our Problem VI. Thus of the two solutions which we just gave of this Problem, the first corresponds to the method of Problem LXIII, and the second corresponds to that of Problem LXVIII of the Work cited of Moivre; but our solutions have the advantage of being more direct, more general and more analytic than those of that Author.

Problem V above can also be brought back to the duration of games; but it is necessary to suppose that one of the players having first b tokens, the other having none, and that the game ends only when the latter one will have won the b tokens of his adversary.

PROBLEM VII.

64. *Let a number a of urns be arranged in sequence, and of which each contains n tickets in part white and in part black at will; let us draw at once from each of these urns a ticket at random and let us put next the ticket drawn from each urn into the following urn; by observing to put in the first urn the ticket drawn from the last; we demand what will be probably the number of black tickets in each urn after a number b of parallel drawings.*

Let $y_{x,t}$ be the number of black tickets that there will be probably in the x^{th} urn after t drawings; it is easy to see that after a new drawing this number will be probably increased by $\frac{y_{x-1,t}}{n}$, and diminished by $\frac{y_{x,t}}{n}$, so that we will have the equation

$$y_{x,t+1} = y_{x,t} + \frac{1}{n}y_{x-1,t} - \frac{1}{n}y_{x,t},$$

which is reduced to this form

$$y_{x,t} + (n-1)y_{x+1,t} - ny_{x+1,t+1} = 0.$$

Here the given quantities are the values of $y_{x,t}$ when $t = 0$ and when $x = 1, 2, 3, \ldots, n$, which indicate the numbers of black tickets that there are in each urn before the first drawing; so that one of the conditions of the Problem is that the terms $y_{x,0}$ be all given from $x = 0$ to $x = n$ inclusively: the other condition to which it is necessary to satisfy is that the tickets drawn from the last urn return to the first; and it is clear that for this there is only to suppose that the a^{th} urn precedes the first, that is to say that

that urn is also the 0^{th}; so that the value of $y_{x,t}$ which corresponds to $x = a$ is always identified with that which corresponds to $x = 0$; which will give this other condition $y_{a,t} = y_{0,t}$, whatever be t.

Now if we return to the differential equation found above in formula (F) of no. 7, we have

$$A = 1, \quad B = n - 1, \quad B' = 0, \quad C' = -n;$$

which reenters into the second case of no. 11; so that, because of

$$p = -\frac{B}{C'} = 1 - \frac{1}{n}, \quad q = \frac{A}{B} = \frac{1}{n-1},$$

we will have immediately

$$y_{x,t} = \left(1 - \frac{1}{n}\right)^t \left[y_{x,0} + \frac{t}{n-1} y_{x-1,0} + \frac{t(t-1)}{2(n-2)} y_{x-2,0} + \cdots\right],$$

the number of terms being $s + 1$.

Now as we must have $y_{0,t} = y_{a,t}$, whatever be t, it is clear that in order to satisfy this condition, it will be necessary that we have

$$y_{0,0} = y_{a,0}, \quad y_{-1,0} = y_{a-1,0}, \quad y_{-2,0} = y_{a-2,0}, \ldots$$

and, in general,

$$y_{-s,0} = y_{a-s,0},$$

s being any positive number or zero. Thus, as the values of $y_{x,0}$ are supposed known from $x = 1$ to $x = a$ inclusively, we will know all the values of $y_{x,0}$ which can enter into the preceding expression for $y_{x,t}$.

COROLLARY.

65. If we do not wish that the tickets drawn from the last urn reenter into the first, but that we always put into that one a white ticket after each extraction, there will be then only to suppose that the 0^{th} urn which is counted as preceding the first urn, contains only some white tickets, which will give $y_{0,t} = 0$, t being whatever; and we will see easily that in order to satisfy this condition, it will be necessary to suppose

$$y_{0,0} = 0, \quad y_{-1,o} = 0, \quad y_{-2,0} = 0, \ldots,$$

and, in general,

$$y_{-s,0} = 0,$$

s being any positive number or zero. Thus it will be necessary, in this case, to take only x terms of the general expression of $y_{x,t}$ by neglecting all the following.

In general, if we suppose that each ticket drawn from the first urn is replaced by a ticket drawn at random following any law which varies, as we wish, at each drawing, in a manner that the probability that this ticket is black is any given function of t that we will designate by (t), we will consider that, as the probability that the ticket which enters into the x^{th} urn at the t^{th} drawing is black is represented by $\frac{y_{x-1,t}}{n}$ in the preceding solution, the probability (t) which corresponds to the first urn for which $x = 1$ will be $\frac{y_{0,t}}{n}$; so that we will have $y_{0,t} = n(t)$; consequently we will know the first vertical row of the Table of no. 6; and thence we will be able, by the formulas of no. 11, to deduce the values of $y_{-s,0}$.