RECHERCHES

SUR

L'INTEGRATION DES ÉQUATIONS DIFFÉRENTIELLES

AUX DIFFÉRENCES FINIES

ET SUR

LEUR USAGE DANS LA THÉORIE DES HASARDS

Mr. de la Place, Professor at the École Royale militaire*

Mémoires de l'Académie Royale des Sciences de Paris
(Savants étrangers), 1773, T. VII (1776) pp. 37-162.
Oeuvres Complète **8**, pp. 69-144.

Read to the Academy 10 February 1773.

I.

The first researches that one has made on the summation of arithmetic progressions and on geometric progressions contained the germ of the integral Calculus in finite differences in one and two variables; here is how: an arithmetic progression is a sequence of terms which increase equally, and it was necessary to find the sum according to this condition; it is clear that each term of the sequence is the finite difference of the sum of the preceding terms, to that same sum augmented by this term; one proposed therefore to find this sum according to the nature of its finite difference; thus by whatever manner that one is arrived there, one has truely integrated a quantity in the finite differences. The geometers who have come next have pushed further these researches; they have determined the sum of the squares and of the superior and entire powers of the natural numbers; they have arrived there first by some indirect methods: they did not perceive that that which they sought returned to finding a quantity of which the finite difference was known; but as soon as they had made this reflection, they have resolved directly, not only the cases already known, but many others more extended. In general, $\phi(x)$ representing any function whatsoever of the variable x, of which the finite difference

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 18, 2010 is supposed constant, they have proposed to find a quantity of which the finite difference is equal to that function, and this is the object of the integral Calculus in the finite differences in a single variable.

Similarly, the research of the general term of a geometric progression returns to finding the x^{th} term of a sequence[1] such that each term is to the one which precedes it in constant ratio. Let y_{x-1} be the $(x-1)^{\text{st}}$ term and y_x be the x^{th} term: the law of the sequence requires that one have $y_x = py_{x-1}$, whatever be x, p being constant. Now it is clear that, in whatever manner that one is arrived to find y_x, one has veritably integrated the equation in the finite differences $y_x = py_{x-1}$. Next, one has generalized this research by proposing to find the general term of the sequences such that each of their terms is equal to many of the preceding multiplied by some constants any whatsoever; these sequences have been named for this *récurrentes*. One has arrived first to find their general term by some indirect ways, although quite ingenious; one did not perceive that this returned to integrating a linear equation in finite differences; but, when one had made this reflection, one tried to apply to these equations the methods known for the linear equations in the infinitely small differences, with the modifications that the assumption of finite differences requires, and one resolved in this manner some cases much more extended than those which were already.

Mr. Moivre is, I believe, the first who had determined the general term of the recurrent sequences; but Mr. de Lagrange is the first who is aware that this research depends on the integration of a linear equation in finite differences, and who had applied the good method of undetermined coefficients of Mr. d'Alembert (*see* Vol. I of the *Mémoires de Turin*). I myself have proposed next to deepen this interesting calculus, in a Memoir printed in Volume IV of those of Turin;[2] and next, having had occasion to reflect further there, I have made on this new researches of which I will render account shortly. I must observe here that Mr. the marquis de Condorcet has given excellent things on this matter, in his *Traité du Calcul intégral*, and in the *Mémoires de l'Académie*.

It was until then only a question of equations in ordinary finite differences and of the sequences which depend on them; but the solution of many problems on the chances has led me to a new kind of sequence which I have named *récurro-récurrentes*, and of which I believe to have given first the theory and indicated the usage in the Science of probabilities (*see* T. VI of *Savants étranges*.[3]) The equations on which these sequences depend are nearly, in the finite differences, that which the equations in the partial differences are in the infinitely small differences; that which I have given on these equations is only a trial: in deepening them, I have seen that they were quite important in the Theory of chances, and that they gave a method to treat them much more generally that one had done yet: this is that which engages me to consider them anew; but, the new researches that I have made on this object supposing those that I

[1] *Translator's note*: The word suite is used to refer to both a sequence and a series. It is rendered according to its usage.

[2] Recherches sur le calcul intégral aux differences infiniment petites, & aux différences finies. *Mélanges de philosophie et de mathématiques de la Société royale de Turin, pour les années 1766-1769 (Miscellanea Taurensia IV)*, 273-345, 1771.

[3] Mémoire sur les suites récurro-récurrentes et sur leur usages dans la théorie des hasards, *Mémoires de l'Académie Royale des Sciences de Paris (Savants étranges)* 6, 1774, p. 353-371.

have already given, I am going to begin again here all this matter.

II.

One can imagine thus the equations in finite differences; I imagine the sequence

$$y_1, y_2, y_3, y_4, y_5, \dots, y_x$$

formed following a law such as one has constantly

$$X_x = M_x y_x + N_x \Delta y_x + P_x \Delta^2 y_x + \dots + S_x \Delta^n y_x; \qquad \text{(A)}$$

the numbers 1, 2, 3, ..., x, placed at the base of y, indicating the rank which y occupies in the sequence, or, that which returns to the same, the index of the series; the quantities X_x, M_x, N_x, ... are some functions any whatsoever of the variable x, of which the difference is supposed constant and equal to unity. The characteristic Δ serves to express the finite difference of the quantity before which it is placed, as in the infinitesimal Analysis the letter d expresses the infinitely small difference of the quantities. This put, the preceding equation is an equation in finite differences, which can generally represent the equations of this kind, where the variable y_x and its differences are under a linear form.

Although I have supposed the constant difference of x equal to unity, this diminishes nothing from the generality of the preceding equation (A); because, if this difference, instead of being 1, is equal to q, one will make $\frac{x}{q} = x'$, and y_x being a function of x will become a function of qx'; I name $y_{x'}$ this last function. Now one has, by hypothesis,

$$\begin{aligned}\Delta y_x = y_{x+q} - y_x =& f(x+q) - f(x) \\ =& f[q(x'+a)] - f(qx') = y_{x'+1} - y_{x'} = \Delta y_{x'},\end{aligned}$$

the constant difference of x' being 1. Similarly,

$$\Delta^2 y_x = y_{x+2q} - 2y_{x+q} + y_x = y_{x'+2} - 2y_{x'+1} + y_{x'} = \Delta^2 y_{x'},$$

and thus of the remaining. Equation (A) will be therefore transformed into the following

$$X_{x'} = M_{x'} y_{x'} + N_{x'} \Delta y_{x'} + \dots + S_{x'} \Delta^n y_{x'},$$

in which the difference of x' is equal to unity.

One can form easily other differential equations, in which y_x and its differences would enter in any manner whatsoever; but those which are contained in equation (A) are the only ones which it is truly interesting to consider.

Before researching to integrate them, I am going to recall here a principle quite useful in the analysis of the infinitely small differences, and which applies equally and with the same advantage to finite differences; here is in what it consists:

Each function of x which, containing n arbitrary irreducible constants, satisfying for y_x in a differential equation of order n, between x and y_x, is the complete expression of y_x.

By *irreducible constants*, I intend that they are such that two or many can not be reduced to one alone; it follows thence that, if a function containing n irreducible arbitrary constants satisfy as y_x in a differential equation of order $n-1$, this equation is surely identical; because, if it was not, the most general function of x which was able to satisfy for y_x would contain only $n-1$ irreducible arbitrary constants.

For the convenience of the calculus, I will suppose that the quantities noted in this manner, ${}^1H$, ${}^2H$, $\dots$, or ${}^1M$, ${}^2M$, $\dots$, express some different quantities and which can have no relation among themselves; but these here, H_1, H_2, H_3, $\dots$, H_x or M_1, M_2, M_3, $ldots$, M_x represent the different terms of a sequence formed according to one law any whatsoever, the numbers 1, 2, 3, $\dots$, x designating the rank of the H or of the M in the sequence. This put, since one has

$$\begin{aligned}
&\Delta y_x = y_{x+1} - y_x,\\
&\Delta y^2 y_x = y_{x+2} - 2y_{x+1} + y_x,\\
&\Delta^3 y_x = y_{x+2} - 3y_{x+2} + 3y_{x+1} - y_x,\\
&\dots\dots\dots\dots\dots\dots\dots\dots ,
\end{aligned}$$

I am able to give to equation (A) this form

$$\begin{aligned}
X_x = &+y_x(M_x - N_x + P_x - \dots)\\
&+y_{x+1}(N_x - 2P_x + \dots)\\
&+\dots\dots\dots\dots\dots\\
&+y_{x+n}S_x.
\end{aligned}$$

whence it results that each linear equation in finite differences can be generally represented by this here

$$\text{(B)} \qquad y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + {}^2H_x y_{x-3} + \dots + {}^{n-1}H_x y_{x-n} + X_x;$$

the equation

$$y_x = H_x y_{x-1} + X_x$$

is of the first order, this here

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + X_x$$

is of the second order, and thus in sequence.

As in the series I will have need of characteristics in order to designate the finite difference of the quantities, their finite integrals, the product of all the terms of a sequence, I will serve myself for this with the following.

The characteristic Δ placed before a quantity will designate for it, as above, the finite difference: thus ΔH_x will express the finite difference of H_x; the characteristic Σ placed before a quantity will designate for it the finite integral: thus H_x will signify the finite integral of H_x; finally the characteristic ∇ will designate the product of all the terms of a sequence: thus ∇H_x will represent the product $H_1 H_2 H_3 \dots H_x$ of all the terms of the sequence $H_1, H_2, H_3, \dots, H_x$.

III.

PROBLEM I. — *The differential equation of the first order*

$$y_x = H_x y_{x-1} + X_x$$

being given, one proposes to integrate it.

I make in this equation $y_x = u_x \nabla H_x$; it becomes

$$u_x \nabla H_x = H_x u_{x-1} \nabla H_{x-1} + X_x;$$

but one has

$$H_x \nabla H_{x-1} = \nabla H_x,$$

hence

$$u_x = u_{x-1} + \frac{X_x}{\nabla H_x} \qquad \text{or} \qquad \Delta u_{x-1} = \frac{X_x}{\nabla H_x};$$

and, as this equation holds whatever be x, one will have

$$\Delta u_x = \frac{X_{x+1}}{\nabla H_{x+1}},$$

hence, by integrating,

$$u_x = A + \sum \frac{X_{x+1}}{\nabla H_{x+1}},$$

A being an arbitrary constant. One has therefore

$$y_x = \nabla H_x \left(A + \sum \frac{X_{x+1}}{\nabla H_{x+1}} \right).$$

If H_x was constant and equal to p, one would have

$$\nabla H_x = p^x \qquad \text{and} \qquad y_x = p^x \left(A + \sum \frac{X_{x+1}}{p^{x+1}} \right).$$

IV.

PROBLEM II. — *The differentio-differential equation*

(B) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + {}^2H_x y_{x-3} + \ldots + {}^{n-1}H_x y_{x-n} + X_x$$

being given, one proposes to integrate it.

I make

(C) $$y_x = \alpha_x y_{x-1} + T_x,$$

α_x and T_x being two new variables, and I conclude from it the following equations:

$$\begin{aligned} y_{x-1} &= \alpha_{x-1} y_{x-2} + T_{x-1}, \\ y_{x-2} &= \alpha_{x-2} y_{x-3} + T_{x-2}, \\ y_{x-3} &= \alpha_{x-3} y_{x-4} + T_{x-3}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ y_{x-n+1} &= \alpha_{x-n+1} y_{x-n} + T_{x-n+1}; \end{aligned}$$

I multiply the first of these equations by $-{}^1\beta$, the second by $-{}^2\beta$, the third by $-{}^3\beta$, … and I add them with equation (C): this which gives me

$$\begin{aligned}y_x =&(\alpha_x + {}^1\beta)y_{x-1} + (-{}^1\beta\alpha_{x-1} + {}^2\beta)y_{x-2}\\&+(-{}^2\beta\alpha_{x-2} + {}^3\beta)y_{x-3} + \ldots - {}^{n-1}\beta\alpha_{x-n+1}y_{x-n}\\&+T_x - {}^1\beta T_{x-1} - {}^2\beta T_{x-2} - \ldots - {}^{n-1}\beta T_{x-n+1}.\end{aligned}$$

By comparing this equation with equation (B), one will have

1°

$$T_x = {}^1\beta T_{x-1} + {}^2\beta T_{x-2} + \ldots + {}^{n-1}\beta T_{x-n+1} + X_x;$$

2° The following equations:

$$\begin{aligned}{}^1\beta + \alpha_x =&H_x,\\{}^2\beta - {}^1\beta\alpha_{x-1} =&{}^1H_x,\\{}^3\beta - {}^2\beta\alpha_{x-2} =&{}^2H_x\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\-{}^{n-1}\beta\alpha_{x-n+1} =&{}^{n-1}H_x.\end{aligned}$$

Thence one will conclude

$$\begin{aligned}{}^1\beta =&H_x - \alpha_x,\\{}^2\beta =&{}^1H_x + \alpha_{x-1}H_x - \alpha_x\alpha_{x-1},\\{}^3\beta =&{}^2H_x + \alpha_{x-2}\,{}^1H_x + \alpha_{x-1}\alpha_{x-2}H_x - \alpha_x\alpha_{x-1}\alpha_{x-2},\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\{}^{n-1}\beta =&{}^{n-2}H_x + \alpha_{x-n+2}\,{}^{n-3}H_x + \alpha_{x-n+3}\alpha_{x-n+2}\,{}^{n-4}H_x + \ldots\\&-\alpha_x\alpha_{x-1}\ldots\alpha_{x-n+2} = -\frac{{}^{n-1}H_x}{\alpha_{x-n+1}},\end{aligned}$$

because of the equation

$$-{}^{n-1}\beta\alpha_{x-n+1} = {}^{n-1}H_x;$$

one will have therefore, in order to resolve the problem, the following two equations:

$$\text{(D)}\qquad \begin{cases}T_x =(H_x - \alpha_x)T_{x-1} + ({}^1H_x + \alpha_{x-1}H_x - \alpha_x\alpha_{x-1})T_{x-2} + \ldots\\ \qquad -\dfrac{{}^{n-1}H_x}{\alpha_{x-n+1}}T_{x-n+1} + X_x,\end{cases}$$

$$\text{(E)}\qquad 0 = t - \frac{H_x}{\alpha_x} - \frac{{}^1H_x}{\alpha_x\alpha_{x-1}} - \frac{{}^2H_x}{\alpha_x\alpha_{x-1}\alpha_{x-2}} - \ldots - \frac{{}^{n-1}H_x}{\alpha_x\ldots\alpha_{x-n+1}}.$$

Equations (D) and (E) are of a degree inferior to the proposed, and equation (D) is of the same form; now it is not necessary to integrate generally these equations in order to integrate equation (B) of the problem; it suffices to know for α_x a quantity which satisfies equation (E). I name δ_x this value; one will substitute it into equation (D), which I name (D′) after this substitution, and one will seek the complete integral of equation (D′); next, by means of the equation $y_x = \delta_x y_{x-1} + T_x$, one will conclude, by integrating by problem I,

$$y_x = \nabla\delta_x \left(A + \sum \frac{T_{x+1}}{\nabla\delta_{x+1}} \right),$$

A being an arbitrary constant.

This equation is the complete integral of equation (B), because, equation (D′) being necessarily of order $n-1$, the complete expression of T_x contains $n-1$ irreducible arbitrary constants; hence, $\nabla\delta_x \left(A + \sum \frac{T_{x+1}}{\nabla\delta_{x+1}} \right)$ contains n arbitrary constants. These constants are moreover irreducibles, because $\nabla\delta_x \sum \frac{T_{x+1}}{\nabla\delta_{x+1}}$ contains in it $n-1$ irreducibles, and none of them is reducible with the constant A.

The preceding expression of y_x can serve to make known the integral of equation (B) of the problem; because, since equation (D′) is linear, one can suppose that the expression of T_x has this form

$$T_x = \nabla\lambda_x \left({}^1A + \sum \frac{{}^1T_{x+1}}{\nabla\lambda_{x+1}} \right),$$

${}^1T_x$ depending on the integration of a linear equation of order $n-2$; one has therefore

$$y_x = \nabla\delta_x \left[A + {}^1A \sum \frac{\nabla\lambda_{x+1}}{\nabla\delta_{x+1}} + \sum \frac{\sum \frac{{}^1T_{x+1}}{\nabla\lambda_{x+1}}}{\nabla\delta_{x+1}} \right];$$

by continuing to reason thus, one will see that the expression of y_x is of this form

$$y_x = A\nabla\delta_x + {}^1A\nabla{}^1\delta_x + {}^2A\nabla{}^2\delta_x + \ldots + {}^{n-1}A\nabla{}^{n-1}\delta_x + L_x,$$

A, ${}^1A$, ${}^2A, \ldots$ being arbitrary.

If one supposes $X_x = 0$ in equation (B), it is easy to see, by the sequence of operations that I just indicated, that L_x will be null; thus, in this case

$$y_x = A\nabla\delta_x + {}^1A\nabla{}^1\delta_x + \ldots + {}^{n-1}A\nabla{}^{n-1}\delta_x,$$

δ_x satisfying under the assumption for α_x in equation (E); ${}^1\delta_x$, ${}^2\delta_x$, $\ldots$ will satisfy similarly; because, since the equation $y_x = A\nabla{}^1\delta_x$, for example, satisfies equation (B) by supposing $X = 0$, one will have

$$\nabla{}^1\delta_x = H_x\nabla{}^1\delta_{x-1} + {}^1H_x\nabla{}^1\delta_{x-2} + \ldots,$$

hence

$$0 = 1 - \frac{H_x}{{}^1\delta_x} - \frac{{}^1H_x}{{}^1\delta_x\,{}^1\delta_{x-1}} - \cdots$$

V.

I suppose, in equations (D′) and (B), $X_x = 0$; I will have the following two expressions of y_x:

(1) $$y_x = \nabla\delta_x \left(A + \sum \frac{T_{x+1}}{\nabla\delta_{x+1}} \right),$$

(2) $$y_x = A\nabla\delta_x + {}^1A\nabla^1\delta_x + {}^2A\nabla^2\delta_x + \ldots + {}^{n-1}A\nabla^{n-1}\delta_x.$$

These two expressions, different in appearance, must really coincide; I suppose therefore that the complete integral of equation (D′) is

$$T_x = {}^1AR_x + {}^2A\,{}^1R_x + \ldots + {}^{n-1}A\,{}^{n-2}R_x;$$

by substituting this value of T_x into equation (1), one will have

$$y_x = \nabla\delta_x \left(A + {}^1A\frac{R_{x+1}}{\nabla\delta_{x+1}} + {}^2A\frac{{}^1R_{x+1}}{\nabla\delta_{x+1}} + \ldots + {}^{n-1}A\frac{{}^{n-2}R_{x+1}}{\nabla\delta_{x+1}} \right).$$

By comparing this last equation with equation (2), one will have

$$\nabla\delta_x \sum \frac{R_{x+1}}{\nabla\delta_{x+1}} = \nabla^1\delta_x,$$
$$\nabla\delta_x \sum \frac{{}^1R_{x+1}}{\nabla\delta_{x+1}} = \nabla^2\delta_x,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore

$$R_x = \nabla\delta_x\Delta\frac{\nabla^1\delta_{x-1}}{\nabla\delta_{x-1}},$$
$${}^1R_x = \nabla\delta_x\Delta\frac{\nabla^2\delta_{x-1}}{\nabla\delta_{x-1}},$$
$${}^2R_x = \nabla\delta_x\Delta\frac{\nabla^3\delta_{x-1}}{\nabla\delta_{x-1}},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore, if I know how to resolve equation (B) by supposing $X_x = 0$, I will know how to resolve equation (D′) by supposing similarly $X_x = 0$. Let therefore u_x, ${}^1u_x$, ${}^2u_x, \ldots$ be the particular values of y_x in equation (B), so that its complete integral is

$$y_x = Au_x + {}^1A\,{}^1u_x + {}^2A\,{}^2u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x,$$

one will have

$$u_x = \nabla\delta_x, \qquad {}^1u_x = \nabla^1\delta_x, \qquad \ldots,$$

and the complete integral of equation (D′), by supposing $X_x = 0$ in it, will be

$$T_x = {}^1Au_x\Delta\frac{\overset{1}{u}_{x-1}}{u_{x-1}} + {}^2Au_x\Delta\frac{\overset{2}{u}_{x-1}}{u_{x-1}} + \ldots + {}^{n-1}Au_x\Delta\frac{\overset{n-1}{u}_{x-1}}{u_{x-1}}.$$

Presently, if I know how to integrate equation (D′) by supposing X_x anything, I will be able, under the same assumption, to integrate equation (B), since one has, by that which precedes,

$$y_x = u_x\left(A + \sum\frac{T_{x+1}}{u_{x+1}}\right);$$

therefore the difficulty to integrate the equation

$$\text{(B)} \qquad y_x = H_xy_{x-1} + {}^1H_xy_{x-2} + \ldots + {}^{n-1}H_xy_{x-n} + X_x,$$

when one knows how to integrate this one

$$(b) \qquad y_x = H_xy_{x-1} + {}^1H_xy_{x-2} + \ldots + {}^{n-1}H_xy_{x-n},$$

is reduced to integrate the equation

$$\text{(D}'\text{)} \qquad T_x = (H - \delta_x)T_{x-1} + \ldots - \frac{{}^{n-1}H_x}{\delta_{x-n+1}}T_{x-n+1} + X_x,$$

which is of degree $n-1$, and when one knows how to integrate by supposing $X_x = 0$; one will make similarly the integration of (D′) to depend on the integration of an equation of degree $n-2$, and thus in sequence; whence there results that the equation

$$y_x = H_xy_{x-1} + {}^1H_xy_{x-2} + \ldots + {}^{n-1}H_xy_{x-n} + X_x$$

is integrable in the same cases as this one

$$y_x = H_xy_{x-1} + \ldots + {}^{n-1}H_xy_{x-n}.$$

VI.

The process which I just indicated in order to restore the integral of equation (B) to that of equation (*b*) can serve to demonstrate the liaison which these two integrals have between them; but it would be quite painful to employ it to integrate equation (B). It would be therefore very useful to have immediately the general expression of y_x in equation (B), when one has that of equation (*b*).

I take for this equation

$$y_x = u_x\left(A + \sum\frac{T_{x+1}}{u_{x+1}}\right),$$

T_x being supposed to be the complete expression of T_x in equation (D′). Now, this equation (D′) being of the same form as equation (B), if one names $\overset{1}{u}_x$, ${}^1\overset{1}{u}_x$, ${}^2\overset{1}{u}_x, \ldots$ the particular integrals of T_x in equation (D′), when one supposes $X_x = 0$ there, one will have, in the same manner and whatever be X_x,

$$T_x = \overset{1}{u}_x \left({}^1A + \sum \frac{{}^1T_{x+1}}{\overset{1}{u}_{x+1}} \right),$$

${}^1T_x$ being the complete expression of ${}^1T_x$ in an equation of order $n-2$, which I name (D″) and which results from (D′) in the same manner as this one results from equation (B); one will have similarly

$${}^1T_x = \overset{2}{u}_x \left({}^2A + \sum \frac{{}^2T_{x+1}}{\overset{2}{u}_{x+1}} \right),$$

and thus in sequence until one arrives to the equation of the first order

$${}^{n-2}T_x = S_x\, {}^{n-2}T_{x-1} + X_x,$$

of which the integral is

$${}^{n-2}T_x = \overset{n-1}{u}_x \left({}^{n-1}A + \sum \frac{X_{x+1}}{\overset{n-1}{u}_{x+1}} \right).$$

If one substitutes presently into the expression of y_x the value of T_x into ${}^1T_x$, that of ${}^1T_x$ into ${}^2T_x$, etc., one will have

$$\text{(K)} \quad y_x = u_x \left\{ A + \sum \frac{\overset{1}{u}_{x+1}}{u_{x+1}} \left({}^1A + \sum \frac{\overset{2}{u}_{x+1}}{\overset{1}{u}_{x+1}} \left[{}^2A \ldots + \sum \frac{\overset{n-1}{u}_{x+n-1}}{\overset{n-2}{u}_{x+n-1}} \left({}^{n-1}A + \sum \frac{X_{x+n}}{\overset{n-1}{u}_{x+n}} \right) \cdots \right] \right) \right\}.$$

It is necessary presently to determine $\overset{1}{u}_x, \overset{2}{u}_x, \ldots$; now one has, by the previous Article,

$$\overset{1}{u}_x = R_x = u_x \Delta \frac{{}^1u_{x-1}}{u_{x-1}},$$

similarly

$${}^1\overset{1}{u}_x = u_x \Delta \frac{{}^2u_{x-1}}{u_{x-1}},$$

$${}^2\overset{1}{u}_x = u_x \Delta \frac{{}^3u_{x-1}}{u_{x-1}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots;$$

one will have likewise

$$\overset{2}{u}_x = \overset{1}{u}_x \Delta \frac{{}^{1}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}},$$

$${}^{1}\overset{2}{u}_x = \overset{1}{u}_x \Delta \frac{{}^{2}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}},$$

$${}^{2}\overset{2}{u}_x = \overset{1}{u}_x \Delta \frac{{}^{3}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}},$$

$$\dots\dots\dots\dots\dots\dots;$$

formula (K) will become

$$\text{(O)} \quad y_x = u_x \left\{ A + \sum \Delta \frac{{}^{1}u_x}{u_x} \left({}^{1}A + \sum \Delta \frac{{}^{1}\overset{1}{u}_{x+1}}{\overset{1}{u}_{x+1}} \left[{}^{2}A \dots + \sum \Delta \frac{{}^{1}\overset{n-2}{u}_{x+n-2}}{\overset{n-2}{u}_{x+n-2}} \left({}^{n-1}A + \sum \frac{X_{x+n}}{\overset{n-1}{u}_{x+n}} \right) \cdots \right] \right) \right\};$$

if one knows only the number $n-1$ of particular integrals of y_x, in the equation

$$y_x = H_x y_{x-1} + {}^{1}H_x y_{x-2} + \dots + {}^{n-1}H_x y_{x-n},$$

the integration will be of difficulty no longer; I suppose that this is the integral ${}^{n-1}u_x$ which is unknown; since one knows u_x, ${}^{1}u_x$, $\dots$, ${}^{n-2}u_x$, one will know $\overset{1}{u}_x$, $\overset{2}{u}_x$, $\dots$ until $\overset{n-1}{u}_x$ exclusively. In order to determine $\overset{n-1}{u}_x$, it is necessary to integrate the equation

$${}^{n-2}T_x = S_x\, {}^{n-2}T_{x-1} + X_x,$$

by supposing $X_x = 0$, this which will be easy by Problem I if one knows S_x. In order to find it, I observe that, in equation (D′), the coefficient of T_{x-1} is

$$H_x - \delta_x = H_x - \frac{u_x}{u_{x-1}},$$

because of

$$\delta_x = \frac{u_x}{u_{x-1}}.$$

Similarly the one of ${}^{1}T_{x-1}$, in equation (D″), is

$$H_x - \frac{u_x}{u_{x-1}} - \frac{\overset{1}{u}_x}{\overset{1}{u}_{x-1}},$$

and thus in sequence; hence,

$$S_x = H_x - \frac{u_x}{u_{x-1}} - \frac{\overset{1}{u}_x}{\overset{1}{u}_{x-1}} - \dots - \frac{\overset{n-2}{u}_x}{\overset{n-2}{u}_{x-1}}.$$

If, instead of knowing the integral of the equation

$$y_x = H_x y_{x-1} + \ldots + {}^{n-1}H_x y_{x-n},$$

one knows a number n or $n-1$ of values for α_x, in equation (E), the preceding formulas will serve equally, because $\delta_x,\ {}^1\delta_x, \ldots$ being these values, one has

$$u_x = \nabla\delta_x, \qquad {}^1u_x = \nabla\,{}^1\delta_x, \qquad \ldots$$

VII.

Formula (O) has not at all yet the total degree of simplicity that the complete integral of y_x can have, because one has seen (Art. IV) that this integral has the following form

$$y_x = Au_x + {}^1A\,{}^1u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x + L_x;$$

it is necessary therefore to restore equation (O) to this form; for this, I divide equation (O) by u_x, and I conclude from it, by differentiating it,

$$\Delta\frac{y_{x-1}}{u_{x-1}} = \Delta\frac{{}^1u_{x-1}}{u_{x-1}}\left\{{}^1A + \sum\Delta\frac{{}^1\overset{1}{u}_x}{\overset{1}{u}_x}\left[{}^2A\ldots + \sum\Delta\frac{{}^1\overset{n-2}{u}_{x+n-3}}{\overset{n-2}{u}_{x+n-3}}\left({}^{n-1}A + \sum\frac{X_{x+n-1}}{\overset{n-1}{u}_{x+n-1}}\right)\ldots\right]\right\},$$

whence one will conclude, by dividing by $\Delta\frac{{}^1u_{x-1}}{u_{x-1}}$ and differentiating,

$$\Delta\frac{\Delta\frac{y_{x-2}}{u_{x-2}}}{\Delta\frac{{}^1u_{x-2}}{u_{x-2}}} = \Delta\frac{{}^1\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}[{}^2A + \ldots].$$

One will have therefore, by continuing to differentiate thus, an equation of this form

$${}^{n-1}A + \sum\frac{X_{x-1}}{\overset{n-1}{u}_{x-1}} = \gamma_x y_x + {}^1\gamma_x y_{x-1} + {}^2\gamma_x y_{x-2} + \ldots + {}^{n-1}\gamma_x y_{x-n+1},$$

$\gamma_x,\ {}^1\gamma_x, \ldots$ being some functions of $u_x,\ {}^1u_x, \ldots$ and of their finite differences. I observe now that, in order to form the values of $\overset{1}{u}_x, \overset{2}{u}_x, \overset{3}{u}_x, \ldots$, I have considered (preceding Article) the quantities $u_x,\ {}^1u_x,\ {}^2u_x, \ldots$ in this order

$$u_x,\ {}^1u_x,\ {}^2u_x, \ldots, {}^{n-1}u_x;$$

but if, instead of that, I had considered them in the following order

$${}^1u_x, u_x,\ {}^2u_x, \ldots, {}^{n-1}u_x,$$

I would arrive to the following equation

$${}^{n-1}A + \sum\frac{X_{x+1}}{\left(\overset{n-1}{u}_{x+1}\right)} = (\gamma_x)y_x + ({}^1\gamma_x)y_{x-1} + \ldots + ({}^{n-1}\gamma_x)y_{x-n+1},$$

$\left(\overset{n-1}{u}_x\right)$, (γ_x), ... being that which $\overset{n-1}{u}_x$, γ_x, ... become when one changes u_x into ${}^1u_x$, and ${}^1u_x$ into u_x. If I had supposed $X_{x+1} = 0$, I would have arrived to the two equations

$$
\begin{aligned}
{}^{n-1}A &= \gamma_x y_x + {}^1\gamma_x y_{x-1} + \ldots + {}^{n-1}\gamma_x y_{x-n+1},\\
{}^{n-1}A &= (\gamma_x) y_x + ({}^1\gamma_x) y_{x-1} + \ldots + ({}^{n-1}\gamma_x) y_{x-n+1},
\end{aligned}
$$

in which the constant ${}^{n-1}A$ is clearly the same, since I have supposed, in order to form the one and the other equation, that the complete value of y_x is

$$y_x = Au_x + {}^1A\,{}^1u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x.$$

One will have therefore, by comparing these two equations,

$$
\begin{aligned}
&\gamma_x y_x + {}^1\gamma_x y_{x-1} + \ldots + {}^{n-1}\gamma_x y_{x-n+1}\\
&\quad = (\gamma_x) y_x + ({}^1\gamma_x) y_{x-1} + \ldots + ({}^{n-1}\gamma_x) y_{x-n+1},
\end{aligned}
$$

an equation which must be an identity; because, if it were not, this equation being differential of order $n-1$ would have however for the complete integral

$$y_x = Au_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x,$$

an equation which contains n arbitrary constants, this which would be absurd (Art. II).

One has therefore

$${}^{n-1}A + \sum \frac{X_{x+1}}{\left(\overset{n-1}{u}_{x+1}\right)} = {}^{n-1}A + \sum \frac{X_{x+1}}{\overset{n-1}{u}_{x+1}},$$

hence

$$\left(\overset{n-1}{u}_{x+1}\right) = \overset{n-1}{u}_{x+1}.$$

Thus the expression of $\overset{n-1}{u}_x$ remains always the same, whether one changes u_x into ${}^1u_x$, and ${}^1u_x$ into u_x; one will be assured in the same manner that if in $\overset{n-1}{u}_x$ one changes u_x into ${}^2u_x$, and ${}^2u_x$ into u_x; or ${}^1u_x$ into ${}^2u_x$, and ${}^2u_x$ into ${}^1u_x$, and generally ${}^ku_x$ into ${}^iu_x$, and ${}^iu_x$ into ${}^ku_x$, k and i being less than $n-1$, the expression $\overset{n-1}{u}_x$ will always remain the same, and that thus, whatever order that one gives to the quantities u_x, ${}^1u_x$, ${}^2u_x, \ldots$ in order to form $\overset{n-1}{u}_x$, this expression will remain always the same, provided that ${}^{n-1}u_x$ is considered as the last of these quantities.

I make $\overset{n-1}{u}_{x+1} = {}^{n-1}z_{x+1}$; next, instead of considering ${}^{n-1}u_x$ as the last of the quantities u_x, ${}^1u_x, \ldots$ I suppose actually that ${}^{n-2}u_x$ is this last; let ${}^{n-2}z_{x+1}$ be that which becomes then ${}^{n-1}z_{x+1}$, that is to say when one changes ${}^{n-2}u_x$ into ${}^{n-1}u_x$, and ${}^{n-1}u_x$ into ${}^{n-2}u_x$. One will have, by a process similar to the preceding,

$${}^{n-2}A + \sum \frac{X_{x+1}}{{}^{n-2}z_{x+1}} = \underline{\gamma}_x y_x + {}^1\underline{\gamma}_x y_{x-1} + \ldots + {}^{n-1}\underline{\gamma}_x y_{x-n+1},$$

$\underline{\gamma}$, ${}^{1}\underline{\gamma}_x$, … being that which γ_x, ${}^{1}\gamma_x$, … become when one changes ${}^{n-1}u_x$ into ${}^{n-2}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$; one will have similarly

$$
{}^{n-3}A + \sum \frac{X_{x+1}}{{}^{n-3}z_{x+1}} = \underline{\underline{\gamma}}_x\, y_x + {}^{1}\underline{\underline{\gamma}}_x\, y_{x-1} + \ldots + {}^{n-1}\underline{\underline{\gamma}}_x\, y_{x-n+1},
$$

${}^{n-3}z_{x+1}$, $\underline{\underline{\gamma}}_x$, ${}^{1}\underline{\underline{\gamma}}_x$ being that which ${}^{n-1}z_{x+1}$, γ_x, ${}^{1}\gamma_x$, … become when one changes ${}^{n-1}u_x$ into ${}^{n-3}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$. This set, by disposing in the following order all the equations that one can form thus

$$
(>)\quad \begin{cases}
{}^{n-1}A + \sum \dfrac{X_{x+1}}{{}^{n-1}z_{x+1}} = \gamma_x y_x + {}^{1}\gamma_x y_{x-1} + {}^{2}\gamma_x y_{x-2} + \ldots + {}^{n-1}\gamma_x y_{x-n+1}, \\
{}^{n-2}A + \sum \dfrac{X_{x+1}}{{}^{n-2}z_{x+1}} = \underline{\gamma}_x y_x + {}^{1}\underline{\gamma}_x y_{x-1} + {}^{2}\underline{\gamma}_x y_{x-2} + \ldots + {}^{n-1}\underline{\gamma}_x y_{x-n+1}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
A + \sum \dfrac{X_{x+1}}{z_{x+1}} = \dfrac{\gamma_x}{n-1} y_x + \dfrac{{}^{1}\gamma_x}{n-1} y_{x-1} + \dfrac{{}^{2}\gamma_x}{n-1} y_{x-2} + \ldots + \dfrac{{}^{n-1}\gamma_x}{n-1} y_{x-n+1},
\end{cases}
$$

and adding them altogether, after having multiplied the first by ${}^{n-1}u_x$, the second by ${}^{n-2}u_x$, etc., finally the last by u_x, one will have an equation of this form

$$
\begin{aligned}
\lambda_x y_x + \ldots + {}^{n-1}\lambda_x y_{x-n+1} = {} & u_x\left(A + \sum \frac{X_{x+1}}{z_{x+1}}\right) \\
& + {}^{1}u_x\left({}^{1}A + \sum \frac{X_{x+1}}{{}^{1}z_{x+1}}\right) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \\
& + {}^{n-1}u_x\left({}^{n-1}A + \sum \frac{X_{x+1}}{{}^{n-1}z_{x+1}}\right),
\end{aligned}
$$

this which gives, by making $X_{x+1} = 0$,

$$
\lambda_x y_x + {}^{1}\lambda_x y_{x-1} + \ldots + {}^{n-1}\lambda_x y_{x-n+1} = A u_x + {}^{1}A\, {}^{1}u_x + \ldots + {}^{n-1}A\, {}^{n-1}u_x;
$$

but one has in this case

$$
y_x = A u_x + {}^{1}A\, {}^{1}u_x + \ldots,
$$

hence

$$
y_x = \lambda_x y_x + {}^{1}\lambda_x y_{x-1} + \ldots + {}^{n-1}\lambda_x y_{x-n+1}.
$$

Now this equation must be an identity, because otherwise, although of order $n-1$, its integral would contain the n arbitrary constants which the complete expression of y_x contains; one has therefore for the complete integral of equation (B) of Problem II, whatever be X_x,

$$\begin{aligned} y_x = & u_x\left(A+\sum\frac{X_{x+1}}{z_{x+1}}\right) \\ & + {}^1u_x\left({}^1A+\sum\frac{X_{x+1}}{{}^1z_{x+1}}\right) \\ & + \dots\dots\dots\dots\dots\dots \\ & + {}^{n-1}u_x\left({}^{n-1}A+\sum\frac{X_{x+1}}{{}^{n-1}z_{x+1}}\right), \end{aligned}$$

Thence results this quite simple rule, in order to have the complete integral of the equation

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n} + X_x,$$

when one knows how to integrate this here

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n}.$$

Let

$$y_x = Au_x + {}^1A\,{}^1u_x + {}^2A\,{}^2u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x$$

be the integral of this last, and let one make

$$\begin{array}{lll} \overset{1}{u}_x = u_x\Delta\dfrac{{}^1u_{x-1}}{u_{x-1}}, & \overset{2}{u}_x = \overset{1}{u}_x\Delta\dfrac{{}^1\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \overset{3}{u}_x = \overset{1}{u}_x\Delta\dfrac{{}^1\overset{2}{u}_{x-1}}{\overset{2}{u}_{x-1}}, \\ {}^1\overset{1}{u}_x = u_x\Delta\dfrac{{}^2u_{x-1}}{u_{x-1}}, & {}^1\overset{2}{u}_x = \overset{1}{u}_x\Delta\dfrac{{}^2\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \dots\dots\dots\dots\dots\dots, \\ {}^2\overset{1}{u}_x = u_x\Delta\dfrac{{}^3u_{x-1}}{u_{x-1}}, & {}^2\overset{2}{u}_x = \overset{1}{u}_x\Delta\dfrac{{}^3\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \\ \dots\dots\dots\dots\dots\dots, & \dots\dots\dots\dots\dots\dots, & \end{array}$$

until one arrives to form $\overset{n-1}{u}_x$, let $\overset{n-1}{u}_x = {}^{n-1}z_x$. If, in the expression of ${}^{n-1}z_x$, one changes ${}^{n-1}u_x$ into ${}^{n-2}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$, one will form ${}^{n-2}z_x$; if, in the same expression of ${}^{n-1}z_x$, one changes ${}^{n-1}u_x$ into ${}^{n-3}u_x$, and reciprocally ${}^{n-3}u_x$ into ${}^{n-1}u_x$, one will form ${}^{n-3}z_x$, and thus in sequence; the complete integral of equation

(B) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \cdots + {}^{n-1}H_x y_{x-n} + X_x$$

will be

$$(H)\qquad \begin{cases} y_x = u_x \left(A + \sum \dfrac{X_{x+1}}{z_{x+1}} \right) \\ \quad + {}^1u_x \left({}^1A + \sum \dfrac{X_{x+1}}{{}^1z_{x+1}} \right) \\ \quad + \dots\dots\dots\dots\dots\dots\dots \\ \quad + {}^{n-1}u_x \left({}^{n-1}A + \sum \dfrac{X_{x+1}}{{}^{n-1}z_{x+1}} \right). \end{cases}$$

VIII.

I take now the equations (>) of the preceding Article; they give

$$\begin{gathered} {}^{n-1}A + \sum \frac{X_{x+2}}{{}^{n-1}z_{x+2}} = \gamma_{x+1} y_{x+1} + \ldots + {}^{n-1}\gamma_{x+1} y_{x-n+2}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ A + \sum \frac{X_{x+2}}{z_{x+2}} = \frac{\gamma_{x+1}}{n-1} y_{x+1} + \ldots + \frac{{}^{n-1}\gamma_{x+1}}{n-1} y_{x-n+2}; \end{gathered}$$

if one multiplies the first by ${}^{n-1}u_x$, the second by ${}^{n-2}u_x$, …, one will have, by adding them together, an equation of this form

$$\lambda_x y_{x+1} + {}^1\lambda_x y_{x+2} + \ldots + {}^{n-1}\lambda_x y_{x-n+2} = A u_x + {}^1A\, {}^1u_x + \ldots + {}^{n-1}A\, {}^{n-1}u_x;$$

therefore

$$\lambda_x y_{x+1} + {}^1\lambda_x y_{x+2} + \ldots + {}^{n-1}\lambda_x y_{x-n+2} = y_x,$$

an equation which must be an identity; hence,

$$\begin{aligned} y_x = {} & u_x \left(A + \sum \frac{X_{x+2}}{z_{x+2}} \right) \\ & + {}^1u_x \left({}^1A + \sum \frac{X_{x+2}}{{}^1z_{x+2}} \right) \\ & + \dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

One will find similarly

$$\begin{aligned} y_x = {} & u_x \left(A + \sum \frac{X_{x+3}}{z_{x+3}} \right) \\ & + {}^1u_x \left({}^1A + \sum \frac{X_{x+3}}{{}^1z_{x+3}} \right) \\ & + \dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

and thus in sequence until one arrives to this last equation inclusively,

$$y_x = u_x\left(A + \sum \frac{X_{x+n}}{z_{x+n}}\right) + {}^1u_x\left({}^1A + \sum \frac{X_{x+n}}{{}^1z_{x+n}}\right) + \dots\dots\dots\dots$$

All these equations being the complete integral of equation (B) are identically the same; in comparing them together, one will form the following equations

$$\frac{u_x}{z_{x+1}} + \frac{{}^1u_x}{{}^1z_{x+1}} + \dots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+1}} = 0.$$

$$\frac{u_x}{z_{x+2}} + \frac{{}^1u_x}{{}^1z_{x+2}} + \dots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+2}} = 0.$$

$$\dots\dots\dots\dots\dots\dots,$$

$$\frac{u_x}{z_{x+n-1}} + \frac{{}^1u_x}{{}^1z_{x+n-1}} + \dots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+n-1}} = 0.$$

IX.

The integration of equation (B) of Problem II being reduced to the integration of this same equation when $X_x = 0$, there is no longer a question to resolve the problem but to integrate this here, but this appears very difficult in general; thus I will limit myself to the particular cases. Here is one quite expanded of it, in which the integration succeeds, and which embraces all the cases already known; it is the one in which one has

$$(\mathrm{B}') \qquad y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \dots + {}^{n-1}C\phi_x\phi_{x-1}\dots\phi_{x-n+1}y_{x-n}.$$

If $\phi_x = 1$, one will have the equation of the recurrent sequences.

Equation (E) of Article IV becomes in this case

$$(\mathrm{E}') \qquad 0 = 1 - \frac{C\phi_x}{\alpha_x} - \frac{{}^1C\phi_x\phi_{x-1}}{\alpha_x\alpha_{x-1}} - \dots - \frac{{}^{n-1}C\phi_x\phi_{x-1}\dots\phi_{x-n+1}}{\alpha_x\dots\alpha_{x-n+1}}.$$

Now (Art. IV), it suffices in order to integrate equation (B′) to know a number n of values for α_x in equation (E′). Let therefore $\alpha_x = a\phi_x$, a being constant, and equation (E′) will give

$$(h) \qquad a^n = Ca^{n-1} + {}^1Ca^{n-2} + {}^2Ca^{n-3} + \dots + {}^{n-1}C;$$

whence one will have a number n of values for a, and consequently for α_x, since $\alpha_x = a\phi_x$.

Let p, ${}^1p$, ${}^2p$, $\dots$, ${}^{n-1}p$ be the different values of a in equation (h). One will have (Art. IV)

$$\delta_x = p\phi_x, \quad {}^1\delta_x = {}^1p\phi_x, \quad {}^2\delta_x = {}^2p\phi_x, \quad \dots$$

Now one has (Art. V)

$$\begin{aligned} u_x &= \nabla \delta_x = \phi_1 \phi_2 \phi_3 \ldots \phi_x p^x, \\ {}^1u_x &= \nabla^1 \delta_x = \phi_1 \phi_2 \phi_3 \ldots \phi_x {}^1p^x, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

The complete integral of equation (B′) is therefore

$$y_x = \phi_1 \phi_2 \phi_3 \ldots \phi_x (Ap^x + {}^1A\,{}^1p^x + \ldots + {}^{n-1}A\,{}^{n-1}p^x).$$

One will determine the arbitrary constants A, ${}^1A$, ${}^2A, \ldots$ by means of n values of y_x, under as many particular assumptions for x. Let

$$y_1 = M, \quad y_2 = {}^1M, \quad \ldots, \quad y_n = {}^{n-1}M;$$

and one will have

$$\begin{aligned} \frac{M}{\phi_1} &= Ap + {}^1A\,{}^1p + {}^2A\,{}^2p + \ldots + {}^{n-1}A\,{}^{n-1}p, \\ \frac{{}^1M}{\phi_1\phi_2} &= Ap^2 + {}^1A\,{}^1p^2 + {}^2A\,{}^2p^2 + \ldots + {}^{n-1}A\,{}^{n-1}p^2, \\ \frac{{}^2M}{\phi_1\phi_2\phi_3} &= Ap^3 + {}^1A\,{}^1p^3 + {}^2A\,{}^2p^3 + \ldots + {}^{n-1}A\,{}^{n-1}p^3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \frac{{}^{n-1}M}{\phi_1\phi_2\ldots\phi_n} &= Ap^n + {}^1A\,{}^1p^n + {}^2A\,{}^2p^n + \ldots + {}^{n-1}A\,{}^{n-1}p^n, \end{aligned}$$

In order to resolve these equations, one can make use of the ordinary methods of elimination: but here is one of them which appears to me simpler.

I multiply the first equation by ${}^{n-1}p$, and I subtract it from the second; I multiply similarly the second by ${}^{n-1}p$, and I subtract it from the third, and thus in sequence, this which produces the following equations:

$$\begin{aligned} &\frac{{}^1M}{\phi_1\phi_2} - \frac{M}{\phi_1}\,{}^{n-1}p = Ap(p - {}^{n-1}p) + {}^1A\,{}^1p({}^1p - {}^{n-1}p) + \ldots + {}^{n-2}A\,{}^{n-2}p({}^{n-2}p - {}^{n-1}p), \\ &\frac{{}^2M}{\phi_1\phi_2\phi_3} - \frac{{}^1M}{\phi_1\phi_2}\,{}^{n-1}p = Ap^2(p - {}^{n-1}p) + {}^1A\,{}^1p^2({}^1p - {}^{n-1}p) + \ldots + {}^{n-2}A\,{}^{n-2}p^2({}^{n-2}p - {}^{n-1}p), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\frac{{}^{n-1}M}{\phi_1\ldots\phi_n} - \frac{{}^{n-2}M}{\phi_1\ldots\phi_{n-1}}\,{}^{n-1}p = Ap^{n-1}(p - {}^{n-1}p) + \ldots + {}^{n-2}A\,{}^{n-2}p^{n-1}({}^{n-2}p - {}^{n-1}p), \end{aligned}$$

I multiply again the first of these equations by ${}^{n-2}p$, and I subtract it from the second; I multiply similarly the second by ${}^{n-2}p$, and I subtract it from the third, this which gives

$$\begin{aligned}
&\frac{{}^2M}{\phi_1\phi_2\phi_3} - \frac{{}^1M}{\phi_1\phi_2}({}^{n-1}p - {}^{n-2}p) + \frac{M}{\phi_1}{}^{n-1}p^{n-2}p \\
=&Ap(p - {}^{n-1}p)(p - {}^{n-2}p) \\
&+ {}^1A{}^1p({}^1p - {}^{n-1}p)({}^1p - {}^{n-2}p) \\
&+ \dots\dots\dots\dots\dots\dots\dots \\
&+ {}^{n-3}A{}^{n-3}p({}^{n-3}p - {}^{n-1}p)({}^{n-3}p - {}^{n-2}p), \\
&\frac{{}^3M}{\phi_1\phi_2\phi_3\phi_4} - \frac{{}^2M}{\phi_1\phi_2\phi_3}({}^{n-1}p - {}^{n-2}p) + \frac{{}^1M}{\phi_1\phi_2}{}^{n-1}p^{n-3}p \\
=&Ap^2(p - {}^{n-1}p)(p - {}^{n-2}p) \\
&+ \dots\dots\dots\dots\dots\dots\dots \\
&+ {}^{n-3}A{}^{n-3}p^2({}^{n-3}p - {}^{n-1}p)({}^{n-3}p - {}^{n-2}p), \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots ;
\end{aligned}$$

by operating on these last equations, as on the previous, one will have

$$\begin{aligned}
&\frac{{}^3M}{\phi_1\phi_2\phi_3\phi_4} - \frac{{}^2M}{\phi_1\phi_2\phi_3}({}^{n-1}p - {}^{n-2}p + {}^{n-3}p) \\
&\quad + \frac{{}^1M}{\phi_1\phi_2}[({}^{n-2}p + {}^{n-1}p){}^{n-3}p + {}^{n-1}p^{n-2}p] - \frac{M}{\phi_1}{}^{n-1}p^{n-2}p^{n-3}p \\
&= Ap(p - {}^{n-1}p)(p - {}^{n-2}p)(p - {}^{n-3}p) + \dots,
\end{aligned}$$

and thus in sequence.

Thence it is easy to conclude that, if one names:
f the sum of the quantities ${}^1p$, ${}^2p$, ${}^3p$, …, ${}^{n-1}p$,
h the sum of their products two by two,
i the sum of their products three by three,
q the sum of their products four by four, etc.,
${}^1f$ the sum of the quantities p, ${}^2p$, ${}^3p$, …, ${}^{n-1}p$,
${}^1h$ the sum of their products two by two,
${}^1i$ the sum of their products three by three, etc.,
and thus in sequence, one will have

$$\begin{aligned}
A &= \frac{{}^{n-1}M - \phi_n f{}^{n-2}M + \phi_n\phi_{n-1}h{}^{n-3}M - \phi_n\phi_{n-1}\phi_{n-2}i{}^{n-4}M + \dots}{\phi_1\phi_2\phi_3\dots\phi_n p(p - {}^1p)(p - {}^2p)(p - {}^3p)\dots}, \\
{}^1A &= \frac{{}^{n-1}M - \phi_n{}^1f{}^{n-2}M + \phi_n\phi_{n-1}{}^1h{}^{n-3}M - \dots}{\phi_1\phi_2\phi_3\dots\phi_n{}^1p({}^1p - p)({}^1p - {}^2p)({}^1p - {}^3p)\dots} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

One can determine in a quite simple manner the quantities $f, h, i, q, {}^1f, {}^1h, {}^1i, {}^1q, \ldots$; I take for this the equation

$$(h) \qquad a^n - Ca^{n-1} - {}^1C^{n-2} - \ldots - {}^{n-1}C = 0;$$

I divide it by $a-p$, and the resulting equation will be

$$a^{n-1} - fa^{n-2} - ha^{n-3} - ia^{n-4} + qa^{n-5} + \ldots = 0.$$

I multiply this result by $a-p$, and I will have the following equation

$$a^n - (p+f)a^{n-1} + (pf+h)a^{n-2} - (ph+i)a^{n-3} + \ldots = 0;$$

I compare it with equation (h), and I conclude from it

$$\begin{aligned} f &= +C - p, \\ h &= -{}^1C - pf, \\ i &= +{}^2C - ph, \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

and, consequently,

$$\begin{aligned} {}^1f &= +C - {}^1p, \\ {}^1h &= -{}^1C - {}^1p.{}^1f, \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

I have supposed until here that all the roots of equation (h) are unequal, but it can happen that one or many of these roots are equal among themselves; here is in this case the method that it is necessary to follow.

I suppose that one has $p = {}^1p$; one will make ${}^1p = p + dp$, and the equation

$$y_x = \phi_1\phi_2\phi_3\ldots\phi_x(Ap^x + {}^1A\,{}^1p^x + {}^2A\,{}^2p^x + \ldots + {}^{n-1}A\,{}^{n-1}p^x)$$

will give, by reducing $(p+dp)^x$ into series,

$$y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[A + {}^1A\left(1 + \frac{xdp}{p} + \frac{x(x-1)}{1.2}\frac{dp^2}{p^2} + \ldots\right)\right] + {}^2A\,{}^2p^x + \ldots\right\}.$$

Let

$$A + {}^1A = B \quad \text{and} \quad {}^1A\frac{dp}{p} = D,$$

B and D being some arbitrary and finite constants; ${}^1A$ will be therefore infinitely great of order $\frac{1}{dp}$; ${}^1A\frac{dp^2}{p^2}$, ${}^1A\frac{dp^3}{p^3}, \ldots$ will be infinitely small. Hence

$$y_x = \phi_1\phi_2\ldots\phi_x[p^x(B+Dx) + {}^2A\,{}^2p^x + {}^3A\,{}^3p^x + \ldots].$$

If, moreover, one has $p = {}^2p$, one will make ${}^2p = p + dp$ in this expression of y_x, and one will have

$$y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[B + {}^2A + \left(D + {}^2A\frac{dp}{p}\right)x + {}^2A\frac{dp^2}{p^2}\frac{x(x-1)}{1.2} + \ldots\right] + {}^3A\,{}^3p^x + \ldots\right\}.$$

Let

$$^2A + B = {}^1B, \qquad D + {}^2A\frac{dp}{p} = {}^1D \qquad \text{and} \qquad {}^2\mathrm{A}\frac{dp^2}{p^2} = {}^1E,$$

^1B, ^1D and ^1E being some arbitrary and finite constants; one will have

$$y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[{}^1B + {}^1Dx + {}^1E\frac{x(x-1)}{1.2} + \ldots\right] + {}^3A\,{}^3p^x + \ldots\right\};$$

if moreover one had $p = {}^3p$, one would have

$$y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[{}^2B + {}^2Dx + {}^2E\frac{x(x-1)}{1.2} + {}^2F\frac{x(x-1)(x-2)}{1.2.3}\right] + {}^4A\,{}^4p^x + \ldots\right\},$$

and thus in sequence; one would determine the arbitrary constants, at least of n particular values of y_x.

If equation (h) has two imaginary roots p and ^1p, one will make

$$p = a + b\sqrt{-1} \qquad \text{and} \qquad {}^1p = a - b\sqrt{-1}.$$

Let

$$\frac{a}{\sqrt{aa+bb}} = \cos q \qquad \text{and} \qquad \frac{b}{\sqrt{aa+bb}} = \sin q;$$

one will have

$$\begin{aligned} Ap^x + {}^1A\,{}^1p^x &= (aa+bb)^{\frac{x}{2}}[A(\cos q + \sqrt{-1}\sin q)^x + {}^1A(\cos q - \sqrt{-1}\sin q)^x] \\ &= (aa+bb)^{\frac{x}{2}}[(A + {}^1A)\cos qx + (A - {}^1A)\sqrt{-1}\sin qx)^x] \end{aligned}$$

because

$$(\cos q \pm \sqrt{-1}\sin q)^x = \cos qx \pm \sqrt{-1}\sin qx.$$

Let

$$A + {}^1A = B \quad \text{and} \quad (A - {}^1A)\sqrt{-1} = {}^1B,$$

B and ^1B being reals; one will have

$$Ap^x + {}^1A\,{}^1p^x = (aa+bb)^{\frac{x}{2}}(B\cos qx + {}^1B\sin qx);$$

one will have therefore then

$$y^x = \phi_1\phi_2\ldots\phi_x\left[(aa+bb)^{\frac{x}{2}}(B\cos qx + {}^1B\sin qx) + {}^2A\,{}^2p^x + \ldots\right];$$

it will be the same process if there were a greater number of imaginaries.

If one supposes, in the preceding calculations, $\phi_x = 1$, one will have the case of the recurrent sequences. Thence results this theorem:

If one names Y_x the general term of a recurrent sequence, such that one has

$$Y_x = CY_{x-1} + {}^1CY_{x-2} + \ldots + {}^{n-1}CY_{x-n},$$

the general term of a sequence such that one has

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \ldots + {}^{n-1}C\phi_x \ldots \phi_{x-n+1}y_{x-n},$$

and in which the arbitrary constants which arrive by integrating are the same as in the preceding, will be

$$y_x = \phi_1\phi_2 \ldots \phi_x Y_x.$$

This is it of which it is easy to be assured besides; because, if one substitutes this value of y_x into the equation

$$y_x = C\phi_x y_{x-1} + \ldots,$$

one will have

$$\phi_1\phi_2 \ldots \phi_x Y_x = C\phi_1\phi_2 \ldots \phi_x Y_{x-1} + \ldots,$$

hence

$$Y_x = CY_{x-1} + {}^1CY_{x-2} + \ldots,$$

an equation which holds by assumption.

X.

When one has, by the preceding article, the integral of the equation

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \ldots + {}^{n-1}C\phi_x \ldots \phi_{x-n+1}y_{x-n} + X_x,$$

by supposing $X_x = 0$, it is easy to conclude this same integral, X_x being anything. For this, I observe that, since, X_x being null, one has

$$y_x = \phi_1\phi_2 \ldots \phi_x(Ap^x + {}^1A\,{}^1p^x + \ldots\, {}^{n-1}A\,{}^{n-1}p^x),$$

one will have, by Article V,

$$\begin{aligned} u_x &= \phi_1\phi_2\phi_3 \ldots \phi_x p^x, \\ {}^1u_x &= \phi_1\phi_2\phi_3 \ldots \phi_x\,{}^1p^x, \\ {}^2u_x &= \phi_1\phi_2\phi_3 \ldots \phi_x\,{}^2p^x, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

whence one will conclude, by Article VII,

$$\begin{aligned}
\overset{1}{u}_x &= \phi_1\phi_2\dots\phi_x p^x \Delta \frac{{}^1p^{x-1}}{p^{x-1}} = \phi_1\phi_2\dots\phi_x({}^1p-p)\,{}^1p^{x-1},\\
{}^1\overset{1}{u}_x &= \phi_1\phi_2\dots\phi_x({}^2p-p)^2p^{x-1},\\
{}^2\overset{1}{u}_x &= \phi_1\phi_2\dots\phi_x({}^3p-p)^3p^{x-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots,\\
\overset{2}{u}_x &= \phi_1\phi_2\dots\phi_x({}^2p-p)({}^2p-{}^1p)^2p^{x-2},\\
{}^1\overset{2}{u}_x &= \phi_1\phi_2\dots\phi_x({}^3p-p)({}^3p-{}^1p)^3p^{x-2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
\overset{3}{u}_x &= \phi_1\phi_2\dots\phi_x({}^3p-p)({}^3p-{}^1p)({}^3p-{}^2p)^3p^{x-3},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

and thus in sequence, hence

$$\overset{n-1}{u}_{x+1} = {}^{n-1}z_{x+1} = \phi_1\phi_2\dots\phi_{x+1}({}^{n-1}p-p)({}^{n-1}p-{}^1p)({}^{n-1}p-{}^2p)\dots{}^{n-1}p^{x-n+2};$$

similarly

$$\begin{gathered}
{}^{n-2}z_{x+1} = \phi_1\phi_2\dots\phi_{x+1}({}^{n-2}p-p)({}^{n-2}p-{}^1p)\dots{}^{n-2}p^{x-n+2},\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots;
\end{gathered}$$

whence one will conclude, by substituting these values into formula (H) of article VII and making $X_x = \phi_1\phi_2\dots\phi_x\,{}^1X_x$ for brevity,

$$\begin{aligned}
y_x = &\frac{\phi_1\phi_2\dots\phi_x}{(p-{}^1p)(p-{}^2p)(p-{}^3p)\dots}p^{x+n-1}\left(G+\sum\frac{{}^1X_{x+1}}{p^{x+1}}\right)\\
&+\frac{\phi_1\phi_2\dots\phi_x}{({}^1p-p)({}^1p-{}^2p)\dots}{}^1p^{x+n-1}\left({}^1G+\sum\frac{{}^1X_{x+1}}{{}^1p^{x+1}}\right)\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

If $p={}^1p$, one will make ${}^1p=p+dp$. Let $K=\frac{1}{(p-{}^2p)(p-{}^3p)\dots}$, and one will have

$$\begin{aligned}
y_x = &\phi_1\phi_2\dots\phi_x p^{x+n-1}\left\{B+Dx-\frac{K}{p}\sum\frac{{}^1X_{x+1}}{p^{x+1}}(x+1)+\left[\frac{dK}{dp}+\frac{K}{p}(x+n-1)\right]\sum\frac{{}^1X_{x+1}}{p^{x+1}}\right\}\\
&+\frac{\phi_1\phi_2\dots\phi_x}{({}^2p-p)^2({}^2p-{}^3p)\dots}{}^2p^{x+n-1}\left({}^2G+\sum\frac{{}^1X_{x+1}}{{}^2p^{x+1}}\right),
\end{aligned}$$

B and D being two arbitrary constants.

If, moreover, one has $p = {}^2p$, one will make, in this last expression of y_x, ${}^2p = p + dp$, and thus in sequence.

One can therefore integrate generally all the differential equations contained in the following formula

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \ldots + \mathrm{X}_x;$$

whence it results that, if one designates by θ_x any function whatsoever of x, the following equation

$$\theta_x y_x = C\theta_{x-1}\phi_x y_{x-1} + {}^1C\theta_{x-2}\phi_x\phi_{x-1}y_{x-2} + \ldots + \mathrm{X}_x$$

is generally integrable, since by making $\theta_x y_x = t_x$ this equation is of the same form as the preceding.

XI.

Here is now another kind of linear differential equations, of which the order depends on the variable x; let, for example,

$$\begin{aligned} y_x = & a_{x-1}y_{x-1} + b_{x-2}y_{x-2} + f_{x-3}y_{x-3} + X_x \\ & + a_{x-4}y_{x-4} + b_{x-5}y_{x-5} + f_{x-6}y_{x-6} \\ & + a_{x-7}y_{x-7} + b_{x-8}y_{x-8} + \ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_3y_3 + b_2y_2 + f_1y_1. \end{aligned}$$

It is easy to bring these equations back to the form of equation (B) of problem II, because one has

$$\begin{aligned} y_{x-3} = & a_{x-4}y_{x-4} + b_{x-5}y_{x-5} + f_{x-6}y_{x-6} + X_{x-3} \\ & + a_{x-7}y_{x-7} + b_{x-8}y_{x-8} + \ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_3y_3 + b_2y_2 + f_1y_1. \end{aligned}$$

If one subtracts this last equation from the preceding, one will have

$$y_x = a_{x-1}y_{x-1} + b_{x-2}y_{x-2} + (f_{x-3} + 1)y_{x-3} + X_x - X_{x-3},$$

an equation contained in equation (B).

XII.

Presently here is a quite extended use of the integral Calculus in the finite differences, in order to determine directly the general expression of the quantities subject to a certain law which serves to form them, an expression that until here it seems to me that one has always sought to draw by way of induction, a method not only indirect, but which, moreover, must be often at fault.

In order to make myself better understood, I take the following example:

Let x be the sine of an angle z and u its cosine; one has generally, as one knows,

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

whence one draws

$$\begin{aligned}
&\sin z = x,\\
&\sin 2z = x(2u),\\
&\sin 3z = x(4u^2-1),\\
&\sin 4z = x(8u^3-4u),\\
&\sin 5z = x(16u^4-12u^2+1),\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

It is necessary now to determine the general expression of $\sin nz$.

One can arrive by way of induction, by continuing further these expressions and seeking to discover the law of the different coefficients of the powers of u; but it will happen, if it is not in this example, at least in an infinity of others, that this law will be very complicated and very difficult to grasp: it matters consequently to have a general and sure method in order to find it in all the possible cases.

Let, for this, the differential equation be

$$(\nabla)\qquad y_x = \begin{cases} y_n = y_{n-1}(a_n u + b_n) \\ \qquad + y_{n-2}({}^1a_n u^2 + {}^1b_n u + {}^1c_n) \\ \qquad + y_{n-3}({}^2a_n u^3 + {}^2b_n u^2 + {}^2c_n u + {}^2f_n) \\ \qquad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

I suppose that one has

$$\begin{aligned}
&y_1 = \alpha u + \beta,\\
&y_2 = \delta u^2 + \gamma u + \Omega,\\
&y_3 = \varpi u^3 + \pi u^2 + \theta u + \sigma,\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

Here is how I conclude the general expression of y_n.

I make

$$y_n = A_n u^n + B_n u^{n-1} + C_n u^{n-2} + \dots,$$

hence,

$$\begin{aligned}
y_{n-1} &= A_{n-1}u^{n-1} + B_{n-1}u^{n-2} + C_{n-1}u^{n-3} + \dots,\\
y_{n-2} &= A_{n-2}u^{n-2} + B_{n-2}u^{n-3} + C_{n-2}u^{n-4} + \dots,
\end{aligned}$$

and thus in sequence; if one substitutes these values of $y_{n-1}, y_{n-2}, \dots$ into equation (∇), one will have

$$\begin{aligned}
y_n =& u^n(a_nA_{n-1} + {}^1a_nA_{n-2} + {}^2a_nA_{n-3} + \dots \\
&+ u^{n-1}(a_nB_{n-1} + {}^1a_nB_{n-2} + {}^2a_nB_{n-3} + \dots \\
&\qquad + b_nA_{n-1} + {}^1b_nA_{n-2} + {}^2b_nA_{n-3} + \dots) \\
&+ u^{n-2}(a_nC_{n-1} + {}^1a_nC_{n-2} + {}^2a_nC_{n-3} + \dots \\
&\qquad + b_nB_{n-1} + {}^1b_nB_{n-2} + {}^2b_nB_{n-3} + \dots \\
&\qquad + {}^1c_nA_{n-2} + {}^2c_nA_{n-3} + {}^2c_nA_{n-4} + \dots) \\
&\qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

By comparing this expression of y_n with the preceding, one will have the following equations

$$\begin{aligned}
A_n =& a_nA_{n-1} + {}^1a_nA_{n-2} + {}^2a_nA_{n-3} + \dots, \\
B_n =& a_nB_{n-1} + {}^1a_nB_{n-2} + {}^2a_nB_{n-3} + \dots \\
&\quad + b_nA_{n-1} + {}^1b_nA_{n-2} + {}^2b_nA_{n-3} + \dots, \\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

by means of which one will determine, by the preceding methods, A_n, B_n, …, and one will have thus the general expression of y_n.

I suppose that one wishes to have the general expression of $\sin nz$; it is easy to see, by that which precedes, that it will have this form

$$\sin nz = x(A_nu^{n-1} + B_nu^{n-3} + C_nu^{n-5} + D_nu^{n-7} + \dots);$$

therefore

$$\begin{aligned}
\sin(n-1)z &= x(A_{n-1}u^{n-2} + B_{n-1}u^{n-4} + C_{n-1}u^{n-6} + \dots) \\
\sin(n-2)z &= x(A_{n-2}u^{n-3} + B_{n-2}u^{n-5} + C_{n-2}u^{n-7} + \dots).
\end{aligned}$$

If one substitutes these values of $\sin(n-1)z$ and $\sin(n-2)z$ into the equation

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

one will have

$$\sin nz = x(2A_{n-1}u^{n-1} + 2B_{n-1}u^{n-3} + 2C_{n-1}u^{n-5} + \dots - A_{n-2}u^{n-3} - B_{n-2}u^{n-5} - \dots)$$

and, if one compares this expression with the preceding, one will have

$$(\Lambda) \qquad \begin{cases} A_n = 2A_{n-1}, \\ B_n = 2B_{n-1} - A_{n-2}, \\ C_n = 2C_{n-1} - B_{n-2}, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

By means of these equations one will determine $A_n, B_n, C_n, \dots$, but one must make here an observation in which it is necessary to pay attention to all the researches which depend on the integral Calculus in the finite differences; that which renders its use very delicate. This observation consists in this that the preceding equations (Λ) begin to exist not at all immediately, that is to say when n has one same value in these equations. In order to demonstrate, I observe that the fundamental equation

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

by means of which I have concluded $\sin 2z, \sin 3z, \sin 4z, \ldots$, suppose known the first two sines $\sin 0z$ and $\sin 1z$; it can therefore begin to take place only when $n=2$; hence also, equations (Λ) can begin to exist only when $n=2$. The first of these equations begin to exist when $n=2$, in which case one has $A_2 = 2A_1$; thus, the smallest index of A_n, that is to say the least value that n can have in this expression, is unity; the second equation can therefore begin to take place only when $n=3$, in which case one has $B_3 = 2B_2 - A_1$; hence, the least index of B_n is 2; the third equation can therefore begin to take place only when $n=4$, in which case one has $C_4 = 2C_3 - B_2$; hence, the smallest index of C_n is 3, and thus in sequence. This put:

If one integrates the first equation, one will have

$$A_n = 2^n H,$$

H being arbitrary; now, putting $n=1, A_n=1$, whence $H=\frac{1}{2}$, one has $A_n = 2^{n-1}$, hence $A_{n-2} = 2^{n-3}$. If one substitutes this value of A_{n-2} into the second equation and if next one integrates it; one will have

$$B_n = -2^{n-3}(n+H);$$

since the differential equation in B_n commences to exist when $n=3$, the arbitrary constant H must be determined by the value of B_n, when $n=2$; now, u not being able to have a negative exponent in the expression of $\sin nz$, it follows that $B_2 = 0$, hence $H = -2$; therefore

$$B_n = -2^{n-3}(n-2) \quad \text{and} \quad B_{n-2} = -2^{n-5}(n-4).$$

If one substitutes this value of B_{n-2} into the third equation, and if next one integrates it, one will have

$$C_n = 2^{n-5}\left(\frac{n^2-7n}{2} + H\right)$$

now, putting $n=3$, $C_n = 0$, whence $H=6$, one has $C_n = 2^{n-5}\frac{(n-3)(n-4)}{1.2}$, and thus to infinity. Therefore

$$\sin nz = x\left[2^{n-1}u^{n-1} - \frac{n-2}{1}2^{n-3}u^{n-3} + \frac{(n-3)(n-4)}{1.2}2^{n-5}u^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.2.3}2^{n-7}u^{n-7} + \ldots\right].$$

Let next $z =$ angle $\sin x$; one will have, by differentiating,

$$\frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}},$$

and I wish to have the general expression of $\frac{d^n z}{dx^n}$, dx being supposed constant. For this, let $u = \frac{1}{\sqrt{1-x^2}}$; one will have

$$\begin{aligned}
\frac{du}{dx} &= \frac{x}{(1-x^2)^{\frac{3}{2}}},\\
\frac{d^2u}{dx^2} &= \frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}},\\
\frac{d^3u}{dx^3} &= \frac{6x^3+9x}{(1-x^2)^{\frac{7}{2}}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

It is easy to see, by considering the law of these expressions of du, d^2u, …, that the general expression of $\frac{d^n u}{dx^n}$ has the following form

$$\frac{d^n u}{dx^n} = \frac{A_n x^n + B_n x^{n-2} + C_n x^{n-4} + D_n x^{n-6} + \ldots}{(1-x^2)^{n+\frac{1}{2}}};$$

by differentiating this expression, one has

$$\frac{d^{n+1}u}{dx^{n+1}} = \frac{\begin{array}{r|r|r|r} (n+1)A_n x^{n+1} + (n+3)B_n & x^{n-1} + (n+5)C_n & x^{n-3} + (n+7)D_n & x^{n-5} + \ldots \\ +nA_n & +(n-2)B_n & +(n-4)C_n & +\ldots \end{array}}{(1-x^2)^{n+\frac{3}{2}}}$$

but one has

$$\frac{d^{n+1}u}{dx^{n+1}} = \frac{A_{n+1}x^{n+1} + B_{n+1}x^{n-1} + C_{n+1}x^{n-3} + D_{n+1}x^{n-5} + \ldots}{(1-x^2)^{n+\frac{3}{2}}};$$

by comparing these two expressions of $\frac{d^{n+1}u}{dx^{n+1}}$, one will have the following equations:

$$\begin{aligned}
A_{n+1} &= (n+1)A_n,\\
B_{n+1} &= (n+3)B_n + nA_n,\\
C_{n+1} &= (n+5)C_n + (n-2)B_n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

All these equations begin to exist immediately and when $n = 1$; this put, the first gives

$$A_n = 1.2.3\ldots n;$$

the second gives

$$B_n = 1.2.3\ldots n(n+1)(n+2)\left[H + \sum \frac{n}{(n+1)(n+2)(n+3)}\right],$$

or

$$B_n = 1.2.3\ldots n(n+1)(n+2)\left[Q + \frac{1}{2}\frac{1}{(n+1)(n+2)} - \frac{1}{n+2}\right].$$

One will determine the constant Q by this condition that B_n is zero when $n = 1$; one has therefore $Q = \frac{1}{2.2}$. Therefore

$$B_n = 1.2.3\ldots n\,\frac{1}{2}\,\frac{n(n-1)}{1.2}.$$

The third equation gives, by integrating and adding the appropriate constants,

$$C_n = 1.2.3\ldots n\,\frac{1.3}{2.4}\,\frac{n(n-1)(n-2)(n-3)}{1.2.3.4};$$

one will find similarly

$$D_n = 1.2.3\ldots n\,\frac{1.3.5}{2.4.6}\,\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6},$$

and thus in sequence. Hence

$$\begin{aligned}\frac{d^n z}{dx^n} = \frac{1.2.3\ldots(n-1)}{(1-x^2)^{n-\frac{1}{2}}}\Bigg[x^{n-1} &+ \frac{1}{2}\,\frac{(n-1)(n-2)}{1.2}\,x^{n-3}\\ &+ \frac{1.3}{2.4}\,\frac{(n-1)(n-2)(n-3)(n-4)}{1.2.3.4}x^{n-5}\\ &+ \frac{1.3.5}{2.4.6}\,\frac{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{1.2.3.4.5.6}x^{n-7}\\ &+ \frac{1.3.5.7}{2.4.6.8}\,\frac{(n-1)(n-2)\ldots(n-8)}{1.2.3\ldots 8}x^{n-9}\\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Bigg].\end{aligned}$$

I have supposed, in the two preceding examples, the law of the exponents known, because it was very easy to perceive; but, if it happened that it was complicated, this which must be extremely rare, one will be able to determine it by the preceding method.

XIII.

Here is yet a remarkable usage of the integral Calculus in the finite differences in order to determine the nature of the functions according to some given conditions, this which is often useful, principally in the Calculus of partial differences.[4]

One proposes to find a function of x such that by making successively $x = \phi(x)$ and $x = \psi(x)$, one has

$$(\sigma) \qquad f[\phi(x)] = H_x f[\psi(x)] + X_x,$$

[4]I had found this method at the end of 1772, on the occasion of some problems which Mr. Monge, skillful professor of Mathematics at the schools of the Genoese at Mézières, proposed to me; I did part of it for him then; at the same time, I sent it to Mr. de la Grange, and I have presented it to the Academy in the month of February 1773. Since this time, Mr. the marquis de Condorcet has had printed in the Volume of the Academy for the year 1771 a quite beautiful Memoir on this object; but the route which I have differs from his in this that he does not propose, as I do it, to restore the question to the differential equations of which the difference is constant and equal to unity. *Translator's note*: On 10 March and 17 March 1773, as reported in the Procès-Verbaux of the Paris Academy, Laplace read the paper "Recherches sur l'integration des differentielles aux différences finies et sur leur application à l'analyse des hasards."

$\phi(x)$, $\psi(x)$, H_x being some given functions of x.

For this let

$$u_z = \psi(x) \quad \text{and} \quad u_{z+1} = \phi(x).$$

From the first of these equations, I conclude

$$x = \Gamma(u_z) \quad \text{and} \quad \phi(x) = H(u_z),$$

$\Gamma(u_z)$ and $H(u_z)$ representing some known functions of u_z; hence,

$$u_{z+1} = H(u_z),$$

a differential equation of which the constant difference is equal to unity, and which one can integrate in many cases.

The integral of this equation will give u_z as function of z, and the equation $x = \Gamma(u_z)$ will give x as function of z. Substituting this value of x in H_x and X_x, the quantities will become some functions of z, which I designate by L_z and Z_z. Moreover, one has

$$f[\phi(x)] = f(u_{z+1}) \quad \text{and} \quad f[\psi(x)] = f(u_z);$$

equation (σ) will become therefore, by supposing $f(u_z) = y_z$,

$$y_{z+1} = L_z y_z + Z_z,$$

an equation integrable by Problem I.

One must observe here, consistent with a remark due to Mr. Euler, that the constants which come by integrating the finite differential equations of which the variable is z, and of which the constant difference is unity, can be supposed some functions any whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, π expressing the ratio of the circumference to the diameter.

Presently, if one puts back into the expression of y_z instead of z its value in x, one will have $f[\psi(x)]$, and, if one changes $\psi(x)$ into x, one will have the function of x, which satisfies the Problem. The following examples clarify this method:

The question is to find a function of x such that by changing successively x into x^q and into mx, one has

$$f(x^q) = f(mx) + p,$$

m and p being constants.

I make $u_z = mx$, and $u_{z+1} = x^q$; hence,

$$u_{z+1} = \left(\frac{u_z}{m}\right)^q.$$

In order to integrate this equation, I make $u_1 = a$; therefore $u_2 = \frac{a^q}{m^q}$, $u_3 = \frac{a^{q^2}}{m^{q^2+q}}, \ldots$ Let $u_z = \frac{a^{g_z}}{m^{f_z}}$; therefore

$$u_{z+1} = \frac{a^{qg_z}}{m^{qf_z+q}} = \frac{a^{g_{z+1}}}{m^{f_{z+1}}}.$$

Therefore

$$g_{z+1} = qg_z,$$

this which gives

$$g_z = Aq^z.$$

Now, putting $z=2$, $g_z=q$, whence $A=\frac{1}{q}$, one has $g_z=q^{z-1}$. Moreover, one has $f_{z+1}=qf_z+q$. Therefore $f_z=Aq^z+\frac{q}{1-q}$. Now, putting $z=2$, $f_z=q$; therefore $A=\frac{1}{q-1}$ and $f_z=\frac{1}{q-1}(q^z-q)$; therefore

$$u_z = \frac{a^{q^{z-1}}}{m^{\frac{1}{q-1}(q^z-q)}}.$$

This expression of u_z is complete, since a is arbitrary; now the equation

$$f(x^q) = f(mx) + p$$

will become

$$y_{z+1} = y_z + p.$$

Therefore

$$y_z = C + pz = f(mx).$$

It is necessary presently to have the value of z in x; now, since one has $u_z = mx$, one will have

$$mx = \frac{a^{q^{z-1}}}{m^{\frac{1}{q-1}(q^z-q)}},$$

whence one draws[5]

$$lmx = q^z \frac{la}{q} - \frac{1}{q-1}(q^z-q)\, lm$$

or

$$q^z\left(\frac{la}{q} - \frac{lm}{q-1}\right) = l\frac{mx}{m^{\frac{q}{q-1}}};$$

let $\frac{la}{q} - \frac{lm}{q-1} = K$, and one will find

$$z = \frac{ll\frac{mx}{m^{\frac{q}{q-1}}}}{lq} - \frac{lK}{lq},$$

hence

$$y_z = A + p\frac{ll\frac{mx}{m^{\frac{q}{q-1}}}}{lq},$$

A being an arbitrary constant which can be any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$. Let $\Gamma(\sin 2\pi z, \cos 2\pi z)$ be this function; by substituting instead of z its value, one will have

$$A = \Gamma\left(\sin 2\pi\frac{ll\frac{mx}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi\frac{ll\frac{mx}{m^{\frac{q}{q-1}}}}{lq}\right).$$

[5] *Translator's note*: Laplace uses l to denote the natural logarithm. It appears as l in this document.

Therefore

$$y_z = f(mx) = \Gamma\left(\sin 2\pi \frac{ll\frac{\frac{mx}{q}}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi \frac{ll\frac{\frac{mx}{q}}{m^{\frac{q}{q-1}}}}{lq}\right) + p\frac{ll\frac{\frac{mx}{q}}{m^{\frac{q}{q-1}}}}{lq};$$

thus the function of x demanded is

$$f(x) = \Gamma\left(\sin 2\pi \frac{ll\frac{\frac{x}{q}}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi \frac{ll\frac{\frac{x}{q}}{m^{\frac{q}{q-1}}}}{lq}\right) + p\frac{ll\frac{\frac{x}{q}}{m^{\frac{q}{q-1}}}}{lq}.$$

It is a question again to find $f(x)$ such that

$$[f(x)]^2 = f(2x) + 2.$$

One could first think that it is impossible to satisfy this equation, at least to suppose $f(x)$ equal to a constant; this is indeed that which some able geometers have believed (*see* the second Volume of the *Mémoires de Turin*, p. 320); but one is going to see there are an infinity of other ways to satisfy it.

Let

$$u_z = x \qquad \text{and} \qquad u_{z+1} = 2x;$$

therefore

$$u_{z+1} = 2u_z \qquad \text{and} \qquad u_z = A2^z = x.$$

Moreover, one has

$$f(2x) = f(u_{z+1}), \qquad \text{which I designate by} \qquad t_{z+1},$$

and

$$f(x) = f(u_z) = t_z;$$

and one will have

$$t_{z+1} = t_z^2 - 2.$$

In order to integrate this equation, I suppose $t_1 = a + \frac{1}{a}$, therefore

$$t_2 = a^2 + \frac{1}{a^2}, \quad t_3 = a^4 + \frac{1}{a^4}, \quad \ldots,$$

and generally

$$t_z = a^{2^{z-1}} + \frac{1}{a^{2^{z-1}}},$$

a complete expression of t_x, since a is arbitrary; now one has $2^{z-1} = \frac{x}{2A}$, therefore

$$t_z = a^{\frac{x}{2A}} + a^{-\frac{x}{2A}}, \quad \text{or} \quad t_z = b^x + b^{-x},$$

b being an arbitrary constant; now this constant can be supposed any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, and since $z = H + \frac{lx}{l2}$, H being any constant whatsoever, one will have

$$b = f(\sin 2\pi \frac{lx}{l2}, \cos 2\pi \frac{lx}{l2}),$$

hence the function of x demanded is

$$\left[f(\sin 2\pi\frac{lx}{l2},\cos 2\pi\frac{lx}{l2})\right]^{x}+\left[f(\sin 2\pi\frac{lx}{l2},\cos 2\pi\frac{lx}{l2})\right]^{-x}$$

It is a question again to find $f(x-y\sqrt{-1})$, such that one has

$$f(x+y\sqrt{-1})-f(x-y\sqrt{-1})=2M\sqrt{-1}.$$

By supposing $y=g+hx$, one will have

$$f[g\sqrt{-1}+x(1+h\sqrt{-1})]-f[x(1-h\sqrt{-1})-g\sqrt{-1}]=2M\sqrt{-1}.$$

Let

$$x(1+h\sqrt{-1})+g\sqrt{-1}=u_{z+1},$$
$$x(1-h\sqrt{-1})-g\sqrt{-1}=u_{z;}$$

one will have therefore

$$x=\frac{u_z+g\sqrt{-1}}{1-h\sqrt{-1}};$$

therefore

$$u_{z+1}=\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}}u_z+\frac{2g\sqrt{-1}}{1-h\sqrt{-1}},$$

an equation of which the integral is

$$u_z=A\left(\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}}\right)^2-\frac{g}{h}=x(1-h\sqrt{-1})-g\sqrt{-1};$$

hence,

$$zl\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}}=l(g+hx)+K.$$

Now, if one names $\varpi\pi$ the angle of which the tangent is h, and π the ratio of the semi-circumference to the radius, one will have

$$l\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}}=2\sqrt{-1}\varpi\pi;$$

therefore

$$z=\frac{l(g+hx)}{2\sqrt{-1}\varpi\pi}+K'.$$

Now one has

$$f(u_{z+1})-f(u_z)=2M\sqrt{-1};$$

and, by representing $f(u_z)$ by t_z,

$$t_{z+1}=t_z+2M\sqrt{-1},$$

therefore

$$t_z = H + 2Mz\sqrt{-1};$$

substituting instead of z its value, one will have

$$t_z = M\frac{l(g+hx)}{\varpi\pi} + L,$$

L being an arbitrary constant, which can be any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, or of $\sin\frac{l(g+hx)}{\varpi\sqrt{-1}}$ and of $\cos\frac{l(g+hx)}{\varpi\sqrt{-1}}$, and consequently of $e^{\frac{l(g+hx)}{\varpi}}$; now, $e^{l(g+hx)} = g+hx$; therefore L can be a function of $(g+hx)^{\frac{1}{\varpi}}$; hence

$$f(x - y\sqrt{-1}) = M\frac{l(g+hx)}{\varpi\pi} + \Gamma\left[(g+hx)\frac{1}{\varpi}\right].$$

XIV.

On the equations in finite differences, when one has many equations among many variables.

I suppose that one has the following two equations among the three variables y_x, ${}^1y_x$ and x

(1) $$y_x + A_x y_{x-1} = B_x\,{}^1y_x + C_x\,{}^1y_{x-1},$$

(2) $$y_x + {}^1A_x y_{x-1} = {}^1B_x\,{}^1y_x + {}^1C_x\,{}^1y_{x-1}.$$

The simplest way to integrate them is to reduce them by elimination to two other equations, the one between y_x and x, the other between ${}^1y_x$ and x; for this, I multiply the first by ${}^1C_x$, the second by C_x, and I subtract the one from the other; this which gives

$$({}^1C_x - C_x)y_x + ({}^1C_xA_x - C_x\,{}^1A_x)y_{x-1} = ({}^1C_xB_x - C_x\,{}^1B_x)\,{}^1y_x,$$

hence

(3) $$\begin{cases} ({}^1C_{x-1} - C_{x-1})y_{x-1} + ({}^1C_{x-1}A_{x-1} - C_{x-1}\,{}^1A_{x-1})y_{x-1} \\ \quad = ({}^1C_{x-1}B_{x-1} - C_{x-1}\,{}^1B_{x-1})\,{}^1y_{x-1}. \end{cases}$$

I multiply equation (1) by α, equation(2) by ${}^1\alpha$, and I add them with equation (3), this which gives

$$\begin{aligned}(\alpha + {}^1\alpha)y_x + (\alpha A_x + {}^1\alpha\,{}^1A_x + {}^1C_{x-1} - C_{x-1})y_{x-1} + ({}^1C_{x-1}A_{x-1} - C_{x-1}\,{}^1A_{x-1})y_{x-2} \\ = (\alpha B + {}^1\alpha\,{}^1B)\,{}^1y_x + (\alpha C_x + {}^1\alpha\,{}^1C_x + {}^1C_{x-1}B_{x-1} - C_{x-1}\,{}^1B_{x-1})\,{}^1y_{x-1};\end{aligned}$$

I make ${}^1y_x$ and ${}^1y_{x-1}$ vanish by means of the equations

$$\begin{aligned}&\alpha B_x + {}^1\alpha\,{}^1B_x = 0,\\ &\alpha C_x + {}^1\alpha\,{}^1C_x + {}^1C_{x-1}B_{x-1} - C_{x-1}\,{}^1B_{x-1} = 0,\end{aligned}$$

and I have in this manner a differential equation between y_x and x alone; by an entirely similar process, one will find one of them between ${}^1y_x$ and x; and it would be the same thing if one has a greater number of equations and of variables.

It is easy to see that, if there was in each equation some terms such as $T_x, X_x, \ldots, T_x, X_x$ being some functions any whatsoever of x, they would be integrable in the same cases where they are it, these terms not being there.

When one has $n-1$ equations among n variables, these being able to have an infinity of different relations among them, the integration of these equations presents thus a great number of curious researches; but there is a case which merits a particular attention, in this that it is encountered sometimes and principally in the analyses of chances; it is the case in which these equations return to themselves.