QUATRIÈME SUPPLÉMENT.

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§1. U being any function whatever of a variable t, if we develop it according to the powers of t, the coefficient of t^x, in this development, will be a function of x that I will designate by y_x; U is that which I have named *generating function* of y_x. If we multiply U by a function T of t, similarly developed according to the ascending powers of t, the product UT will be a new generating function of a function of x, derived from the function y_x according to a law which will depend on the function T. If T is equal to $\frac{1}{t} - 1$, it is easy to see that the derived will be $y_{x+1} - y_x$, or the finite difference of y_x. Let us designate generally, whatever be T, this derived by δy_x. If we multiply the product UT by T, the derived of the product UT^2 will be a derived of δy_x similar to the derived of δy_x in y_x; we will be able therefore to designate by $\delta^2 y_x$ this second derived; whence it is clear generally that UT^n will be the generating function of $\delta^n y_x$. [617]

If we multiply U by another function Z of t, similarly developed according to the ascending powers of t, and if we designate by the characteristic $\triangle$ that which we have named δ relative to the function T, UZ^n will be the generating function of $\triangle^n y_x$.

We are able to imagine T as a function of Z. By developing this function into series with respect to the ascending powers of Z, we will have an expression of T of this form

$$T = A^{(0)} + A^{(1)}Z + A^{(2)}Z^2 + \cdots$$

By multiplying this equation by U and passing again from the generating functions to the coefficients, we will have [618]

$$\delta y_x = A^{(0)}y_x + A^{(1)}\triangle y_x + A^{(2)}\triangle^2 y_x + \cdots$$

We see thus that the same equation, which holds between T and Z, holds between their characteristics δ and $\triangle$, provided that, in the development of this equation according to the powers of δ and of $\triangle$, we substitute, instead of any power δ^r, $\delta^r y_x$; instead of a power $\triangle^{r'}$, $\triangle^{r'} y_x$; instead of a product such as $\delta^r\triangle^{r'}$, $\delta^r\triangle^{r'} y_x$; and that we multiply by y_x the terms independent of δ and $\triangle$. Thus, by supposing T equal to $\frac{1}{t} - 1$, $Z = \frac{1}{t^i} - 1$, δy_x will be the finite difference of y_x, x varying by unity; $\triangle y_x$ will be the finite difference of y_x, x varying with i; we have next

$$Z = (1 + T)^i - 1,$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 2, 2013 and, consequently,

$$Z^n = [(1+T)^i - 1]^n;$$

that which gives

$$\triangle^n = [(1+\delta)^i - 1]^n,$$

provided that after the development we place y_x after the powers of the characteristics. This equation will hold furthermore by making n negative, but then the differences are changed into integrals. The consideration of the generating functions show thus, in the most natural and most simple manner, the analogy of the powers and of the differences. We are able to consider this theory as the calculus of characteristics.

If we have $0 = \delta y_x$, we will have an equation in the finite differences: UT becomes then a polynomial which contains only powers of t smaller than the highest of t in T. Let us designate by Q the polynomial in t the most general of this nature; we will have

$$U = \frac{Q}{T}.$$

The coefficient of t^x in the development of U will be the integral y_x of the equation $0 = \delta y_x$; by this reason, I name U generating function of this equation.

If we imagine U a function of two variables t and t', the coefficient of the product [619] $t^x t'^{x'}$, in the development of U, will be a function of x and of x' that I designate by $y_{x,x'}$; T being a function developed in the same variables t and t', the product UT will be the generating function of a derived of $y_{x,x'}$, that I will designate by $\delta y_{x,x'}$; and it is easy to conclude from it that UT^n will be the generating function of $\delta^n y_{x,x'}$.

If we have $0 = \delta y_{x,x'}$, we will have an equation in the partial finite differences. Let us represent this equation by the following

$$\begin{aligned} 0 = {} & a y_{x,x'} + b y_{x,x'+1} + c y_{x,x'+2} + \cdots, \\ & + a' y_{x+1,x'} + b' y_{x+1,x'+1} + \cdots, \\ & + a'' y_{x+1,x'} + \cdots \\ & + \cdots; \end{aligned}$$

it is easy to see that the generating function of the proposed equation will be

$$\frac{A + Bt' + Ct'^2 + \cdots + Ht'^{n'-1} + A' + B't + C't^2 + \cdots + H't^{n-1}}{\left\{ \begin{array}{l} \quad a t^n t'^{n'} + b t^n t'^{n'-1} + a t^n t'^{n'-2} + \cdots \\ + a' t^{n-1} t'^{n'} + b' t^{n-1} t'^{n'-1} + \cdots \\ + a'' t^{n-2} t'^{n'} + \cdots \\ + \cdots\cdots, \end{array} \right\}}$$

n and n' being the greatest increases of x and of x', in the proposed equation in partial differences; A, B, C, $\ldots$, H are arbitrary functions of t; A', B', C', $\ldots$, H' are arbitrary functions of t'. We will determine all these functions by means of the generating functions of

$$y_{0,x'}, y_{1,x'}, y_{2,x'}, \ldots, y_{n-1,x'},$$
$$y_{x,0}, \; y_{x,1}, \; y_{x,2}, \; \ldots, y_{x,n'-1}.$$

One of the principal advantages of this manner to integrate the equations in partial differences consists in this that, the algebraic analysis furnishing diverse ways to develop the functions, we are able to choose the one which agrees best to the proposed question. The solution of the following problems, by the count de Laplace, my son, [620]
and the considerations that he has joined will spread a new day on the calculus of generating functions.

§2. A player A draws from an urn, containing some white and black balls, one ball which he returns after the trial, with the probability p to bring forth a white ball and the probability q to extract from it a black; a second player B draws next, from another urn, a ball which he returns equally after the drawing, with the probabilities p' of a white ball and q' of a black. These two players continue thus to extract alternately, each from their respective urn, a ball which they always take care to return. If one of the players brings forth a white ball, he counts a point; if, on the contrary, he makes a black ball exit, he counts nothing, and the turn of the player passes simply to the other. The players having settled, by the conditions of their game, the number of points that each must attain first in order to win the game, and having commenced to play, there is lacking yet to player A the number x points in order to win, and x' to player B; and the turn to play belongs to player A. We demand, in this position, what is the probability of each player to win the game.

Let $z_{x,x'}$ be the probability of second player B, and let us represent by $Y_{x,x'}$ his probability, if he were the first to play. Player A, by beginning, is able to bring forth a white ball, and the probability of B becomes $Y_{x-1,x'}$; or the first player makes a black ball exit, and then counts nothing, and the probability of the second is changed into $Y_{x,x'}$; but the probability of the first case is p, that of the second q; we will have therefore the equation

$$z_{x,x'} = pY_{x-1,x'} + qY_{x,x'};$$

by a similar reasoning, we will have further this one

$$Y_{x,x'} = p'z_{x,x'-1} + q'z_{x,x'};$$

whence we deduce

$$Y_{x-1,x'} = p'z_{x-1,x'-1} + q'z_{x-1,x'},$$

and consequently[1] [621]

$$z_{x,x'} = p(p'z_{x-1,x'-1} + q'z_{x-1,x'}) + q(p'z_{x,x'-1} + q'z_{x,x'})$$

[1] We arrive again to this equation in partial differences by considering together the two successive drawings of A and B as one trial, and by examining the different cases which are able to be presented after this trial played; now they are in number of four: 1° either the two players bring forth each one white ball, an event of which the probability is pp'; then the probability $z_{x,x'}$ will be changed into this one $z_{x-1,x'-1}$; 2° or the first player extracts a white ball and the second a black; under this hypothesis, which has for probability pq', $z_{x,x'}$ will become $z_{x-1,x'}$; 3° or on the contrary the first player makes a black ball exit and the second a white; under this hypothesis, which has for probability $p'q$, $z_{x,x'}$ will become $z_{x,x'-1}$; 4° or finally each player draws a black ball, an event of which the probability is qq', and then the probability $z_{x,x'}$ remains the same. We will have therefore, by the known principles of probabilities, the equation

$$z_{x,x'} = pp'z_{x-1,x'-1} + pq'z_{x-1,x'} + p'qz_{x,x'-1} + qq'z_{x,x'}.$$

On obtains the generating function of $z_{x,x'}$, in this equation in partial differences, by applying to this case the general rule which has just been exposed.

or

$$z_{x,x'} = \frac{pq'}{1-qq'} z_{x-1,x'} + \frac{p'q}{1-qq'} z_{x,x'-1} + \frac{pp'}{1-qq'} z_{x-1,x'-1},$$

and by making

$$\frac{pq'}{1-qq'} = m, \quad \frac{p'q}{1-qq'} = m', \quad \frac{pp'}{1-qq'} = n,$$

it will become

$$z_{x,x'} = mz_{x-1,x'} + m'z_{x,x'-1} + nz_{x-1,x'-1}.$$

The generating function of $z_{x,x'}$, in this equation in partial differences, is

$$\frac{A + A'}{1 - mt - m't' - ntt'},$$

A being an arbitrary function of t, and A' another arbitrary function of t'; I observe first that by attributing to the function A' the term independent of t in the function A, the generating function above is able to be game under this form

$$\frac{A_1 t + A'_1}{1 - mt - m't' - ntt'},$$

A_1 and A'_1 being new arbitrary functions of t and of t' that it is the question to determine. Now, if we pay attention that $z_{0,x'}$ is null, whatever be x', the probability of [622]
player A is changed then to certitude, we see that the coefficient of t^0 in the development of the generating function with respect to the powers of t must be null, and we will have

$$\frac{A'_1}{1 - m't'} = 0 \qquad \text{or} \qquad A'_1 = 0.$$

Moreover, $z_{x,0}$ is null when x is zero, and equal to unity when x is either 1 or 2, or 3, . . ., since then the probability of player B is changed into certitude; the generating function of $z_{x,0}$ is therefore $\frac{t}{1-t}$; it is the coefficient of t'^0 in the development of the generating function according to the powers of t'; we will have therefore

$$\frac{A_1 t}{1 - mt} = \frac{t}{1-t};$$

that which gives

$$A_1 t = \frac{t(1 - mt)}{1 - t};$$

consequently the generating function of $z_{x,x'}$ is

$$(a) \qquad \frac{t(1 - mt)}{(1-t)(1 - mt - m't' - ntt')};$$

by putting it under this form

$$\frac{t}{1-t} \frac{1}{1 - \left(\frac{m'+nt}{1-mt}\right) t'}$$

and the development with respect to the powers of t', we have

$$\frac{t}{1-t}\left[1+\left(\frac{m'+nt}{1-mt}\right)t'+\left(\frac{m'+nt}{1-mt}\right)^2 t'^2+\left(\frac{m'+nt}{1-mt}\right)^3 t'^3+\cdots\right].$$

The coefficient of $t'^{x'}$ in this series is

$$\frac{t}{1-t}\left(\frac{m'+nt}{1-mt}\right)^{x'},$$

and the one of t^x in the development of this last function will be the expression of $z_{x,x'}$. [623]
Now, if we reduce first the expression $t\left(\frac{m'+nt}{1-mt}\right)^{x'}$ into a series ordered according to the powers of t, and if we multiply it next by the development of $\frac{1}{1-t}$, it is easy to see that the coefficient of t^x in this product is that which the series becomes by making $t=1$ in it and stopping ourselves at the power x of t; and we will find, for the value of this coefficient or of $z_{x,x'}$,

$$z_{x,x'}=m'^{x'}\left\{\begin{array}{l}1+\frac{x'}{1}\frac{n}{m'}+\frac{x'(x'-1)}{1.2}\frac{n^2}{m'^2}+\frac{x'(x'-1)(x'-2)}{1.2.3}\frac{n^3}{m'^3}+\cdots+\frac{x'(x'-1)\cdots(x'-x+2)}{1.2.\ldots(x-1)}\frac{n^{x-1}}{m'^{x-1}}\\ +\frac{x'}{1}m\left[1+\frac{x'}{1}\frac{n}{m'}+\frac{x'(x'-1)}{1.2}\frac{n^2}{m'^2}+\cdots+\frac{x'(x'-1)\cdots(x'-x+3)}{1.2.\ldots(x-2)}\frac{n^{x-2}}{m'^{x-2}}\right]\\ +\frac{x'(x'-1)}{1.2}m^2\left[1+\frac{x'}{1}\frac{n}{m'}+\cdots+\frac{x'(x'-1)\cdots(x'-x+4)}{1.2.\ldots(x-3)}\frac{n^{x-3}}{m'^{x-3}}\right]\\ +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ +\frac{x'(x'+1)\cdots(x'+x-2)}{1.2.\ldots(x-1)}m^{x-1}\end{array}\right\}$$

By designating by $y_{x,x'}$ the probability of player A, we will be led, by the same reasonings, to a similar equation in the partial differences,

$$y_{x,x'}=my_{x-1,x'}+m'y_{x,x'-1}+ny_{x-1,x'-1},$$

which gives similarly for the variable $y_{x,x'}$ a generating function of the form

$$\frac{A_1t+A_1'}{1-mt-m't'-ntt'},$$

A_1 and A_1' being, as above, arbitrary functions of t and of t' what we will determine by the same considerations. In fact the generating function of $y_{0,x'}$ is $\frac{1}{1-t'}$, that of $y_{x,0}$ is unity: we will form therefore the equations

$$\frac{A_1'}{1-m't'}=\frac{1}{1-t'};$$

whence we deduce

$$A_1'=\frac{1-m't'}{1-t'}$$

and [624]

$$\frac{A_1 t + 1}{1 - mt} = 1;$$

whence we conclude

$$A_1 t = -mt.$$

The generating function of $y_{x,x'}$ will be therefore

$$(b) \qquad \frac{\frac{1-m't'}{1-t'} - mt}{1 - mt - m't' - ntt'},$$

which, developed according to the powers of t and of t', will give, by the coefficient of $t^x t'^{x'}$, the expression of $y_{x,x'}$ which will be of a form similar to that of $z_{x,x'}$, although a little more complicated.

By adding the two generating functions (a) and (b), their sum is reduced to that here

$$\frac{1}{(1-t)(1-t')},$$

in which the coefficient of $t^x t'^{x'}$ is unity; thus we have

$$y_{x,x'} + z_{x,x'} = 1;$$

and effectively, the game must be necessarily won by one of the players, because both are certain to be able to extract each from their urn the determined numbers of white balls.

Now, let us suppose $p = 0$ and consequently $q = 1$, we have

$$m = 0, \quad m' = 1 \quad \text{and} \quad n = 0;$$

then the expression of $z_{x,x'}$ becomes unity; that which is evident, since the player B, having no more chances to lose, must always end by winning.

If, to the contrary, we suppose $p = 1$ and $q = 0$, that is if the first player A counts a point before each drawing of player B, then

$$m = q', \quad m' = 0 \quad \text{and} \quad n = p';$$

x' being greater than x or equal, the expression $z_{x,x'}$ is reduced to zero; and, in fact, it [625]
is evidently impossible that, in this case, player B is able to win the game; but, when x is greater than x', the value of $z_{x,x'}$ takes this form

$$z_{x,x'} = p'^{x'} \left[1 + \frac{x'}{1} q' + \frac{x'(x'+1)}{1.2} q'^2 + \cdots + \frac{x'(x'+1)\cdots(x-2)}{1.2\ldots(x-x'-1)} q'^{x-x'-1} \right].$$

Under this assumption, player B is able to win only so much as he will bring forth x' white balls before $x - x'$ black balls; otherwise, he is anticipated by player A who counts a point at each trial: this expression of $z_{x,x'}$ is therefore the probability that player B will have drawn x' white balls before having extracted from it $x - x'$ blacks, and, consequently, the probability to win, if he made the wager with player A, who would count then a point with the exit of each black ball while he counts one of them at the exit of a white, to attain x' points before his adversary has $x - x'$ of them; that which is the *problem of points*. ([2])

If we examine with attention the form of the general expression which gives $z_{x,x'}$, [626]
we will recognize that this problem is able yet to be resolved, and even with simplicity, by means of the theory of combinations: in fact, let a be the number of white balls contained in the urn of player A, and b the one of the blacks; a' the number of white balls of player B, and b' the one of the blacks; by considering, as we have already done, the set of two successive drawings of A and B as one trial,

aa' will be the number of combinations in which the players bring forth each one white ball;

ab' the one of the combinations which will give one white ball to A and one black to B;

$a'b$ the one of the combinations which will give, to the contrary, one black ball to A and one white to B;

bb' the one of the combinations in which both players draw a black ball;

And the sum $aa' + ab' + a'b + bb'$ will form the collection of all the combinations which are able to take place in a trial. The combinations where the players bring forth each one black ball bring no change to their position, we are able to set it aside, and then we occupy ourselves only with the trials where there will be brought forth at least one white ball. It is clear that in $x + x'$ similar trials one of the players has necessarily won, and the game must be decided: now the number of all the equally possible combinations, according to which these $x + x'$ trials are able to be presented, will be

$$(aa' + ab' + a'b)^{x+x'};$$

[2]The generating function of $z_{x,x'}$ is reduced in this case to

$$\frac{t(1-q't)}{(1-t)(1-q't-p'tt')},$$

and the equation in the corresponding partial differences will be

$$z_{x,x'} = q'z_{x-1,x'} + p'z_{x-1,x'-1},$$

in which $z_{x,x'}$ is a function of x and of x' which we will designate by $\phi(x,x')$; if we make $x - x' = s$, we will have

$$\phi(x,x') = \phi(s + x', x'),$$

and, if we represent by $z_{s,x'}$ this last function, there results from it

$$z_{x,x'} = z_{s,x'}, \quad z_{x-1,x'} = z_{s-1,x'}, \quad z_{x-1,x'-1} = z_{s,x'-1};$$

and the equation in the partial differences is changed into that here

$$z_{s,x'} = q'z_{s-1,x'} + p'z_{s,x'-1},$$

an equation to which the problem of points would lead directly under the conditions enunciated above. By paying attention that, in consequence of this transformation, $z_{s,0} = 1$ and $z_{0,x'} = 0$, and that $z_{0,0}$ is not able to take place, it is easy to see that the generating function of $z_{s,x'}$ will be

$$\frac{t(1-q't)}{(1-t)(1-q't-p't')},$$

in the development of which the coefficient of $t^s t'^{x'}$ will be the expression of $z_{s,x'}$.

the question is reduced therefore to choose in all these combinations those which make player B win, that is those in which this player will have x' white balls before player A has brought forth x of them. In order to fix the ideas, let us suppose x' greater than x; we are able to form the following hypotheses: either player B will have won at the x^{th} trial, that is by drawing without interruption a white ball at each trial, and then the number of the preceding combinations which are corresponding to this case is evidently [627]

$$a'^{x'}\Big[b^{x'}+\frac{x'}{1}ab^{x'-1}+\frac{x'(x'-1)}{1.2}a^2b^{x'-2}+\cdots +\frac{x'(x'-1)\cdots(x'-x+2)}{1.2\ldots(x-1)}a^{x-1}b^{x'-x+1}\Big](aa'+ab'+a'b)^x;$$

and by dividing it by $(aa'+ab'+a'b)^{x+x'}$, the total number of combinations, we will have, for the probability of this hypothesis,

$$\frac{a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'}}\left[1+\frac{x'}{1}\frac{a}{b}+\frac{x'(x'-1)}{1.2}\frac{a^2}{b^2}+\cdots+\frac{x'(x'-1)\cdots(x'-x+2)}{1.2\ldots(x-1)}\frac{a^{x-1}}{b^{x-1}}\right];$$

or the player B will have won at the $(x'+1)^{\text{st}}$ trial, that is by having drawn only a single black ball, for example at the commencement, and then the number of combinations favorable to this event is

$$b'a'^{x'}\Big[b^{x'}+\frac{x'}{1}ab^{x'-1}+\frac{x'(x'-1)}{1.2}a^2b^{x'-2}+\cdots +\frac{x'(x'-1)\cdots(x'-x+3)}{1.2\ldots(x-2)}a^{x-2}b^{x'-x+2}\Big](aa'+ab'+a'b)^{x-1};$$

but this number is the same, if the black ball is brought forth in the first trial or in the second, ..., or in the x^{th} trial; it is necessary therefore to multiply it by x' in order to have all the combinations relative to this hypothesis, of which the probability is, by this means,

$$\frac{x'}{1}\frac{ab'a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+1}}\left[1+\frac{x'}{1}\frac{a}{b}+\frac{x'(x'-1)}{1.2}\frac{a^2}{b^2}+\cdots+\frac{x'(x'-1)\cdots(x'-x+3)}{1.2\ldots(x-2)}\frac{a^{x-2}}{b^{x-2}}\right];$$

or player B will have won at the $(x'+2)^{\text{nd}}$ trial, and we will see in the same manner that the probability of this hypothesis will be

$$\frac{x'(x'+1)}{1.2}\frac{a^2b'^2a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+2}}\left[1+\frac{x'}{1}\frac{a}{b}+\cdots+\frac{x'(x'-1)\cdots(x'-x+4)}{1.2\ldots(x-3)}\frac{a^{x-3}}{b^{x-3}}\right];$$

By continuing thus, we will have the probabilities of all the successive hypotheses which are able to be presented under the supposition of the gain of the game by player B, until that where he would win only at the $(x'+x-1)^{\text{st}}$ trial, an event of which the [628] probability will be

$$\frac{x'(x'+1)\cdots(x'+x-2)}{1.2\ldots(x-1)}\frac{a^{x-1}b'^{x-1}a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+x-1}};$$

and effectively, in this case, there are not able to be trials where the players bring forth at the same time a white ball.

The sum of all these probabilities will give evidently that of player B in order to win the game.

If we pay attention that

$$\frac{ab'}{aa'+ab'+a'b}=m, \quad \frac{a'b}{aa'+ab'+a'b}=m', \quad \text{and} \quad \frac{q}{b}=\frac{n}{m'},$$

we recover the expression of $z_{x,x'}$.

Let us imagine presently that there are in the urns some white balls bearing the n° 1, and other balls, of the same color, which bear the n° 2; each ball diminishing by its numeral, by its exit, the number of points which are lacking yet to the player to which it is favorable. The problem is no longer susceptible to be resolved generally by means of combinations, instead the calculation of the generating functions will continue to furnish a general expression of which the development will contain the complete solution of the question and will be able, in certain cases, to be executed by laws easy to know, as we will have occasion to see.

Let p be the probability player A to extract a ball labeled 1, p_1 that to extract a ball labeled 2, and q that to bring forth a black ball; p', p'_1 and q' the corresponding probabilities for player B; and let always $z_{x,x'}$ be the probability of this last player in order to win the game. By following the same march as above, we will be led to the equation in partial differences

$$\begin{aligned} z_{x,x'} = & m z_{x-1,x'} + m_1 z_{x-2,x'} + m' z_{x,x'-1} + m'_1 z_{x,x'-2} \\ & + n z_{x-1,x'-1} + n_1 z_{x-2,x'-1} + n' z_{x-1,x'-2} + n'_1 z_{x-2,x'-2} \end{aligned}$$

in which we make [629]

$$\frac{pq'}{1-qq'}=m, \qquad \frac{p_1q'}{1-qq'}=m_1, \qquad \frac{p'q}{1-qq'}=m', \qquad \frac{p'_1q}{1-qq'}=m'_1,$$
$$\frac{pp'}{1-qq'}=n, \qquad \frac{p_1p'}{1-qq'}=n_1, \qquad \frac{pp'_1}{1-qq'}=n', \qquad \frac{p_1p'_1}{1-qq'}=n'_1;$$

the generating function of the variable $z_{x,x'}$ given by this equation, will be

$$(c) \qquad \frac{A+Bt'+A'+B't}{1-mt-m_1t^2-m't'-m'_1t'^2-ntt'-n_1t^2t'-n'tt'^2-n'_1t^2t'^2},$$

A and B being arbitrary functions of t, A' and B' arbitrary functions of t', which will be determined by means of the generating functions of

$$z_{0,x'}, \quad z_{x,0}, \quad z_{1,x'}, \quad z_{x,1}$$

which are themselves it by the conditions of the game.

We find, as previously, that the generating function of $z_{0,z'}$ is zero and that of $z_{x,0}$, $\frac{t}{1-t}$.

From the general equation, we deduce the equation in finite differences

$$z_{1,x'} = m' z_{1,x'-1} + m'_1 z_{1,x'-2},$$

which holds for all the values of x' from $x' = 2$ inclusively, and which gives consequently, for the generating function of $z_{1,x'}$,

$$\frac{a + bt'}{1 - m't' - m'_1 t'^2},$$

a and b being constants that we determine by means of the values of $z_{1,0}$ and $z_{1,1}$; and as $z_{1,0}$ is equal to unity, $z_{1,1}$ is equal to $m' + m'_1$, and is at the same time the coefficient of t' in the development of the generating function; there results from it

$$a = 1 \quad \text{and} \quad b = m'_1;$$

the generating function of $z_{1,x'}$ is therefore

$$\frac{1 + m'_1 t'}{1 - m't' - m'_1 t'^2}.$$

Now, if in the preceding equation we put $1 - y_{x,x'}$ in the place of $z_{x,x'}$, $y_{x,x'}$ being always the probability of the first player A, it is reformed in the same manner with respect to this last variable, and we will deduce from it similarly the equation in the finite differences [630]

$$y_{x,1} = m y_{x-1,1} + m_1 y_{x-2,1}.$$

But we will see at the same time that it begins to hold only when x surpasses 2; because, x being 2, we will have

$$y_{2,1} = m y_{1,1} + m_1 y_{0,1} + n_1 + n'_1.$$

It is necessary therefore to employ it only by departing from $x = 3$, and then the generating function of $y_{x,1}$ is of the form

$$\frac{a + bt + ct^2}{1 - mt - m_1 t^2},$$

a, b and c being constants that we will determine, as previously, by means of the values of $y_{1,0}$, $y_{1,1}$ and $y_{1,2}$; now $y_{1,0}$ is unity; $y_{1,1}$ is equal to $1 - m' - m'_1$, and is the coefficient of t in the development of the generating function; $y_{2,1}$ has for value, as we have just seen,

$$m(1 - m' - m'_1) + m_1 + n_1 + n'_1;$$

this is the coefficient of t^2 in the development of the function. We will conclude from it

$$a = 1, \quad b = 1 - m - m' - m'_1, \quad \text{and} \quad c = n_1 + n'_1,$$

and the generating function of $y_{x,1}$ will be therefore

$$\frac{1 + (1 - m - m' - m'_1)\, t + (n_1 + n'_1)\, t^2}{1 - mt - m_1 t^2};$$

consequently that of $z_{x,1}$ is

$$\frac{1}{1-t}-\frac{1+(1-m-m'-m_1')\,t+(n_1+n_1')\,t^2}{1-mt-m_1t^2}$$
$$=\frac{(m'+m_1')\,t+(n_1+n_1')\,t^2+(n_1+n_1')\,t^3}{(1-t)(1-mt-m_1t^2)}.$$

Let us resume actually the generating function (c); we are able always to restore it to this form [631]

$$\frac{A_1t+B_1t^2t'+A_1'+B_1'tt'}{1-mt-m_1t^2-m't'-m_1't'^2-ntt'-n_1t^2t'-n'tt'^2-n_1't^2t'^2},$$

A_1 and B_1 being the arbitrary functions of t, A_1' and B_1' the arbitrary functions of t'; which we determine easily, by equating first the coefficient of t^0 in the development of this function to the generating function of $z_{0,x'}$ or zero, next the one of t'^0 to the generating function of $z_{x,0}$ or $\frac{t}{1-t}$, since the one of t to the generating function of $z_{1,x'}$, and finally the one of t' to the generating function of $z_{x,1}$, that which will give successively

$$A_1'=0,\quad A_1=\frac{1-mt-m_1t^2}{1-t},\quad B_1'=m_1',\quad B_1=\frac{m_1'+n'+n_1't}{1-t},$$

and, consequently, for the generating function of $z_{x,x'}$,

$$(d)\qquad \frac{(1-mt-m_1t^2)t+m_1'tt'+n't^2t'+n_1't^3t'}{(1-t)(1-mt-m_1t^2-m't'-m_1't'^2-ntt'-n_1t^2t'-n'tt'^2-n_1't^2t'^2)}.$$

If we suppose p and p' null, then we have

$$m=0,\quad m'=0,\quad n=0,\quad n_1=0,\quad \text{and}\quad n'=0,$$

and the function (d) takes this form

$$\frac{tt'(m_1'+n_1't^2)}{(1-t)(1-m_1t^2)\left[1-\left(\frac{m_1'+n_1't^2}{1-m_1t^2}\right)t'^2\right]}+\frac{t}{(1-t)\left[1-\left(\frac{m_1'+n_1't^2}{1-m_1t^2}\right)t'^2\right]},$$

under which it is susceptible of the same developments as the function (a). There is to note that we will recover the same coefficient for

$$t^{2r}t'^{2r'},\quad t^{2r-1}t'^{2r'},\quad t^{2r}t'^{2r'-1},\quad t^{2r-1}t'^{2r'-1};$$

that which is seen *a priori*, by paying attention that the players always count two points with each white ball that they make exit.

Let us suppose that player A alone has some balls labeled 1 and 2, and that the other player has only some white balls marked 1, or which count to him only one point on exiting; then [632]

$$p_1'=0$$

and, hence,

$$m'_1 = 0, \quad n' = 0, \quad n'_1 = 0;$$

the function (d) becomes

$$\frac{t(1 - mt - m_1 t^2)}{(1-t)(1 - mt - m_1 t^2 - m't' - ntt' - n_1 t^2 t')} = \frac{t}{1-t} \frac{1}{1 - \left[\frac{m' + (n + n_1 t)t}{1 - (m + m_1 t)t}\right] t'};$$

by developing it according to the powers of t', the coefficient of $t'^{x'}$ will be

$$\frac{t\,[m' + (n + n_1 t)t]^{x'}}{(1-t)\,[1 - (m + m_1 t)t]^{x'}},$$

an expression that the concern is to develop with respect to the powers of t in order to have the coefficient of t^x; now this coefficient will be the sum of all the coefficients of the powers of t inferior or equal to t^{x-1}, in the development of the expression

$$\frac{[m' + (n + n_1 t)t]^{x'}}{[1 - (m + m_1 t)t]^{x'}},$$

which, by omitting the terms where the powers of t outside the binomials are superior to t^{x-1}, is able to be put under this form

$$m'^{x'} \left\{ \begin{array}{l} 1 + \frac{x'}{1}\left(\frac{n + n_1 t}{m'}\right) t + \frac{x'(x'-1)}{1.2}\left(\frac{n + n_1 t}{m'}\right)^2 t^2 + \cdots + \frac{x'(x'-1)\cdots(x'-x+2)}{1.2\ldots(x-1)}\left(\frac{n + n_1 t}{m'}\right)^{x-1} t^{x-1} \\ + \frac{x'}{1}(m + m_1 t)t\left[1 + \frac{x'}{1}\left(\frac{n + n_1 t}{m'}\right) t + \cdots + \frac{x'(x'-1)\cdots(x'-x+3)}{1.2\ldots(x-2)}\left(\frac{n + n_1 t}{m'}\right)^{x-2} t^{x-2}\right] \\ + \frac{x'(x'+1)}{1.2}(m + m_1 t)^2 t^2 \left[1 + \cdots + \frac{x'(x'-1)\cdots(x'-x+4)}{1.2\ldots(x-3)}\left(\frac{n + n_1 t}{m'}\right)^{x-3} t^{x-3}\right] \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \frac{x'(x'-1)\cdots(x'-x+2)}{1.2\ldots(x-1)}(m + m_1 t)^{x-1} t^{x-1}. \end{array} \right\}$$

If we reject further from this series all the powers of t superior to t^{x-1}, which will result from the developments of the binomials, and if, in that which remains, we make $t = 1$, we will have the expression of $z_{x,x'}$. [633]

Let us examine further the case where player A would be certain to extract at each trial a ball which would count to that player one point, that is where we would have

$$p = 1, \quad p_1 = 0, \quad q = 0,$$

and consequently

$$\begin{array}{llll} m = q', & m_1 = 0, & m' = 0, & m'_1 = 0, \\ n = p', & n_1 = 0, & n' = p'_1, & n'_1 = 0. \end{array}$$

The generating function of $z_{x,x'}$ or the function (d) would be reduced to

$$\frac{t(1-q't)+p'_1t^2t'}{(1-t)(1-q't-p'tt'-p'_1tt'^2)},$$

and that of $y_{x,x'}$ would be, hence,

$$\begin{aligned}&\frac{1}{(1-t)(1-t')}-\frac{t(1-q't)+p'_1t^2t'}{(1-t)(1-q't-p'tt'-p'_1tt'^2)},\\ &=\frac{1}{1-t'}+\frac{tt'}{(1-t)(1-q't-p'tt'-p'_1tt'^2)}.\end{aligned}$$

In this last expression, the first term represents the generating function of $y_{0,x'}$, which is equal to unity whatever be x', and the second will give, by developing it with respect to the powers of t and of t', all the other values of $y_{x,x'}$; now the coefficient of t^x will be

$$\frac{t'[q'+(p'+p'_1t')t']^{x-1}}{1-t'};$$

whence it results that, if we reject from the development of the series

$$q'^{x-1}\left[t'+\frac{(x-1)}{1}\left(\frac{p'+p'_1t'}{q'}\right)t'^2+\frac{(x-1)(x-2)}{1.2}\left(\frac{p'+p'_1t'}{q'}\right)^2t'^3+\cdots\right]$$

all the powers of t' superior to $t'^{x'}$, and if we made in that which remains $t'=1$, we will have, by supposing x' even and equal to $2r+2$, the coefficient of $t^xt'^{x'}$, or [634]

$$y_{x,x'}=q'^{x-1}\left\{\begin{aligned}&1+\frac{(x-1)}{1}\left(\frac{p'+p'_1}{q'}\right)+\frac{(x-1)(x-2)}{1.2}\left(\frac{p'+p'_1}{q'}\right)^2+\cdots+\frac{(x-1)(x-2)\cdots(x-r)}{1.2\ldots r}\left(\frac{p'+p'_1}{q'}\right)^r\\ &+\frac{(x-1)(x-2)\cdots(x-r-1)}{1.2\ldots(r+1)}\frac{p'^{r+1}}{q'^{r+1}}\left[1+\frac{(r+1)}{1}\frac{p'_1}{p'}+\frac{(r+1)r}{1.2}\frac{p'^2_1}{p'^2}+\cdots+\frac{(r+1)r\cdots 2}{1.2\ldots r}\frac{p'^r_1}{p'^r}\right]\\ &+\frac{(x-1)(x-2)\cdots(x-r-2)}{1.2\ldots(r+2)}\frac{p'^{r+2}}{q'^{r+2}}\left[1+\frac{(r+2)}{1}\frac{p'_1}{p'}+\cdots+\frac{(r+2)(r+1)r\cdots 4}{1.2\ldots(r+1)}\frac{p'^{r-1}_1}{p'^{r-1}}\right]\\ &+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+\frac{(x-1)(x-2)\cdots(x-2r-1)}{1.2\ldots(2r+1)}\frac{p'^{2r+1}}{q'^{2r+1}}\end{aligned}\right\}$$

and, in the case of x' odd or equal to $2r+1$,

$$y_{x,x'}=q'^{x-1}\left\{\begin{aligned}&1+\frac{(x-1)}{1}\left(\frac{p'+p'_1}{q'}\right)+\frac{(x-1)(x-2)}{1.2}\left(\frac{p'+p'_1}{q'}\right)^2+\cdots+\frac{(x-1)(x-2)\cdots(x-r)}{1.2\ldots r}\left(\frac{p'+p'_1}{q'}\right)^r\\ &+\frac{(x-1)(x-2)\cdots(x-r-1)}{1.2\ldots(r+1)}\frac{p'^{r+1}}{q'^{r+1}}\left[1+\frac{(r+1)}{1}\frac{p'_1}{p'}+\frac{(r+1)r}{1.2}\frac{p'^2_1}{p'^2}+\cdots+\frac{(r+1)r\cdots 3}{1.2\ldots(r-1)}\frac{p'^{r-1}_1}{p'^{r-1}}\right]\\ &+\frac{(x-1)(x-2)\cdots(x-r-2)}{1.2\ldots(r+2)}\frac{p'^{r+2}}{q'^{r+2}}\left[1+\frac{(r+2)}{1}\frac{p'_1}{p'}+\cdots+\frac{(r+2)(r+1)r\cdots 5}{1.2\ldots(r-2)}\frac{p'^{r-2}_1}{p'^{r-2}}\right]\\ &+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+\frac{(x-1)(x-2)\cdots(x-2r)}{1.2\ldots 2r}\frac{p'^{2r}}{q'^{2r}}\end{aligned}\right\}$$

It is clear that player B is able to expect to win only as long as x is greater than $r+1$, or that x' equal $2r+2$ or $2r+1$; and effectively, beyond this supposition, the preceding values of $y_{x,x'}$ become all equal to unity.

We will note also that player A has necessarily won the game when player B will have drawn $x-r-1$ black balls before having attained x' points; but this last player is able yet to have lost before having drawn the totality of this number of black balls, that which makes that this question is not at all susceptible to return into that which is treated in the analytic Theory, after the *problem of points*, as previously a similar supposition has led us to this last problem.

§3. The problem of points having been the object of the researches of two great geometers of the XVII$^{\text{th}}$ century ([3]), and to some extent the first of this kind subject to analytic methods, one will be perhaps curious to see how this same problem is deduced again, as corollary, from another question of probability, of which the solution will offer besides a new application of the method of generating functions. [635]

We draw successively from an urn, which contains a determined quantity of white and black balls, a ball that we do not return after the trial, and we demand, after a certain number of known drawings, what is the probability to complete the drawing of such given number of white balls before that of such other number, given equally, of black balls.

Let a and a' be the numbers of white and black balls contained originally in the urn, n the number of white balls that we are proposed to attain before having extracted another number n' of black balls; and let us suppose that after having drawn successively from the urn a ball without returning it, we have brought forth $n-x$ white balls and $n'-x'$ black balls, x and x' being then the number of white and black balls that there remain to make exit in order to decide the question. Let us represent by $y_{x,x'}$ the probability to bring forth in the following drawings x white balls before x' black balls, or to attain the totality of n white balls before having extracted n' blacks; we will have, according to the known rules of probabilities, the equation

$$y_{x,x'} = \frac{a-n+x}{a+a'-n-n'+x+x'}y_{x-1,x'} + \frac{a'-n'+x'}{a+a'-n-n'+x+x'}y_{x,x'-1}.$$

Let us make

$$a-n+x = s, \quad a'-n'+x' = s' \quad \text{and} \quad y_{x,x'} = u_{s,s'};$$

the preceding equation becomes

$$u_{s,s'} = \frac{s}{s+s'}u_{s-1,s'} + \frac{s'}{s+s'}u_{s,s'-1},$$

and, by supposing [636]

$$u = \frac{1.2.3\ldots s.1.2.3\ldots s'}{1.2.3\ldots(s+s')}z_{s,s'},$$

it is restored to this form

$$z_{s,s'} = z_{s-1,s'} + z_{s,s'-1},$$

[3] Pascal and Fermat.

an equation in the partial differences with constant coefficients, which must hold for all the entire and positive values of s and of s', by departing from $s = a - n$ and from $s' = a' - n'$, and gives consequently for the generating function of $z_{s,s'}$

$$t^{a-n} t'^{a'-n'} \frac{A + A'}{1 - t - t'},$$

A being an arbitrary function of t, and A' an arbitrary function of t'. We are able always to transform this expression into this one

$$t^{a-n} t'^{a'-n'} \frac{A_1 + A'_1 t'}{1 - t - t'},$$

in which A_1 and A'_1 are new arbitrary functions of t and of t'. In order to determine them, we will observe that, $y_{0,0}$ not being able to take place and $y_{x,0}$ being equal to zero, whatever be the entire and positive values of x, we will have

$$0 = u_{s,a'-n'} = \frac{1.2.3 \ldots s.1.2.3 \ldots (a' - n')}{1.2.3 \ldots (a' - n' + s)} z_{s,a'-n'};$$

consequently the generating function of $z_{s,a'-n'}$ will be null, that which gives

$$t^{a-n} t'^{a'-n'} \frac{A_1}{1 - t} = 0, \quad \text{and hence} \quad A_1 = 0.$$

Moreover, $y_{0,x'}$ being equal to unity for all the values of x' from $x' = 1$, we will have similarly

$$1 = u_{a-n,s'} = \frac{1.2.3 \ldots (a - n).1.2.3 \ldots s'}{1.2.3 \ldots (a - n + s')} z_{a-n,s'};$$

whence we deduce, for the value of $z_{a-n,s'}$ or the coefficient of $t^{a-n} t'^{s'}$ in the development of its generating function, [637]

$$z_{a-n,s'} = \frac{(a - n + 1)(a - n + 2) \cdots (a - n + s')}{1.2.3 \ldots s'},$$

that which gives

$$\begin{aligned} t^{a-n} t'^{a'-n'} \frac{A'_1 t'}{1 - t'} =& t^{a-n} t'^{a'-n'} \frac{(a - n + 1) \cdots (a + a' - n - n' + 1)}{1.2.3 \ldots (a' - n' + 1)} \\ & \times \left[\begin{aligned} & t' + \frac{(a + a' - n - n' + 2) t'^2}{a' - n' + 2} + \cdots \\ & + \frac{(a + a' - n - n' + 2) \cdots (a + a' - n - n' + x') t'^{x'}}{(a' - n' + 2) \cdots (a' - n' + x')} + \cdots \end{aligned} \right] \end{aligned}$$

The second member of this equation multiplied by $\frac{1}{1 - \frac{t}{1-t'}}$ will be therefore the generating function of $z_{s,s'}$; by developing it with respect to the powers of t and next with respect to those of t', it is easy to see that the coefficient of t^s or of t^{a-n+x} is

$$t'^{a'-n'}\frac{(a-n+1)\cdots(a+a'-n-n'+1)}{1.2.3\ldots(a'-n'+1)}$$
$$\times\left[t'+\frac{(a+a'-n-n'+2)}{a'-n'+2}t'^2+\cdots\right]$$
$$\times\left[1+\frac{x}{1}t'+\frac{x(x+1)}{1.2}t'^2+\cdots+\frac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}t'^{x'-1}+\cdots\right],$$

and the one of $t'^{s'}$, or of $t'^{a'-n'+x'}$ in this last expression, or $z_{s,s'}$, is equal to

$$\frac{(a-n+1)\cdots(a+a'-n-n'+1)}{1.2.3\ldots(a'-n'+1)}$$
$$\times\left[\begin{array}{l}\dfrac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}+\dfrac{a+a'-n-n'+2}{a'-n'+2}\,\dfrac{x(x+1)\cdots(x+x'-3)}{1.2\ldots(x'-2)}+\cdots\\ \qquad+\dfrac{(a+a'-n-n'+2)\cdots(a+a'-n-n'+x')}{(a'-n'+2)\cdots(a'-n'+x')}\end{array}\right].$$

Now, by multiplying this value of $z_{s,s'}$ by

$$\frac{1.2.3\ldots(a'-n'+x')}{(a'-n'+x+1)\cdots(a+a'-n-n'+x+x')},$$

we will have, after all the reductions, for the expression of $y_{x,x'}$, [638]

$$y_{x,x'}=\frac{(a-n+x)\cdots(a-n+1)}{(a+a'-n-n'+x+x')\cdots(a+a'-n-n'+x+1)}$$
$$\times\left[\begin{array}{l}1+\dfrac{x}{1}\dfrac{a'-n'+x'}{a+a'-n-n'+x'}+\dfrac{x(x+1)}{1.2}\dfrac{(a'-n'+x')(a'-n'+x'-1)}{(a+a'-n-n'+x')(a+a'-n-n'+x-1)}+\cdots\\ +\dfrac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}\dfrac{(a'-n'+x')\cdots(a'-n'+2)}{(a+a'-n-n'+x')\cdots(a+a'-n-n+2)}\end{array}\right]$$

Let us imagine actually $a-n$ and $a'-n'$ in the ratio of p to q, so that we have $a-n=pk$ and $a'-n'=qk$, and let us imagine that k becomes a very great number or infinity; it is clear that the probability of the exit of a white ball or of a black ball in the successive drawings will become constant and will be $\frac{p}{p+q}$ for a white ball and $\frac{q}{p+q}$ for a black, and the probability $y_{x,x'}$ will be reduced to this expression

$$y_{x,x'}=\left(\frac{p}{p+q}\right)^x\left[1+\frac{x}{1}\frac{q}{p+q}+\frac{x(x+1)}{1.2}\left(\frac{q}{p+q}\right)^2+\cdots\right.$$
$$\left.+\frac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}\left(\frac{q}{p+q}\right)^{x'-1}\right];$$

such is the formula to which the *problem of points* leads, and effectively we return to the conditions of this problem by the supposition of k infinite.

If we suppose n equal to a and n' equal to a', $y_{x,x'}$ will express then the probability of the exit of all the white balls remaining in the urn before all the blacks had been depleted, and its expression will be changed into that here

$$\frac{1.2.3\ldots x}{(x+x')\cdots(x'+1)}\left[1+\frac{x}{1}+\frac{x(x+1)}{1.2}+\cdots+\frac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}\right],$$

which is reduced itself to

$$\frac{x'}{x+x'}.$$

The probability of extracting from the urn the totality of the white balls before that [639]
of the blacks is therefore to the contrary probability in inverse ratio of the number of white balls to the one of the blacks.

We arrive to this last result, in an extremely simple manner, by means of combinations; in fact, the probability of the exit of all the balls from the urn, in any order, by color, will be

$$\frac{x(x-1)\cdots 2.1\, x'(x'-1)\cdots 2.1}{(x+x')(x+x'-1)\cdots 3.2.1}=\frac{1.2.3\cdots x'}{(x+1)\cdots(x+x')}.$$

But, in order that the white balls exit in totality first, it is necessary necessarily that a ball of the color black exit last: by combining $x'-1$ with $x'-1$ the $x+x'+1$ ranks of exit which are found before the last, we will form as many different rankings for the balls of the color black, and as many orders of exit by color, which will comprehend all those where one black ball exits in last place; now the number of these combinations is

$$\frac{(x+x'-1)(x+x'-2)\cdots(x+1)}{1.2\ldots(x'-1)},$$

and by multiplying it by the probability common to each order of exit by color, we will have the sought probability equal to

$$\frac{1.2.3\cdots x'}{(x+1)\cdots(x+x')}\frac{(x+1)\cdots(x+x'-1)}{1.2.3\ldots(x'-1)}=\frac{x'}{x+x'}.$$

Remarks on generating functions.

§4. Let u be a generating function in one or many variables; each equation between this function and its variables, linear with respect to u, rational with respect to the variables, will subsist still if we pass from the generating functions to the coefficients, among these same coefficients, and will give place to an equation in the partial differences; but if, in this equation in partial differences, we pass again from the coefficients to the generating functions, we will no longer arrive to an equation rigorously exact, at least if we restore at the same time the functions of the variables which have been able [640] to vanish in the first passage. Thus, in one of the questions that we have treated above, the equation in the partial differences

$$z_{x,x'} = mz_{x-1,x'} + m'z_{x,x'-1} + nz_{x-1,x'-1}$$

would give, by going up again simply from the coefficients to the generating functions, this one

$$u = mut + m'ut' + nutt',$$

which is not at all exact; because it is easy to see that, according to the conditions of the problem, it would be necessary to add to the second member the generating function of $z_{x,0}$ less this same function multiplied by m. This function of t, which it is necessary to restore in the second member of the equation in order to complete it, is precisely the arbitrary function that we have had to determine in the solution of this question. In general, the functions to add in order to have still one equation in the passage from the coefficients to the generating functions are the same as the arbitrary functions which form the numerator of the generating function integral before it is developed.

For lack of having regard to these functions, we are able to fall into some grave errors, by serving ourselves in this manner in order to integrate the equations in the partial differences. For this same reason, the march followed in the solution of problems §§8 and 10 of Book II of the *Théorie analytique des Probabilités* is by no means rigorous, and seems to implicate contradiction in this that it established a liaison among the variables which are and must be always independent. Without entering into the particular considerations which have been able to make it succeed here, and that it is easy to know, we will show that the method of integration exposed at the beginning of this *Supplément* is applied equally to these questions, and resolves them with no less simplicity.

In the problem of §8, we have proposed to determine the lot of a number n of players A, B, C, ... of whom p, q, r, ... represent the respective probabilities, that is their probabilities to win a trial when, in order to win the game, there are lacking x trials to player A, x' trials to player B, x'' trials to player C, etc. By naming $y_{x,x',x'',\dots}$ the [641]
probability of player A to win the game, we have the equation in partial differences

$$y_{x,x',x'',\dots} = py_{x-1,x',x'',\dots} + 1y_{x,x'-1,x'',\dots} + ry_{x,x',x''-1,\dots} + \cdots,$$

which gives for $y_{x,x',x'',\dots}$ this generating function

$$\frac{P + Q + R + \cdots}{1 - pt - qt' - rt'' - \cdots},$$

in which P, Q, R, ... are as many arbitrary functions of the variables t, t', t'', ... as there are of these variables, by comprehending not at all t in the first, t' in the second, t'' in the third, etc. Now, this function is able to be put under this form

$$\frac{P' + Q't + R'tt' + S'tt't'' \cdots}{1 - pt - qt' - rt'' - st''' \cdots},$$

P', Q', R', ...being, as above, arbitrary functions, the first of all the variables with the exception of t, the second of all the variables of it excepting t', the third equally of all the variables except t'', and thus consecutively. In order to determine them, we will observe that, in $y_{x,x',\,x'',\ldots}$, two of the indices $x,\ x',\ x'',\ \ldots$or a greater number are not able to be nulls at the same time, since the game ceases when one of the players has attained his points; moreover, $y_{0,x',\,x'',\ldots}$ is equal to unity, whatever be $x',\ x'',\ \ldots$; the generating function of this expression, or that which gives unity for the coefficient of any product whatsoever $t'^{x'}t''^{x''}t'''^{x'''}\ldots$, is

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\frac{t'''}{1-t'''}\cdots;$$

consequently, we will have

$$P' = \frac{t'}{1-t'}\frac{t''}{1-t''}\frac{t'''}{1-t'''}\cdots(1-qt'-rt''-st'''-\cdots).$$

Each value of $y_{x,x',\,x'',\ldots}$ in which another index than x is null being equal to zero, [642]
the corresponding generating function becomes null also; we will have therefore successively

$$Q' = 0, \quad R' = 0, \quad S' = 0, \quad \ldots$$

Hence, the generating function of $y_{x,x',\,x'',\ldots}$ will be

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\cdots\frac{1-qt'-rt''-\cdots}{1-pt-qt'-rt''-\cdots},$$

and the coefficient of t^x, in the development of this function with respect to the powers of t,

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\cdots\frac{p^x}{(1-qt'-rt''-\cdots)^x};$$

whence it is easy to deduce the coefficient of $t'^{x'}t''^{x''}\ldots$, or

$$y_{x,x',\,x'',\ldots} = p^x\left\{\begin{array}{l} 1+\frac{x}{1}(q+r+\cdots) \\ \quad+\frac{x(x+1)}{1.2}(q+r+\cdots)^2 \\ \quad+\frac{x(x+1)(x+2)}{1.2.3}(q+r+\cdots)^3 \\ \quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\},$$

by taking care to reject the terms in which the power of q surpasses $x'-1$, those in which the power of r surpasses $x''-1$,. . .

In the problem of §10, we consider two players A and B of whom the skills are p and q, and of whom the first has a tokens and the second b tokens; and we suppose that at each trial, the one who loses gives a token to his adversary, and that the game finishes only when one of the players will have lost all his tokens. We demand the probability that one of the players, A for example, will win the game before or at the n^{th} trial.

By representing by $y_{x,x'}$ the probability of this player in order to win the game when he has x tokens and when there are no more than x' trials to play in order to attain the n trials, we arrive, by the first principles of the probabilities, to the equation in the partial differences [643]

$$r_{x,x'} = py_{x+1,x'-1} + qy_{x-1,x'-1},$$

which gives, for the generating function of $y_{x,x'}$,

$$\frac{A + A' + B't}{qt^2t' - t + pt'},$$

A being an arbitrary function of t, A' and B' two arbitrary functions of t'. In order to determine them more conveniently, we will transform this generating function into this one

$$\frac{A_1t + A'_1 + B'_1tt'}{qt^2t' - t + pt'},$$

in which A_1, A'_1 and B'_1 are, as above, arbitrary functions of t and of t'. Now $\frac{A'_1}{pt'}$ is the coefficient of t^0 in the development of the function with respect to the powers of t, or the generating function of $y_{0,x'}$; but, by the conditions of the problem, $y_{0,x'}$ is null whatever be x'; consequently its generating function is also it; A'_1 is therefore equal to zero.

The coefficient of t'^0, in the development of the generating function with respect to t', is $-A_1$, that which is at the same time the generating function of $y_{x,0}$, a quantity which is null so long as x is less than the sum of the tokens or $a+b$, and which becomes unity when $x = a + b$; A_1 is therefore a function of t which has for factor t^{a+b}, and of which we are able to take no account in the numerator of the generating function, because it must give only powers of t superior to t^{a+b}, and we have seen it only to have a generating function composed of the powers inferior to t, since x is able to be extended only from $x = 0$ to $x = a + b$.

The generating function of $y_{x,x'}$, thus limited between these values, is reduced therefore to

$$\frac{B'_1tt'}{qt^2t' - t + pt'},$$

which we are able to put easily under this form [644]

$$(\Pi)\quad \left\{\begin{aligned} &\frac{B'_1}{p}t\frac{1}{\left(1 - \frac{\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}}{2p}t\right)\left(1 - \frac{\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}}{2p}t\right)} \\ &= \frac{B'_1}{p}\frac{t}{\sqrt{\frac{1}{t'^2} - 4pq}}\left\{\frac{\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}}{1 - \frac{\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}}{2p}t} - \frac{\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}}{1 - \frac{\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}}{2p}t}\right\}\end{aligned}\right.;$$

whence we deduce, for the coefficient of t^{a+b}, the expression

$$\frac{B'_1}{p}\frac{1}{(2p)^{a+b-1}}\frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^{a+b} - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^{a+b}}{2\sqrt{\frac{1}{t'^2} - 4pq}}.$$

But this coefficient is the generating function of $y_{a+b,x'}$, a quantity which is equal to unity; because it is certain that player A has won the game when he has won all the tokens of B: moreover, x' must be here zero or an even number, since the number of trials in which A is able to win the game is equal to b plus an even number; and, in fact, he must win all the tokens of B, and again win again each token that he has lost, that which requires two trials. The series

$$y_{a+b,0}t'^0 + y_{a+b,2}t'^2 + y_{a+b,4}t'^4 + \cdots ,$$

which represents the coefficient of t^{a+b}, is therefore equal to $\frac{1}{1-t'^2}$, and we conclude from it

$$\frac{B'_1}{p}\frac{(2p)^{a+b-1}}{1-t'^2}\frac{2\sqrt{\frac{1}{t'^2}-4pq}}{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}}.$$

Now the coefficient of t^a, deduced from the development of the function (Π), always with respect to the powers of t, will be [645]

$$\frac{B'_1}{p}\frac{1}{(2p)^{a-1}}\frac{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}}{2\sqrt{\frac{1}{t'^2}-4pq}},$$

and by substituting for $\frac{B'_1}{p}$ its value, we will have this coefficient or the generating function of $y_{x,x'}$ equal to

$$\frac{2^b p^b}{1-t'^2}\frac{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}}{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}}$$

or

$$\frac{2^b p^b t'^b}{1-t'^2}\frac{\left(1+\sqrt{1-4pqt'^2}\right)^{a}-\left(1-\sqrt{1-4pqt'^2}\right)^{a}}{\left(1+\sqrt{1-4pqt'^2}\right)^{a+b}-\left(1-\sqrt{1-4pqt'^2}\right)^{a+b}},$$

that which is formula (o) of the *Théorie analytique*.