

PROBLEMS TO RESOLVE
SECOND PROBLEM
ON THE GAME CALLED HER

PIERRE RENARD DE MONTMORT
EXTRACTED FROM
ESSAY D'ANALYSE SUR LES JEUX DE HAZARD
2ND EDITION OF 1713, §218, PP. 278–279

218. One draws first the places, one sees next to whom will have the deck. We suppose that this is Pierre, & we name the other Players Paul & Jacques.

One agrees to put a certain sum into the game; each of the Players takes for this sum an equal number of tokens, & the one wins all the money of the game there who remains with one or more tokens, the other Players having none. Here is how the game is conducted.

Pierre takes an entire deck composed of fifty-two cards, & gives one of them to each of the Players, by beginning with his right, & at the end of each play the one who is found to have the lowest card, loses a token which he puts into the middle of the table.

Paul who is the first to the right of Pierre, has right if he is not content with his card, to exchange it with Jacques, who is able to refuse to him only in the sole case that he has a King, then Jacques says *cuckoo*. By this term the one who has a King warns the Players that his neighbor to the left having wished to be undone of his card has been stopped by his. It is likewise with Jacques in regard to Paul, & with Paul with regard to Pierre.

It is necessary only to remark, 1^o, that if Pierre is not content with his card, either that which he is given first, or that which he has been constrained to receive from Jacques, he can, having no person with whom to exchange, attempt to take a better card by cutting at random among those which remain in the deck. 2^o That if it happens that Pierre having for example a five, does not wish to hold it, & if by cutting he draws for example a Jack, his card will become a Jack, & thus of every other card, with the exception of the King, because Pierre drawing a King is returned to his card such as it is, & he is found as if he was at first held to his card.

All this changing of cards being made, each Player uncovers his, & the one who is found to have the lowest beginning with the ace, puts a token into the game.

If it is encountered that two or more Players have the same card, & that it is the lowest, the one who has the priority, that is to say the one who is the nearest to the right of Pierre loses & pays. This which shows that we must always hold when we have given to the Player who is at the left a card similar to the one which we receive from him, likewise if we have given to him one lower.

The Player who has lost all his tokens exits from the rank, & the others continue the game until that which each, with the exception of one, have lost all their tokens, in which case the one who remains wins the money of all the Players, & this one is called in terms of the Players *to win the pool*.

Date: August 29, 2009.

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH. .

Here is the Problem of which we demand the solution.

Three Players, Pierre, Paul & Jacques are the only Players who remain, & they have no more than one token each. Pierre holds the cards, Paul is to his right, & Jacques next. We demand what is their lot with respect to the different place which they occupy, & with what proportion should the money of the pool be set out again, this will be, for example, ten pistoles, if they would wish to divide it among themselves without ending the game.

Extract of the letter from Mr. de Montmort to Mr. Nicolas Bernoulli

At Montmort 10 April 1711. (pg. 321)

I have undertaken since some time to achieve the solution of the Problems which I propose at the end of my Book; I find that in Her, when there remains no more than two Players Pierre & Paul, the advantage of Paul is greater than $\frac{1}{85}$, & less than $\frac{1}{84}$. This Problem has some difficulties of a singular nature.

Extract of the letter of Mr. Nicolas Bernoulli to Mr. de Montmort

At Basel this 10 November 1711. (pg. 334)

I have also resolved the Problem on Her in the most simple case, here is that which I have found. If we suppose that each of the Players observes the conduct which is the most advantageous to him, it is necessary that Paul is held only by a card which is higher than a seven, & Pierre to one which is higher than an eight, & we will find that under this assumption the lot of Pierre will be to the lot of Paul as 2697 to 2828. If we suppose that Paul is held also at a seven, then Pierre must be held at an eight, & their lots will be again as 2697 to 2828. It is nonetheless more advantageous for Paul to not be held at a seven as to be held at it, this which is an enigma which I leave to you to develop.

Extract of the letter from Mr. de Montmort to Mr. Nicolas Bernoulli

At Paris 1 March 1712. (pg. 337-340)

The Letter which you have taken the pain to write to me, Sir, dated 13 November, is filled up with admirable things. The affairs which I have at Paris have not left me at all since I am there, & will not leave me at all so much that I will be left the liberty of mind necessary to examine all the beauties of your Letter. Thus until by my return to Montmort I have regained this leisure & this tranquility of mind that I esteem so much, & of which I have besides absolutely need in order to follow you in your algebraic meditations. I will limit myself in this here to make you part of the reflections of two of my friends who I have left at Montmort, & who I have strongly invited, in quitting them, to examine the Problems which you propose on Tennis, & that which you say touching Her; I will join those that I have made lightly enough on some points of your Letter.

Now so that you know, Sir, to whom you have to make, & who are these Sirs who full of admiration for your talent, dare however not at all to be submitted to your decisions; you know that one of the two is called Mr. the Abbé de Monsoury. We are neighbors in the country, his Abbey is only at a mile and a half from Montmort. The other is named Mr. Waldegrave. This is an English Gentleman, brother of the late Milord Waldegrave, who had married a natural daughter of King James. When I worked on Her some years ago, I shared with Mr. the Abbé of Monsoury that which I had found; but neither my calculations nor my reasonings could convince him. He upheld to me always that it was impossible to determine the lot of Pierre or the one of Paul, because we could not determine at what card Paul must be held, without knowing to what card Pierre must be held, & *vice versa*, that which made a circle, & rendered in his opinion the solution impossible. I added a

quantity of subtle reasonings which made me a doubter that I had caught the truth. I was there when I had proposed to you to examine this Problem, my goal was to assure myself through you of the goodness of my solution, without having the pain of reporting my ideas on the subject which were entirely effaced. I have seen with pleasure that you have found as me that Paul could do better only to be held to an eight whatever choice as Pierre took; & that Pierre, when Paul is held, could make his choice better only by being held solely at the nine, whatever choice that Paul has taken; & that under this assumption that Pierre & Paul both take the choice which is their most advantageous, the lot of Paul was to the lot of Pierre :: 2828 : 2697. The honor that your decision has made to my solution, has given rise to our Sirs, & on all to Mr. the Abbé of Monsoury, to examine this Problem at the base. Here is that which they have written to me on this subject it is more than a month.

“In order for you to render account, Sir, of the judgment that Mr. the Abbé & me ourselves have dared to pronounce against Mr. Bernoulli on the subject of his solution on Her; it is not true, according to us, that Paul must be held only at the eight, & Pierre at the nine. We claim that it is indifferent to Paul to change or to be held at the seven, & to Pierre to change or to be held at the eight. In order to prove it, I must first expose their lot in all the cases. The one of Paul having a seven, is $\frac{780}{50 \times 51}$ when he changes, & when he is held his lot is $\frac{720}{50 \times 51}$ if Pierre is held at an eight, & $\frac{816}{50 \times 51}$ if Pierre changes at the eight. The lot of Pierre having an eight is $\frac{150}{23 \times 50}$ if he is held, & $\frac{210}{23 \times 50}$ if he changes in the case that Paul never is held at the seven; & $\frac{350}{27 \times 50}$ by being held, & $\frac{314}{27 \times 50}$ by changing in the case that Paul is held at the seven, here are them all in sequence. The lots of Paul $\frac{780 \text{ or } 720 \text{ or } 816}{50 \times 51}$, those of Pierre $\frac{150 \text{ or } 210}{23 \times 50}$ or $\frac{350 \text{ or } 314}{27 \times 50}$.

“720 being more below 780 than 816 is above, it seems that Paul must deduce a reason to change at the seven. I call this weight which carries Paul to change *A*, likewise Pierre of his different lots must deduce a reason to change at the eight, I call this weight *B*. This put, we say that the same weights carry Pierre & Paul equally to the two choices: Therefore, &c. *A* carries Paul to change his seven, & consequently carries Pierre to change his eight; but that which carries Pierre to change his eight, carries also Paul to be held at his seven. Therefore *A* carries Paul equally to change his seven, & to be held. Likewise *B* carries Pierre to change his eight, & consequently Paul has to be held at the seven; but that which carries Paul to guard his seven, carries also Pierre to guard his eight, & consequently carries Paul to change his seven. Therefore *B* carries Paul equally to change & to guard his seven: it is likewise of Pierre. Therefore the same weights carry equally, &c. Therefore it is false that Paul must be held only at the eight, & Pierre only at the nine. Apparently Mr. Bernoulli is contented to regard the fractions which express the different lots of Pierre & of Paul, without paying attention to the probability of that which the other will make.”

They have confirmed in some later Letters that which they advance in this here, & add: *As we do not agree with Mr. Bernoulli that Paul must be held only at the eight & Pierre at the nine; we have not sought the explication of his enigma which we believe founded on a false supposition.*

Extract of the letter of Mr. Nicolas Bernoulli to Mr. de Montmort
At Basel this 2 June 1712. (pg. 348-349)

I am very sensible to the honor which your two Friends, Mr. the Abbé d’Orbais & Mr. de Waldegrave, have made me in examining that which I have written you on Her, & I am quite obliged to you of their sentiments on that subject which you have communicated to me. The lots which they have found for Pierre & Paul are quite correct; but the reasoning

by which they wish to prove that it is indifferent to Pierre to change at the seven or to be held, & to Paul to change or to be held at the eight, cannot convince me; because in examining it more closely we will find that it is a sophism, & that we cannot reason thus: *The weight A carries Paul to change his seven* (when it is uncertain to what Pierre will determine,) & *consequently carry Pierre to change his eight*, (supposing that Pierre knows that Paul changes at a seven;) *but that which carries Pierre to change his eight, carries also Paul to be held at his seven: Therefore A carries Paul equally to change his seven & to be held*; because we suppose two contradictory choices at the time; namely, that Paul knows & that he is ignorant at the same time what choice Pierre will take & Pierre what choice Paul will take. It is quite true that the weight *A* carries Paul to change at the seven. Having therefore made this hypothesis that Paul has the maxim to change at the seven, it follows that Pierre will do better to change at the eight; but we must stop there, & not pass beyond, because it is not permitted to return to Paul & to conclude; therefore Paul must guard his seven, because according to this hypothesis we have already fixed that Paul has the maxim to change at the seven, & that Pierre changes at the eight only on condition that Paul changes at the seven: Therefore Paul is not able to change from maxim & be held at the seven, without that Pierre change also his; so that following the reasoning of these Sirs one would go always in a circle, that which is a demonstration that we can prove nothing from it. Moreover it is clear by the calculus that it is not indifferent to the Players to change at the seven or at the eight, or to be held; because if this were, we would find also the same lots for all these cases there; now one finds by the calculus that their lots are different according as they are held at such or such card. Therefore it is false that the same weights carry Pierre & Paul equally to the two choices. If these Sirs are not content in this response, I will give myself the honor of writing to you at the first occasion more amply on this, & to make you part of the method of which I have served myself to resolve this Problem, & I hope that these Sirs will find nothing to retell. I am quite pressed presently to enter into detail on all these things. I pray you to assure these two Sirs of my very humble respects, & to thank them on my part of the particular esteem which they wish well to have for me.

Extract of the letter from Mr. de Montmort to Mr. Nicolas Bernoulli
At Montmort this 5 September 1712. (pg. 361-362)

Sir,

When I received the Letter which you have made the honor to write me, dated 2 June, I was ill in bed of a gross ongoing fever; I showed it to Mr. the Abbé d'Orbais who had come to see me, he took copy of that which regards the game of Her, & will send it to his second Mr. de Waldegrave, of whom he received response some days after.

Mr. the Abbé d'Orbais, who your reasons have not at all shaken, supporting still the judgment of Mr. de Waldegrave, comes to write me this note, of which I am sure that style will please you.

Read, Sir, this Letter of Mr. de Waldegrave, it is excellent. One cannot respond to Mr. Bernoulli with more justice & precision. Oh! but I had done well leaving this illustrious Geometer my ally to speak, I myself would never be so sharply explained; if you write to him, I pray to you to show him that I have nothing at all to add. Solas admirandi plaudendique partes mihi reliquit. Besides it is time that you took part in this dispute, Mr. de Waldegrave invited you in the Letter which you have shown me. You are too long time balanced by the name of Bernoulli on one side, & by our reasons on the other. There are

no means to permit you a longer time in this situation too prudent in my opinion for one so great Master. I greet very humbly the Ladies, & give you the good night.

Here is therefore a dispute in fashions, it is, this seems to me, quite lovely; & I myself know great pleasure in having given birth to it, the question is subtle & entirely of argument. Our Sirs are charmed by your honesty & by your modesty, they find themselves very honored that a great Geometer as you, at last a Bernoulli, wishes to break a lance with some Novices as themselves: They make both a thousand compliments to you.

Extract of the letter of Mr. Nicolas Bernoulli to Mr. de Montmort
At Brussels this 30 December 1712. (pg. 376-378)

I begin with that which regards our dispute on Her. I have not the same subject, Sir, to admire & to applaud in the responses of Mr. de Waldegrave, as has done Mr. the Abbé de Monsoury. I know very well that all the reasonings that Paul can make in order to be determined in a choice, Pierre can also make them in order to turn this game to the disadvantage of Paul; but notwithstanding this, I say that Paul does not do so well in following the maxim to guard the seven, as in following the maxim to change at the seven, here are the reasons for it: If it were impossible to decide this Problem, Paul having a seven would not know what choice to take; & in order to rid himself he would commit himself to pure chance, for example, he would put into a sack an equal number of white tokens & of black tokens, in the design of holding himself at the seven if he draws a white, & to change at the seven if he draws a black; he would put, I say, an equal number of whites & of blacks; because if he would put an unequal number, he would be more carried for one choice than for the other, that which is against the hypothesis. Pierre having an eight would do the same thing in order to see if he must change or not. Now each Player Paul & Pierre having the maxim of being committed thus to chance, the lot of Paul will be $\frac{774}{50.51}$ which is less than $\frac{780}{50.51}$, which is the lot of Paul when he changes at the seven; whence it follows that Paul takes a bad choice when he commits himself to chance, & that he has a better lot in changing at the seven. Therefore it is decided that Paul must change at the seven: Here is another demonstration of it which is based on the same circle as one opposes me. As we can always demonstrate whatever choice that Paul takes, that it is a bad choice, & that he would have done better if he had taken the contrary choice, it is worth more to take the choice where one risks the least; now Paul changing at the seven risks only $\frac{36}{50.51}$, instead by guarding the seven he risks $\frac{60}{50.51}$; therefore it is worth more to change at the seven. The reasoning which has lead me to the solution which I have sent to you in my Letter of 10 November 1711, is scarcely different from this one that I just made. Before exposing it to you I wish that you accord to me that which follows: I hold in order to demonstrate that if Paul is held or the maxim of being held at the seven, Pierre must be held at the eight; likewise if Paul is held or to the maxim of being held at the eight alone, Pierre must change at the eight. Therefore if one has demonstrated that Paul must either be held or to change at the seven, one has also demonstrated that Pierre must either be held or to change at the eight; & if one is convinced that it is true that Paul must be held at the seven, for example, one is also convinced that it is true that Pierre must be held at the eight; but one cannot make the return of Pierre to Paul; & by taking for foundation that Pierre is held at the eight, one cannot, I say, in any manner draw the conclusion from it; therefore it is true that Paul must change at the seven; because it is certain that Paul must be held at the seven, I must seek no more by another way that which he must do, because I know it already; & a man is ridiculous who knowing that he must make a choice wishes again to doubt & seek by other ways if he must make it or not. This put, here is how I have reasoned: Paul having

the design to follow the maxim to guard the seven, examines that which can happen to him at worst with respect to the choice which Pierre will take; & he finds that the worst is when Pierre is held at the eight; & that then his lot will be to the lot of Pierre as 2828 to 2697, or that by naming the money of the game 1, his expectation will be $\frac{2828}{5525}$. Next he examines also that which can happen to him at worst when he follows the maxim to change at the seven, & he finds that the worst is when Pierre follows the maxim to change at the eight, & that then his lot will be also $\frac{2828}{5525}$. Therefore the lot of Paul, by following the maxim to guard the seven, is at least $\frac{2828}{5525}$, & it is something more when Pierre follows the maxim to be held at another card than at the eight. Likewise the lot of Paul, by following the maxim to change at the seven is more than $\frac{2828}{5525}$, when Pierre has the maxim of holding himself at another card than at the nine: Therefore the lot of Paul is always at least $\frac{2828}{5525}$, & that which must carry him to follow one maxim rather than the other, is the risk which Pierre courts by not encountering it correctly. Now this risk is greater when Paul changes at the seven, than when he is held at the seven: Therefore Paul must change at seven rather than be held; & consequently Pierre must have the maxim to change at the eight; but one cannot return from Pierre to Paul, & to repeat; therefore Paul must have the maxim to guard the seven. This circle offends Logic, & it is impossible that it is good reasoning: a proposition from which one draws an absurdity is false; now to draw its contradictory is to draw an absurdity; therefore the proposition is false, & consequently its contradictory is true. Now in our case its contradictory knows how to be true no more, because of the same circle; therefore this circle is ridiculous & a false reasoning. One deduces from the proposition *A* its contradictory *B*; therefore the proposition *A* is false, & the proposition *B* is true. Likewise one deduces from proposition *B* its contradictory *A*; therefore *B* is false & *A* is true; therefore this circle demonstrates that two contradictory propositions are both true & both false, that which is impossible; therefore there had been a fault in the reasoning, & I have not at all been wrong, when I say that our Sirs suppose two contradictory things at the time in making the circle. In a word when even it would not be true that Paul does better to change than to be held, it would not be their circle which would demonstrate it. I believe that you will fall in accord with this, & you will hold this response sufficient to the objections which Mr. de Waldegrave & Mr. the Abbé de Monsoury have made to me. I salute very humbly these Sirs, & I thank them for the good opinion which they have of me.

Extract of the letter from Mr. de Montmort to Mr. Nicolas Bernoulli

At Paris this 20 August 1713. (pg. 400)

Your reasonings on Her have converted our Sirs not at all; they find them very delicate & very subtle, but they assure not to be convinced; & as I know their straightforwardness & their frankness, I can be their guarantee that they say that which they think.

Extract of the letter from Mr. de Montmort to Mr. Nicolas Bernoulli

At Paris this 15 November 1713

(pg. 403-406) Since you wish, Sir, that I declare to you finally that which I think in this famous dispute on Her, I am going to obey you.

It seems that our Sirs having claimed at the beginning of the dispute that it was indifferent to Paul to change or to be held at the seven, & to Pierre to change or to be held at the eight, by that they would have too much according to me: you demonstrate quite well that this maxim is false; but you know that in the conversations which they have had with you they themselves are explained in supporting that it was impossible to establish any maxim for the one & the other Player, & in this I believe that they have reason. The two arguments

which you produce against this assertion in your Letter of 30 December 1712 is not able to convince me; I am to the contrary persuaded that the solution of the Problem is impossible, that is to say that one is not able to prescribe to Paul the conduct which he must keep when he has a seven, & to Pierre when he has an 8. It is very true that it is worth more for Paul to take the maxim to change at the 7 than to take from them any other fixed & determined. By this reason that whatever other maxim that we wish to determine for Paul, Pierre who will be instructed in it will take one of them which will render the lot of Paul less than $\frac{780}{50.51}$; but it does not follow that Paul must for this renounce the expectation to render his lot better by holding himself at the 7 than by changing from it. I have believed sometime that a certain composition of tokens for Pierre & for Paul would save the circle; but I have found that we always fall back to it. Suppose that we have prescribed to Paul the maxim of putting a white tokens & b black tokens into a pouch, by intending to change at the 7, if drawing a token it is found white, & to be held if drawing a token it is found black. Pierre who will know the maxim of Paul, which maxim will he follow? He will observe that if he puts into a sack c white tokens & d black tokens, for next drawing a token among all is determined to change at the 8 or to be held, according as he will draw a white or black token; he will observe, I say, by examining this general expression of the lot of Paul:

$$\frac{2828ac + 2834bc + 2838ad + 2828bd}{13.17.25.a + b.c + d}$$

1°. That the lot of Paul is the same when a is infinite with respect to b , & c infinite with respect to d ; or when b being infinite with respect to a , d is infinite with respect to c . 2°. That a being infinite with respect to b the lot of Paul will be so much greater as c will be small with respect to d . 3°. That the lot of Paul is never better than when d being very great with respect to c , a is quite great with respect to b , & c . must all this contains the choice that Paul must take only conditionally to the one of Pierre, & the one of Pierre only conditionally to the one of Paul, that which makes a circle. It appears to me that all the reasons which you bring in order to prove that this circle does not take place, & that the return which amends it is not vicious; it seems to me, I say, that these reasons do not prove that which you wish to prove, but only that that which falls in a like circle is impossible: because finally the supposition that Paul must make himself the maxim to change at the 7 necessarily drives for Pierre the maxim to change at the 8; & this maxim thus established for Pierre, carries away a perfect demonstration that Paul will necessarily take the maxim of being held at the 7: this contradiction is deduced legitimately & without notice to the rules of Logic, since one must suppose that one & the other Player is equally subtle, & will take his choice only on the knowledge that he will have of the choice which the other will take. Now as there is not at all here some fixed point, the maxim of one depending on the maxim of the other Player which is not yet known, as soon as one wishes to establish one of them, one deduces from this assumption a contradiction which demonstrates that one had not ought establish it.

The demonstration which you based on the same circle which one opposes you is very subtle, but it is easy to respond to it.

*As one can (you say) demonstrate whatever choice that Paul takes, &c.*¹

There is here equivocation in the beginning, one does not say at all that one can always demonstrate whatever choice that Paul takes that it is a bad choice, one claims only that one can not establish some maxim. Besides, Sir, it does not suffice to know that Paul can increase his lot by 10 in changing, & only by 6, in being held, when in the two cases Pierre

¹See the letter of 30 December 1712.

will take a bad choice. It would be necessary in order that your demonstration be complete, that you can at the same time demonstrate that the probability in order to increase his lot by 10 is to the probability in order to increase his lot by 6 in a greater ratio than 6 to 10, but this is that which you don't know how to prove: it is likewise in your third argument which begins thus *The lot of Paul is always $\frac{2828}{5525}$, &c.*²

For me, Sir, of all the reasons which must appear to engage Paul to take the maxim to change at the 7, I conclude from it only that he will be well in the practice to make himself a law to change more often at the seven than to be held; but how much more often he must change than to be held, & in particular that which he must make (*hic & nunc*) because this is principally there the question: the calculus teaches nothing on this subject, & I hold the decision impossible.

By this evident reason that if you establish that Paul must change at the seven, Pierre must make himself the maxim to change at the eight, in which case Paul who is as skilled as Pierre will know that he will change at the eight, & that thus he agrees with him to take the maxim of holding himself at the seven, &c. we will meet again in the circle.

In a word, Sir, if I know that you are the counsel of Pierre, it is evident that I myself Paul must be held at the seven; & likewise if I am Pierre, & if I know that you are the counsel of Paul, I must change at the eight, in which case you have given a bad counsel to Paul.

The more I think, & the more I am forced to think on this subject as our Sirs. It does not follow for this that you have erred, the consequence would be worth nothing; suppose therefore, Sir, that I am myself in error, you will much oblige me to hold me to it, by giving me the explication & the demonstration of your anagram.

As this matter is quite curious & as you have much meditated, I would be very glad if you wished well to instruct me at the same time on another question which is completely of the same kind.

A father wishes to give the gifts to his son & he says to him, I am going to put in my hand a number of chips even or odd as I will judge it à propos: this done, if you name even, & if it is even in my hand, I will give to you four écus; if you name odd, & if there is even in my hand, you will have nothing. If you name odd, & if there is odd in my hand, you will have one écu; if you name even, & if there is odd in my hand, you will have nothing. I demand, 1 °, what rule it is necessary to prescribe to the father in order that he economizes his money the best that it is possible. 2 °. What rule it is necessary to prescribe to the son in order that he takes the better choice. 3 °. That one determines what advantage the father makes to his son, & to how much one can evaluate his gifts, by supposing that each of the two will take the conduct which is the most advantageous to him. These questions are very simple, but I believe them insolubles; if this is, it is great pity, because this difficulty is encountered in many things of civil life: when two persons, for example, having business together, each wishes his rule on the conduct of the other; it has place also in many games, mainly in Brehan, a game which makes now the delights of the Ladies of Paris: here is a kind of it. When the turns are large, as the turns being for example to the two or to the four cards, the pass is triple or quadruple; the one who is last believes able to say from the game & to steal the pass, with a nearly entire assurance, by reason that he is to believe that if the first or the second had had of the game, they would have opened the pass, wishing not at all to be exposed to lack it. On the other side the more the motives are strong in order to engage the last to go from the game or even to make a gross vade, the more the first & the second are tempted to pass with good play, in the hope to catch the last who would wish to steal: what rule to give that there? It is impossible, it seems to me, to prescribe anything

²See the end of this letter.

assured. All the ability of the more refined Players is reduced to give to those with which they play a false idea of their manner of play, to affect a certain conduct in some coups of small value, in order to change apropos in the gross coups, & to profit skilfully from an error or prevention in which they will have given in design occasion. To this game, as throughout besides, the skillful are sometimes caught; but it is certain that in this game it is good to be, by reason that one has often business with some persons who are not at all or who are less; because among equally skillful & clarivoyent Players, such as we suppose Pierre & Paul in the game of Her: it would be absolutely impossible to prescribe any rule in the case of Brelan, no more than for our case of Her.

(pag. 409-412) In the time that I write this Letter, Sir, I received one from Mr. de Waldegrave; I wish you part of it, because it exhausts, it seems to me, all that which we can say on this matter. I had sent to him what I worked on to explain to you that which I think on Her; he has wished to make a last effort in order to assure his right & to put it in evidence. His Letter is from Chateau de Berviane & of 13 November: here is the extract of it.

“ a being = 3, & b = 5, we see by formula³ that if Pierre has the maxim to change the 8, the lot of Paul is

$$\frac{8484 + 14170}{13.17.25 \times 8} = \frac{11327}{5525 \times 4} = \frac{2831}{5525} + \frac{3}{5525 \times 4}.$$

“And if Pierre has the maxim to guard his eight

$$= \frac{8514 + 14140}{13.17.25 \times 8} = \frac{2831}{5525} + \frac{3}{5525 \times 4}.$$

“Thus it is equal in this supposition to Pierre to change at the eight or not; & generally the lot of Paul is always $\frac{2831}{5525} + \frac{3}{5525 \times 4}$, when $a = 3$ & $b = 5$, whatever value which we can give to the letters c & d , that is to say that his lot will always be $\frac{2831}{5525} + \frac{3}{5525 \times 4}$, whatever choice that Pierre takes when he will have an 8, either to change or to guard his 8 determinedly, or to be committed to an equal or unequal number of tokens.

“It follows thence that the lot of Paul is at least $\frac{2831}{5525} + \frac{3}{5525 \times 4}$, when he holds only to him to take three white tokens & five black; & if Paul holds another conduct, it is that he hopes to render his lot yet better.

“Therefore Mr. Bernoulli has been wrong, & you also, Sir, not to displease you, to say once that the lot of Paul was to the lot of Pierre :: 2828 to 2697. Mr. Bernoulli has not thought apparently of this way to cast, which effectively seems to be not of the ordinary rules of the game; but he appears to have noticed since our dispute that he has not had reason to say that the lot of Paul was $\frac{2828}{5525}$, since in one of his last Letters he put that the worst which can happen to Paul is to have $\frac{2828}{5525}$.

“It is therefore certain & demonstrated that I have had reason to sustain that under the assumption that each of the Players plays the most advantageously which is possible to him, the lots of Paul & of Pierre are not those which Mr. Bernoulli has given, & that you have held true once; since Paul can render his lot greater than $\frac{2828}{5525}$, & have $\frac{2831}{5525} + \frac{3}{4} \times \frac{1}{5525}$, when he will wish to be contented in it, & (that which it is necessary to note) by putting more tokens to be held at the seven than for changing.

“Here is therefore one of the points of the question decided; I am sure that you will agree & Mr. Bernoulli also. In regard to that which I have also sustained that one could not establish some maxim; although it is impossible to me to demonstrate it with the same evidence, I believe not to be less well-founded to sustain it. Mr. Bernoulli says that the

³The previous formula $\frac{2828ac+2834bc+2838ad+2828bd}{13.17.25.a+b.c+d}$

worst which can happen to Paul is to have $\frac{2828}{5525}$, & adds *that that which must determine him to change rather than to be held, is that if Pierre takes a bad choice, the lot of Paul will be greater when he will change at the seven, than when he will be held.*

It is true that under any other assumption of the values of a & b , than that of $a = 3$ & $b = 5$, Paul can render his lot better than in this one, if Pierre takes the bad choice; but also he will render it worse if Pierre takes the good choice; & what way is there to discover the ratio of probability which there is that Pierre will take the good choice to the probability that he will take the bad, this would seem to me absolutely impossible, & would make fall into the circle? It is true also that the more Paul will increase the value of a , with respect to b , the more he can approach his lot of $\frac{2838}{5525}$, which is all that which can happen to him more advantageous if Pierre takes a bad choice, & if in augmenting the value of b , with respect to a , his lot can never pass $\frac{2834}{5525}$, & the more as in the one & the other case, his lot can never be less than $\frac{2828}{5525}$, as Mr. Bernoulli has quite well remarked; but we can not conclude thence that Paul must render a infinite with respect to b ; or that which is the same thing, that he must change always at the seven; because if this consequence were good, Pierre who is supposed as capable as Paul could also conclude simultaneously that Paul would be held, if he has a card above the seven, & consequently would change infallibly at the eight, & by this means the lot of Paul would be only $\frac{2828}{5525}$, which is that which can happen to him the more bad; thus he would have done bad to have taken the maxim to change always at the seven, & here we are in the circle.

“I have forgotten to make you observe that Pierre has one way in order to bound the lot of Paul to $\frac{2831}{5525} + \frac{3}{4} \times \frac{1}{5525}$, by making $c = 5$ & $d = 3$, that which we will see yet with evidence, by substituting into the formula the values c & d . We could believe that as it holds only to Pierre to limit the lot of Paul to $\frac{2831}{5525} + \frac{3}{4} \times \frac{1}{5525}$, likewise that it holds to Paul only to be assured this same lot, Paul must make always $a = 3$ & $b = 5$, & Pierre $c = 5$ & $d = 3$, that which would make a constant maxim for the one & the other Player; but it seems to me that it would be bad to establish by the reasons which we have said so many times, & that this must not prevent Pierre & Paul to finesse, in the expectation of rendering each his lot better.”

(pg. 413) As in these Letters we have spoken of Her, I have judged that it was à propos to put here the calculations, in order to spare to the Reader the pain of making them.⁴

I will add here on the occasion of that which I have said on the subject of Brehan in this Letter some calculations which I myself am amused to make at the plea of a Player of my friends: one finds by the article 25 that one can wager at each coup without advantage or disadvantage.

- 1°. 1 against 38, that he will find himself a Brehan at least.
- 2°. 2 against 7473, that he will find himself at least two of them.
- 3°. 2 against 1726723, that he will find himself three of them.
- 4°. 1 against 974, that he will find himself the fourth Brehan.
- 5°. 1 against 272, the he will find himself the favorite Brehan.

Whence it follows by article 186 that one can wager with advantage that

the first case will arrive in 27 coups	}	at least.
The second in 2591 coups		
The third in 604354 coups		
The fourth in 676 coups		
The fifth in 189 coups		

⁴See the Table on page 11.

	Lot of Paul when he has the maxim to change at the seven, & Pierre the one to change at the eight.	Pierre has the maxim to hold himself at the eight.	Lot of Paul when he has the maxim to be held at the seven, & Pierre the one to hold himself at the eight.	Pierre the one to change at the eight.
King,	1200	1200	1200	1200
Queen,	1052	1058	1058	1052
Jack,	888	902	902	888
Ten,	724	746	746	724
Nine,	560	590	590	560
Eight,	476	434	434	476
Seven,	390	390	360	408
Six,	444	444	444	444
Five,	490	490	490	490
Four,	528	528	528	528
Three,	558	558	558	558
Two,	580	580	580	580
Ace,	594	594	594	594
	$\frac{8484}{13.51.25} = \frac{2828}{5525}$	$\frac{8514}{13.51.25} = \frac{2838}{5525}$	$\frac{8484}{13.51.25} = \frac{2828}{5525}$	$\frac{8502}{13.51.25} = \frac{2834}{5525}$

TABLE I. Table for Her

That is to say, for example, that there will be advantage to wager that in 27 coups there will be some Brehan, & that there will be disadvantage to wager for 26 coups. It is likewise of the other cases, if this is not in regard to the third where there can be found error in the last two digits. A long work would be necessary in order to render them more exact, & this would not be worth the effort.