Mémoire sur le calcul des fonctions génératrices*

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On 8 July 1836, I read to the Academy a memoir on the calculus of generating functions and on some applications of this calculus to the analysis of probabilities.

I rectified, in first place, the inexactitudes overlooked by Laplace in the exposition of the principles of the calculus of generating functions (see *Théorie analytique probabilités* page 9 and following).

We designate by y_x a function of the variable x and by u a function of t generator of y_x; one will have, according to the author of the *Méchanique celeste*,

$$u = y_0 + y_1 t + y_2 t^2 + \cdots + y_x t^x + \text{etc.}$$

But the generator of y_{x+1} will not be, as he said, $\frac{u}{t}$; it will be $\frac{u-y_0}{t}$. The generating function of y_{x+2} will have for value $\frac{u-y_0-y_1 t}{t^2}$, and generally, the generating function of y_{x+i}, instead of being, as Laplace had believed, $\frac{u}{t^i}$, will be expressed by $\frac{u-y_0-y_1 t-y_2 t^2-\cdots-y_{i-1}t^{i-1}}{t^i}$.

It follows from it that the generating function of the finite difference Δy_x will not be $u\left(\frac{1}{t}-1\right)$, but it will be $u\left(\frac{1}{t}-1\right)-\frac{y_0}{t}$; that of $\Delta^2 y_x$ will be found equal to $u\left(\frac{1}{t}-1\right)^2-\frac{y_0}{t^2}-\frac{y_1-2y_0}{t}$, instead of $u\left(\frac{1}{t}-1\right)^2$, and so forth.

We suppose $\Sigma y_x = z_x$ and designate by u' the generating function of z_x, the generating function of $Az_x = y_x$ will be $u'\left(\frac{1}{t}-1\right)-\frac{z_0}{t}=u$, therefore $u'=\frac{ut+z_0}{1-t}$, z_0 is evidently as much as an arbitrary constant. The preceding value of the generator of Σy_x is that which Laplace himself gives; but this same value, which is without doubt exact, is able to show the inexactitude of the generating function of Ay_x, that one finds in the *Théorie analytique probabilités*; in fact, knowing the generator $\frac{ut+z_0}{t-1}$ of Σy_x in order to recover u, that is the generator of y_x, there is, according to Laplace, only to multiply $\frac{ut+z_0}{1-t}$ by $\frac{1-t}{t}$; but by making this multiplication one finds $u+\frac{z_0}{t}$ instead of u. According to us, it does not suffice to multiply $\frac{ut+z_0}{1-t}$ by $\frac{1-t}{t}$, it is necessary further to subtract $\frac{z_0}{t}$ from the product, that which gives for result u.

The equations between the generating functions will hold by passing again to the coefficients of these functions, and reciprocally. But in the passage one must observe the rules above instead of those of Laplace; these last lead often to some indefensible results.

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We suppose, for example, that one has an equation in finite differences

$$0 = ay_x + a_1 y_{x+1} + a_2 y_{x+2} + \cdots + a_n y_{n+x}$$

a, a_1, a_2, $\cdots$, a_n being some constants. By passing to the generating functions in the manner of Laplace, one finds

$$0 = u\left(a + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{u_n}{t^n}\right)$$

or

$$0 = a + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{u_n}{t^n}$$

an equation which determines the variable t. Now this variable must, by its nature, remain completely indeterminate. But if one observes in the passage to the generating functions the rules above, one will find

$$0 = \left(a + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{u_n}{t^n}\right) u - \frac{a_1 y_0 + a_2 y_1 + \cdots + a_n y_{n-1}}{t}$$
$$- \frac{a_2 y_0 + a_3 y_1 + \cdots + a_n y_{n-2}}{t^2} - \cdots - \frac{a_{n-1} y_0 + a_n y_1}{t^{n-1}} - \frac{a_n y_0}{t^n}$$

whence

$$u = \frac{\begin{array}{r} a_n y_0 + (a_{n-1} y_0 + a_n y_1)t + \cdots + (a_2 y_0 + a_3 y_1 + \cdots + a_n y_{n-2} t^{n-2} \\ +(a_2 y_0 + a_2 y_1 + \cdots + a_n y_{n-2} t^{n-2} \end{array}}{a_n + a_{n-1} t + \cdots a_1 t^{n-1} + a t^n}$$

It will be easy to find the generating function of the quantity y_x defined by the equation

$$ay_x + a_1 y_{x+1} + a_2 y_{x+2} + \cdots + a_x y_{n+u} = A + A_1 x + A_2 x^2 + \cdots$$

A, A_1, A_2 $\cdots$ as a, a_1 $\cdots$ being some constants. In fact it will be only to equate the generating function of $ay_n + a_1 y_{n+1} + \cdots + a_n y_{n+u}$ to that of $A + A_1 x + A_2 x^2 + \cdots$ Now the first is

$$\left(a + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{u_n}{t^n}\right) u - \frac{a_1 y_0 + a_2 y_1 + \cdots + a_2 y_n \cdots}{t} - \cdots$$

and the second is easy to find since, generally, the generating function of x^m is

$$\frac{t + B_2 t^2 + B_3 t^3 + \cdots + B_k t^k + B_{m-1} t^{m-1} + t^m}{(1-t)^{m+1}}$$

One has made for brevity

$$
\begin{aligned}
B_2 &= 2^m - (m+1) \\
B_3 &= 3^m - (m+1)2^m + \frac{(m+1)m}{1\cdot 2} \\
B_4 &= 4^m - (m+1)3^m + \frac{(m+1)m}{1\cdot 2}2^m - \frac{(m+1)m(m-1)}{1\cdot 2\cdot 3} \\
B_5 &= 5^m - (m+1)4^m + \frac{(m+1)m}{1\cdot 2}3^m - \frac{(m+1)m(m-1)}{1\cdot 2\cdot 3}2^m \\
&\qquad + \frac{(m+1)m(m-1)(m-2)}{1\cdot 2\cdot 3\cdot 4} \\
B_k &= k^m - (m+1)(k-1)^m + \frac{(m+1)m}{1\cdot 2}(k-2)^m \\
&\qquad - \frac{(m+1)m(m-1)}{1\cdot 2\cdot 3}(k-3)^m + \cdots \\
&\qquad + (-1)^i \frac{(m+1)m(m-1)\cdots(m-i+2)}{1\cdot 2\cdot 3\cdots i}(k-i)^m + \cdots \\
&\qquad + (-1)^{k-1} \frac{(m+1)m(m-1)\cdots(m-k+3)}{1\cdot 2\cdot 3\cdots k-1}
\end{aligned}
$$

I consider next the diverse formulas of interpolation contained in the *Théorie analytique* (page 13 and following), but of which the demonstration must be changed or completed by virtue of that which precedes; I give these demonstrations, and I show in what case one is able to be satisfied to complete those of Laplace, by proving that the terms that I have introduced in the generating functions of y_{n+1}, y_{n+2}, $\cdots$, Δy_n, $\Delta^2 y_n$, $\cdots$ mutually destroy themselves.

I speak also of the generating functions in two variables where there are some remarks to make, analogous to those which are relative to the generating functions in one variable, thus by designating by u a function of t and t' generator of $y_{x,x'}$, the generating function of $y_{x+1,x'}$ will be

$$\frac{u - y_{0,0} - y_{0,1}t' - y_{0,2}{t'}^2 - \text{etc.}}{t}$$

that of $y_{x,x+1}$ will be

$$\frac{u - y_{0,0} - y_{1,0}t - y_{2,0}t^2 - \text{etc.}}{t}$$

and so forth.

By making use of these formulas, the generating function of the quantity $y_{x,x'}$ given by means of an equation such as

$$
\left.
\begin{array}{llll}
a y_{x,x'} & + a_1 y_{x+1,x'} & + a_2 y_{x+2,x'} & + \cdots \\
+\, b y_{x,x'+1} & + b_1 y_{x+1,x'+1} & + b_2 y_{x+2,x'+1} & + \cdots \\
+\, c\, y_{x,x'+2} & + c_1 y_{x+1,x'+2} & + c_2 y_{x+2,x'+2} & + \cdots \\
+ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots
\end{array}
\right\} = 0
$$

or even such that

$$\left.\begin{array}{l} a y_{x,x'} \quad + a_1 y_{x+1,x'} \quad + a_2 y_{x+2,x'} \quad + \cdots \\ + b y_{x,x'+1} + b_1 y_{x+1,x'+1} + b_2 y_{x+2,x'+1} + \cdots \\ + c y_{x,x'+2} + c_1 y_{x+1,x'+2} + c_2 y_{x+2,x'+2} + \cdots \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} = \begin{array}{l} \text{a rational} \\ \text{and entire} \\ \text{function} \\ \text{of } x \text{ and of } x'. \end{array}$$

is found with the greatest facility, by supposing however that a, a_1, a_2, $b, \cdots$ are some constants.

I terminate my memoir by the resolution of many questions of analysis of probabilities, questions which lead to the equations in partial finite differences and where one seeks the probabilities of composite events; that of the simple events being known.