# MÉMOIRE
## *sur l'art d'estimer la probabilité des causes par les effets*[*]

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*Mémoires de l'Académie royale des sciences et belles-lettres... Berlin.*
Classe de Philosophie Spéculative, 1796, pp. 3–24[†]

The art of estimating the probability of causes by the effects is one of so high importance, that one can not put too much care in discussing the principles of it: this is the object of one part of this memoir. The other is intended to demonstrate the use of it.

We have believed useful to reunite these two objects under a single point of view, & to link them in a manner that an attentive reader could grasp the results from it, without following the calculations on which they repose.

These calculations are besides purely elementary: & nonetheless they embrace some usual cases to which the given solutions are not applied until present, which are based on higher calculations.

These here are familiar to mathematicians, but often strange to some philosophers estimable & of a solid judgment. It would be desirous that all the theories which are not of pure mathematics, could be treated under a form which does not repulse this class of readers.

### FIRST SECTION.
*On the principles of this part of the art of conjecture.*

§ 1. *Stochastic*, or the art of conjecture with rigor, having had for first object to estimate the chances of the game, is founded on some principles related to this origin.

In a subject interesting & completely independent of gain, when one has reduced to a simple expression the probability of an event, the mind is carried naturally to convert it into wager, & it is a common way to imagine it clearly.

The symbol of an urn filled with white & black tickets is familiar to mathematicians, & this image is nothing other than the most simple form of a lottery. One can even always substitute for it (& often with advantage) the one of a polyhedron or a die cast at random.

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[†]Read to the Academy, 6 November 1794.

It follows from it that the abstract calculation of the probabilities can always be described (often even it is indeed) by the action of some aleatory instrument.

§ 2. When one draws an entire lottery, the chance in which one is interested (e.g. the gross lot) must arrive necessarily in a determined number of coups, but in an undetermined epoch.[1]

In this situation nothing is more natural than to evaluate the gain resulting from this chance, by a fictive repartition made to all the tickets with equality. In fact, if a single punter was charged himself with all these tickets, the arithmetic mean which results from the fictive repartition, would give exactly the price of each: & one would not perceive a reason in order that the price changes when the tickets are distributed to some different punters.

§ 3. If for an urn full of tickets, one substitutes a polyhedron, (or, that which reverts to the same, if at each coup one returns into the urn the exited ticket), as long as it is only a question to estimate a single coup, the case differs from the preceding only in a single regard. One no longer has certitude that the favorable chance (the gross lot) will arrive, not only in a determined number of coups, but even in any number of coups.

However (seeing the perfect ignorance where one supposes oneself) one has recourse, in order to find the estimation of a single coup, by the same expedient as one employed in the case of a lottery drawn entirely whole (§ 2). Thus one assimilates perfectly these two cases at the moment where one finds this estimation, & one departs manifestly from the same hypothesis.

One reasons exactly as if one admitted that *all the faces of the polyhedron must be brought forth successively & without repetition, until each of them had appeared in its turn*; & consequently, such is really the assumption on which one uses & on which all the results of the calculus are based. One never calculates a coup without assuming that all the conceivable coups are realized; that a die is a lottery drawn entirely whole; that the tickets of this lottery are distributed to an equal number of punters; & that after each drawing their gain is equally apportioned, as by virtue of an act of society & of community.

§ 4. This assumption is quite reasonable & founded on a quite evident reason. Because there is no wise man, to whom it was not indifferent to draw one ticket from an urn, or to make a cast of a die, if the chances on both sides were the same: that is to say, if the number of tickets were equal to the number of faces; & if for each ticket & for each face respectively the gain was equal; finally if on both sides ignorance of the chance was entire & absolute. Whatever be the number of coups, the urn presents a chance which differs not at all from the one of the die, if one returns the ticket into the urn at each drawing. But when one makes only a drawing, it is quite indifferent if one is proposed, or not, to cast this ticket back into the urn for the following drawing.

§ 5. An instant of reflection suffices to apply this hypothesis (§ 3) to all the cases which embrace the Stochastic. In particular, one is assured soon that it offers nothing incompatible with a research, which at first glance seems to exclude it. *With a given*

[1] I say epoch of time or of place. These relations being able here being substituted mutually the one for the other, would always be contained equally under the name of *epoch*, & each of them expressed tacitly by the other in the particular examples.

*die, playing a certain number of coups; what is the probability of not at all bringing forth a determined face?* This question excludes manifestly the assumption that all the faces of the given die would appear successively turn by turn. But one can resolve it stochastically, only by making some operation equivalent to this same assumption on a die different from the given die. Because if in order to resolve it, one raises to a certain power the probability that a single coup gives; to each degree, one changes the denominator. Thus by multiplying the elementary among them, or (as one expresses oneself) by *composing* the probability, one does no other thing than to deploy a greater number of possible cases, among which one determines the favorable cases. Consequently the symbol of the die is easily applicable; & the recognized hypothesis for the case of the simple probability (§ 3) becomes necessarily that on which one reasons. The composed probability, taken here for example! is become a simple probability which one can paint by the symbol of a die cast a single time, & consequently the estimation which one makes of it reposes on the same principle.

§ 6. In all other cases, by an analogous reasoning, one will be assured easily that such is always the basis of all the calculations on the probabilities. The entire Stochastic reposes on this hypothesis that I am going to now enunciate under a more general form.

STOCHASTIC HYPOTHESIS. *When by virtue of a certain determination of causes, many events appear to us equally possible; we pretend that all these events take place successively turn by turn & without repetition.*

When I establish this hypothesis, as that on which one reasons in Stochastic, I do not intend to say that one can substitute for it some other of a different form. But I intend that each other that one could legitimately substitute for it, will be equivalent to this as for the foundation: that is to say, that all the results of calculation that one could deduce legitimately from the substituted hypothesis, could equally be deduced from this.

§ 7. Although the principle that I just established is easy to understand when one gives his attention, & when one is not prevented by some contrary opinion; it will not be useless to remark its perfect coincidence with the one which JAC BERNOULLI has established since the origin of the Stochastic. This author observes that the process of this art, as for the estimation of the expectation, is precisely the one of the *Rule of alligation*. It is a sum of prices divided by the number of precious objects. In each case, one supposes these objects actually existing, & their diverse gains confounded.

§ 8. In examining closely the principle posed above (§ 6), one will see that it is only the precise definition of that which one hears by these words so often employed, *of the equally possible cases*. And this precision does not appear to me less necessary here than in each other exact theory. In mechanics, for example, under how many forms could one not present the fundamental principle of the lever, (or the one of the composition of forces, &c.)? Could not one complicate the number of balanced bodies, their weights, their distances? Could it not be made that this principle, felt intimately, was yet only vaguely determined? And if usage had prevailed by employing it under a form so incommodious, would it be useless to reduce it to its simplest form?

However simple & manifest that a hypothesis of calculus be, it is very important to put it in view. If one neglects this care, it can happen that one disregards it. If that which

I just remarked had been always recognized & sufficiently analyzed since the origin; it was probably cast since the day on the theory which depends on it & prevented some errors.

As the end of this memoir is particular, & is refered uniquely to the art of estimating causes by the effects, I will not make the application of this reflection to some other objects.

In following, according to the order of the times, the researches which have been made on this subject, one can not be prevented to remark the kind of hesitation which rules in the tentative firsts, & the defect of the liaison between this part of Stochastic & that of which it is in some way the inverse, I wish to say, the art to estimate the effects by the causes.

MM. JAC. BERNOULLI, MOYVRE, BAYES & PRICE have successively applied the calculus to the research of the causes. But the principle on which reposes the justice of their results, not being enunciated, leaves an emptiness which is harmful to clarity: & this defect, very sensible to any attentive reader, has rendered timid these same authors; in a way that their results had neither the extent nor the utility that they would have been able to give to them. And if a wise mistrust has guaranteed them of the error, the uncertainty of their march has left the chances to incur to those who would attempt to follow them.

§ 9. MR. DE LA PLACE the first to pose fluently the principle on which reposes all this part of the theory of probabilities. Here is how he has enunciated it:

PRINCIPLE. *If an event can be produced by a number $n$ of different causes, the probabilities of the existence of these causes taken from the event, are among them as the probabilities of the event taken from these causes*.[2]

I find some advantage to give to his principle the following form. This is only a change of expression.

ETIOLOGICAL PRINCIPLE. *If one plays with an unknown die, & if one brings forth a certain face* (e.g. ace), *one has in favor of a determined die* (among all those which one can call equally possible) *the same relative probability, as one would have had absolute, in favor of the face that one has brought forth, if one had played with this die*.

Such is the principle acknowledged by MR. DE LA PLACE, who has rendered clear & sure the estimation of the probability of the causes by the effects, & that, by this reason, I have believed a duty to call *Etiological principle*.

§ 10. This principle, as useful as certain, must it be taken for an axiom, or is it susceptible to being demonstrated?

I do not wish to stop myself to contest his evidence. But I believe useful, (nearly necessary) to demonstrate it: 1°. because it makes the base of a principle part of the important theory of probabilities. 2°. Because the same difficulty that one seems to find to demonstrate it, indicates some secret vice in the common enunciation of the principles of Stochastic. 3°. Finally, because this demonstration is neither long nor difficult.

[2] *Sçavans étrangers*. T. VI. Mr. de la Place leaves to this enunciation a *consequence* which will be placed above (§ 16). I have believed a duty to give here the *principle* alone.

§ 11. It reposes completely on the hypothesis of Stochastic I have developed & established above (§ 6). Here it is under the form of a particular example.

PROBLEM. *Two dice, with a like number of faces, have some faces marked* ace*; but one of them has two times more than the other. One of these dice having been cast, & having brought forth* ace*; it is the question to conjecture which has probably been cast, & what is this relative probability?*

SOLUTION. Since there has been one die cast, & since there are two dice which can equally have been it: it is necessary to suppose them showing both successively all their faces (§ 6).

In this number of faces one will find *ace* arrived one time by one die & two times by the other; or two times by one, four times by the other, etc.

Limiting myself to the first case, I know that one of the three faces marked *ace* has been brought forth. The probability in favor of that which is alone of its kind is $\frac{1}{3}$, that in favor of the other $\frac{2}{3}$. These probabilities are between them as 1 is to 2.

This example is easy to generalize.

Proceeding from it, one is assured of the truth of the etiological principle (§ 9). Its foundation is thus understood. It conforms likewise the necessity of the general principle (§ 6). And the stochastic theory becomes solid & linked.

§ 12. All this is applicable only to the simple probability. But as each composite probability can be presented under the form of a simple probability, & can likewise be treated stochastically only by some operation equivalent to this reduction (§ 5); it is clear that the etiological principle will be applied to all the cases of simple or composite probability without exception.

§ 13. I have said (§ 1) that one could always substitute one die to one urn. And (§ 2) that the epochs of time & of place can always be indifferently substituted the one to the other. These assertions do not permit exception: but certain questions present themselves under a form which renders them less easy to reduce. Of this number are those which one paints by the emblem of many successive drawings executed in one same urn, by not returning at all into the urn the tickets which one extracts from it at each drawing. These questions resolve themselves without developing all the cases. But this development is necessary in order to reduce them to a common emblem.

We suppose that an urn contains $n$ tickets denoted A, B, C, ...N. And that it is a question to draw from it two tickets successively without recasting into the urn the first that one will have extracted from it. Consequently one could represent $n$ dice, each having $n-1$ faces, & marked by a double letter on each face: thus

$$\begin{array}{ll} \text{Dice} & \begin{array}{ccccc} 1^{\text{st}}, & 2^{\text{nd}}, & 3^{\text{rd}}, & \ldots & n^{th} \end{array} \\ \text{Marks of the faces} & \left\{ \begin{array}{ccccc} \text{AB} & \text{BA} & \text{CA} & \ldots & \text{NA} \\ \text{AC,} & \text{BC,} & \text{CB,} & & \text{NB} \\ \text{AD} & \text{BD} & \text{CD} & & \text{NC} \\ \vdots & \vdots & \vdots & & \vdots \\ \text{AN} & \text{BN} & \text{CN} & & \text{N(N-1)} \end{array} \right. \end{array}$$

This process is easy to apply to a great number of drawings. And thence one will deduce each question of this kind to the form of the simple probability; & consequently, the etiological principle could be applied. So that nothing makes exception to the universality of this principle.

§ 14. The statement that I just made of the principles of Stochastic was not indispensable in order to arrive to the demonstration of the etiological principle. But I have believed useful to restore the attention of the philosophers on these general principles, persuaded, as I follow it, that as long as one will not be interested to pose them clearly & to follow them with rigor, one will be arrested in the consequences by some apparent difficulties, which will cast from the obscurity on this part of the science. It is necessary in an abstract theory, that the hypotheses on which one part is enunciated at entry, very fluently & under the most facile form to grasp; & that next all depends clearly & rigorously on this first position: so that one had to contest only on the principles, & not on the propositions or on the results. It is thus that in every other kind of mixed theory the philosophers have used of it, & they can not deviate from this march without peril.

However as in Stochastic one has sometimes appeared to renounce voluntarily to this method, & to push the consequences before having well set the hypotheses; I believe that one refuses to follow the discussion only I just made. And I am going to demonstrate the etiological principle by serving myself of the received expressions, & without employing the precise stochastic hypothesis that I have posed above (§ 6); which probably will run counter by an appearance of paradox & of novelty.

§ 15. THEOREM. *If an event can be produced by any determined number of different causes; the probabilities of the existence of these causes taken from the event, are among them as the probabilities of the event taken from these causes.*

DEMONSTRATION. Reducing to the same denominator the probabilities of the event by its causes: as soon as the event has been produced by an unknown cause; one finds as many possible cases as the sum of the numerators indicate.

But of all these cases here, the number which expresses all the effects of one same cause, is evidently the numerator of the probability of the event from this cause.

Therefore the probability that the event has been produced by a determined cause, has for numerator the one of the probability of the event by this cause; & for denominator a constant number, (namely the sum of the numerators).

Therefore the probability that the event has been produced by a determined cause, is as the numerator of the probability of the event taken from this cause; or as these same probabilities, (since they were reduced to the same denominator).

Emblematically:

Let there be many dice, with a like number $n$ faces, having each a certain number of faces marked *ace*. For example:

| | | | | |
|---|---|---|---|---|
| The dice | I, | II, | III, | . . . N |
| having *ace* | 1, | 2, | 3, | . . . $n$ respectively. |

Since I have brought forth *ace* with one of these dice, the number of the possible cases is the same as the one of all the *ace* faces. And in this example, it is the triangle of the number $n$.

But of all the possible cases,

| | | | | |
|---|---|---|---|---|
| one finds of them | 1, | 2, | 3, | $\dots n$ |
| products by the dice | I, | II, | III, | ...N respectively. |

Thus the probabilities to have brought forth an *ace* by each of these dice are respectively

| | | | | |
|---|---|---|---|---|
| for the dice | I, | II, | III, | ...N |
| probabilities | $1 : \dfrac{nn+n}{2}$, | $2 : \dfrac{nn+n}{2}$, | $3 : \dfrac{nn+n}{2}$, | $\dots n \, \dfrac{nn+n}{2}$.. |

These probabilities are among them as the numerators of the probabilities to bring forth *ace* by these dice respectively: namely

| | | | | |
|---|---|---|---|---|
| Dice | I, | II, | III, | ...N |
| Numerators | 1, | 2, | 3, | $\dots n$ respectively. |

And consequently these probabilities are among them as the same probabilities to bring forth *ace* by these respective dice: namely,

| | | | | |
|---|---|---|---|---|
| In favor of the dice | I, | II, | III, | ...N |
| Relative probabilities | $\dfrac{1}{n}$, | $\dfrac{2}{n}$, | $\dfrac{3}{n}$, | $\dots \dfrac{n}{n}$ respectively. |

§ 16. After from the etiological principle MR. DE LA PLACE joins immediately a quite important & usual consequence, which has in fact need only to be indicated. Here it is such he expresses it.[3]

PRINCIPLE [with its CONSEQUENCE]. *If an event can be produced by a number $n$ of different causes, the probabilities of the existence of these causes taken from the event are among them as the probabilities of the event taken from these causes. And* [consequently] *the probability of the existence of each of them, is equal to the probability of the event taken from this cause, divided by the sum of all the probabilities of the event taken from each of these causes.*

In other terms & conforming to the enunciation above (§ 9).

CONSEQUENCE of the etiological Principle. *When one can determine by the etiological principle the relative probability in favor of each of the equally possibles; one obtains the absolute probability of any one of them, by dividing the corresponding relative by the sum of all the relatives.*

§ 17. The estimation of the probability of the causes, founded on the etiological principle, is not different from that of the effects by the causes, as for the sense of the word *probability*. Each estimation is a mean, & reposes on the stochastic principle (§ 9).

[3] I put between [ ] a few words that I add to his enunciation.

This remark tends to make sense the necessity to link, as we just did it, these two parts of the theory. This liaison, for example, does not permit adopting the distinction which a celebrated mathematician makes to this subject.

He begins by showing that, in the estimation of the effects by the cause, "there is no liaison between the probability & the reality of the events."[4]

Next, having exposed the method to estimate the probability of the causes by the effects, founded on the etiological principle, thus the method to conclude from it the probability of new future effects: he concludes thus. "This is therefore not the real probability which one can obtain by this means, but a mean probability. Thus not only, as in all the calculation of the probabilities, there is no necessary liaison between the probability & the reality of the events; but *there is of them no longer any between the probability given by the calculation & the real probability.*"[5]

He seems therefore to distinguish a real probability deduced from causes, & a mean probability deduced from effects. But these two kinds of probabilities do not differ as for their essence, & are both *means*. The state of ignorance where we are on the productive causes is the same in the two cases, & the means by which one supplied it is also the same. By judging the coups of a future die by the past, I do not determine without doubt the same probability, as if I knew the number & the nature of the faces of the die. In other terms, if it is true that a certain event must take place necessarily a certain number of times in a certain period by the influence of natural causes; when I estimate its probability after past observations, I do not determine justly this true number. But this is immediately from this *that there is no liaison between the probability & the reality of the events.* And there results from it no distinction to make for this particular case. Each event is completely determined by its causes: this is from our ignorance that the apparent indetermination is born. And it is this indetermination which we try to represent by the emblem of the die or of the urn. It is necessary therefore, in order to be consequent, to employ finally this emblem only under a single form in order to represent the final indetermination. And under this form each probability, of some manner which one estimates it, is never but a mean probability, which represents a conception relative to our ignorance.

§ 18. This is that which this illustrious writer seems to recognize besides, when he seeks the nature of the motive which results from the probability. Because he terminates thus this research. "The motive to believe that out of ten million white balls mixed with one black, it will be not at all the black which I will draw on the first coup, is of the same nature as the motive to believe that the sun will not fail to rise tomorrow, & these two opinions differ between them only by the most or the least probability."[6] However I can not permit to pass this example, without remarking another very important distinction. I will be content to indicate it, because I could bring out the limits of our subject, if I myself delivered to one metaphysical discussion on the nature of our judgments.

The analogous persuasion which each man feels, to see a natural event repeated (such as the rising of the sun), is of a different kind from the persuasion represented

[4]Essai sur l'application de 'analyse à la probabilité des décisions rendues à la pluralité des voix. Discours. prélimin. p. X. Translator's note: This work is due to Buffon.

[5]*Ibid*. p. LXXXVI.

[6]*Ibid*. p. XI.

by a fraction in the theory of the probabilities. This can be added to it, but one can exist without the other. They depend on two orders of different faculties. An infant, an animal feels the first, & forms no explicit calculation, not even implicit: there is no necessary dependence between these two persuasions. That which the calculus evaluates is reasoned, & even, to a certain point, artificial. The other is of instinct & natural. It depends on some intellectual faculties of which the analysis is not easy, & probably in very great part from the principle of the liaison of the ideas.

One would not know how to admit without restriction the assertions which I have cited, in which these two persuasions are not distinguished: less yet those by which one identifies the motive to believe resulting from the calculation of the probabilities & the motive which carries us into the *judgments which are confused with the sensations* (e.g. in those which we carry on the visible distance of objects to the eye): much less again will one admit that the motive founded on this calculation is the same which makes us believe in *the existence of bodies*.[7]

The importance of the subject, & the weight of one such authority, will excuse me to have employed some process to discuss it.

## SECOND SECTION

*Précis of the march of the applications.*

The amateurs of the calculus can consult the mathematical memoir contained in this volume, & arranged under the class to which it pertains. The abridged & familiar exposition which I am going to make of it, is destined to those of the readers who are disposed to be content with the march of the mathematician & with the principle results of his calculations, without following it step by step & in all the details of his processes.

§ 19. *First problem.* One die with a given number of faces, but of unknown nature, having been cast a certain number of times; one has brought forth constantly an *ace* face. What is the probability that this die has a given number of *ace* faces?

*Process.* Let be made all the possible assumptions on the composition of the die; that is to say, on the number of its *ace* faces (& hence also of its non-*ace* faces), from unity to the number of its faces.

Let be sought the probabilities to bring forth the given number of *ace* faces corresponding to these assumptions. These probabilities are expressed by some fractions of which the numerators are the supposed numbers of *ace* faces of the die, raised to a power of which the exponent is the number of the *ace* faces brought forth; & of which the denominator is the given number of faces of the die, raised to a power of the same exponent.

The probabilities of the event corresponding to these assumptions, are among them as the numerators of these fractions. Therefore also (§ 9) the probabilities of these assumptions are among them as these numerators. And the probability of each assumption is the corresponding numerator divided by their sum.

*Symbolically.* Let $n$ be the number of faces of the die.

Let $p$ be the number brought forth of *ace* faces.

Let $m$ be the supposed number of *ace* faces of the die.

[7] *Ibid.* X & XI.

Let the sum of the first $n$ natural numbers of which the exponent is $p$, be designated by $\int n^p$.

The probability that the die has $m$ *ace* faces, is . . . $\frac{m^p}{\int n^p}$.

*Examples.* Let $p = 1$, or let one have brought forth an *ace* face; the probability that the die has $m$ faces, is $\frac{m}{\int n} = 2\frac{m}{n.n+1}$. The more $n$ is great, the more this probability approaches being $\frac{2}{n} \times \frac{m}{n}$. Hence, the ratio of the number of proposed *ace* faces to the number of faces of the die being given; this last probability (or the limit of the proposed probability corresponding to the increase of $n$) is in ratio inverse to the number of faces of the die.

Let $p = 2$, or let one have brought forth two *ace* faces. The probability that the die has $m$ *ace* faces, is $\frac{m^2}{\int n^2} = 1.2.3\frac{m^2}{n.n+1.2n+1}$. The more $n$ is great, the more this probability approaches being $\frac{3}{n} \times \frac{m^2}{n^2}$.

*In general.* The more the number of faces of the die is great, the more this probability approaches being $\frac{p+1}{n} \times \frac{m^p}{n^p}$. Hence, the ratio of the proposed number of *ace* faces to the total number of faces being given, the proposed probability approaches so much more to be in inverse ratio to the number of faces of the die, as this number is greater.

20. *Second problem.* All being posed as previously, one demands the probability that playing one coup more, one will bring forth anew an *ace* face.

Having found (by the first problem) the probability of each composition of the die, let each of these probabilities be multiplied by the probability to bring forth an *ace* face, corresponding to each of these compositions. The probability to bring forth a new *ace* face is the sum of these probabilities.

*Symbolically.* The sought probability is $\frac{\int n^{p+1}}{n \int n^p}$.

*Examples.* Let $p = 1$. This probability is $\frac{\frac{n.n+1.2n+1}{1.2.3}}{n\frac{n.n+1}{1.2}} = \frac{1}{3} \cdot \frac{2n+1}{n+1} = \frac{2}{3} - \frac{1}{3(n+1)}$. Hence, the more $n$ is great, the more this probability approaches being $\frac{2}{3}$.

Let $p = 2$. This probability is $\frac{\int n^3}{n \int n^2} = \frac{\left(\frac{n.n+1}{1.2}\right)^2}{n \cdot \frac{n.n+1.2n+1}{1.2.3}} = \frac{3}{2} \cdot \frac{n+1}{2n+1} = \frac{3}{4} \cdot \left(1 + \frac{1}{2n+1}\right)$ Hence, the more $n$ is great, the more this probability approaches being $\frac{3}{4}$.

*Generally.* The more $n$ is great; the more the sought probability approaches being $\frac{p+1}{p+2} = 1 - \frac{1}{p+2}$. Namely, the more the die has faces, the more the probability to bring forth an *ace* face after having brought forth a certain number of *ace* faces, approaches being expressed by a fraction, of which the numerator is the number of *ace* faces brought forth increased by unity, & of which the denominator surpasses by one unit the numerator. The complement of this probability to certitude, is a fraction having for numerator unity, & for denominator the number of *ace* faces brought forth, increased by two units.

§ 21. On finds likewise: That, the probability to bring forth anew *ace* faces alone, in a given number of coups, is expressed by a fraction, of which the numerator is the sum of the powers of the natural numbers from unity to the number of faces of the die, having for exponent the sum of the *ace* faces brought forth & to bring forth, & of which the denominator is the product of the power of the number of faces of the die, having for exponent the number of the *ace* faces to bring forth, by the sum of the powers of the natural numbers, from unity to the number of faces of the die, having for exponent the number of *ace* faces brought forth.

*Symbolically.*

| | |
|---|---|
| Faces of the die | $n$ |
| *Ace* faces brought forth | $p$ |
| *Ace* faces to bring forth | $p'$ |
| Probability | $\dfrac{\int n^{p+p'}}{n^{p'}\int n^{p}}$. |

*Example.* Let $\begin{matrix} p=1 \\ p'=2 \end{matrix}$. This probability is $\frac{\int n^3}{n^2\int n} = \frac{n+1}{2n} = \frac{1}{2}\left(1+\frac{1}{n}\right)$. The limit of this probability is $\frac{1}{2}$.

*In general.* The limit of the proposed probability is $\frac{p+1}{p+p'+1} = 1 - \frac{p'}{p+p'+1}$. The complement to certitude of this limit is expressed by a fraction of which the numerator is the number of *ace* faces to bring forth, & of which the denominator is the sum of the number of *ace* faces brought forth & to bring forth, increased by unity. Hence, this probability differs so much less from certitude, as the number of *ace* faces brought forth is greater relatively to the number of faces to bring forth.

This limit is *half*, as often as the number of *ace* faces to bring forth surpasses by unity the number of *ace* faces brought forth.

§ 22. *Third problem.* In a determined number of casts, one has brought forth some known number of *ace* & *non-ace* faces. One demands the probabilities of the different compositions of the die.

Let all the assumptions on the composition of the die be made likewise as to the number of its *ace* faces, from unity to a number inferior by one unit to the number of its faces.

The probabilities of the event corresponding to these assumptions, are expressed by the following fractions. The numerator is the product of the power of the supposed number of *ace* faces of which the exponent is the number of *ace* faces brought forth, by the power of the supposed number of *non-ace* faces of which the exponent is the number of the *non-ace* faces brought forth. The denominator is the power of the number of faces of the die of which the exponent is the number of coups played. Hence, the probabilities of the event are among them as the numerators of these fractions. Thus also (§ 9) the probabilities of the assumptions are among them as these numerators. And the probability of each assumption is the corresponding numerator divided by its sum.

*Symbolically.*

| | | | |
|---|---|---|---|
| Number of faces of the die | $n$ | | |
| *Ace* faces brought forth | $p$ | *Ace* faces supposed | $m$ |
| *Non-ace* faces brought forth | $q$ | *Non-ace* faces supposed | $n-m$ |

Let the sum of all the products $m^p(n-m)^q$ be expressed by $\int P$: by giving to $m$ all the values from unity to $n-1$.

The probability that the die has $\begin{matrix} m \textit{ ace} \text{ faces} \\ n-m \textit{ non-ace} \text{ faces} \end{matrix}$ is $\frac{m^p(n-m)^q}{\int P}$.

*First remark.* The most probable composition of the die takes place, when $m : n - m = p : q$; namely the most probable ratio of the number of the *ace* & *non-ace* faces of the die, is the one which is equal (or most approaches being equal) to the ratio of the *ace* & *non-ace* faces brought forth.

*Second remark.* The order, according to which the *ace* & *non-ace* faces have been brought forth, influence not at all on the demanded probability.

*Third remark.* It is to the mathematician to present the denominator of this fraction, in the most suitable manner for the calculus, or for the consequences to draw from this expression. In particular, one can always reduce it to the summation of the powers of the natural numbers.

*Fourth remark.* The more the number of faces of the die is great, the more the expression of the denominator approaches being

$$\frac{1.2\ldots q}{p+1.p+2\ldots p+q+1} n^{p+q+1};$$

& hence, the more the sought probability approaches being

$$\frac{1.2\ldots q}{p+1.p+2\ldots p+q+1} \times \frac{1}{n} \times \frac{m^p}{n} \times \frac{n-m^q}{n}.$$

§ 23. *Fourth problem.* All being posed as in the preceding §: one demands the probability that playing anew some given number of coups, one will bring forth some given numbers of *ace* & *non-ace* faces, in a determined order.

Having determined by the preceding problem the probability of each composition of the die: let each of these probabilities be multiplied by the corresponding probability to bring forth in the given number of coups the given number of *ace* & *non-ace* faces. The sum of these products is the sought probability.

*Symbolically.* Admitting the symbols of the preceding problem, let further that $p'$ & $q'$ be the numbers of *ace* & *non-ace* faces to bring forth in $p' + q'$ coups. Let be designated by $\int P'$ the sum of the products

$$m^{p+p'} \times n - m^{q+q'};$$

by giving to $m$ all the integer values from unity to $n - 1$. The sought probability is

$$\frac{\int P}{n^{p'+q'} \int P}.$$

*First remark.* It is again to the mathematician to present the development of this expression under the simplest form. In particular, it is always reducible to the summation of the powers of the natural numbers.

*Second remark.* This expression is that of the probability to bring forth the event proposed in a determined order.

In order to obtain the probability of the event in any order, it is necessary to multiply this expression by the number of ways in which can be disposed $p' + q'$ quantities, of which $p'$ of one kind & $q'$ of the other; namely, by

$$\frac{p'+q'}{1} \cdot \frac{p'+q'-1}{2} \cdots \frac{p'+1}{q'}.$$

*Third remark*. The more the number of faces of the die is great, the more the expression of the numerator approaches being

$$\frac{1.2.3 \ldots q+q'}{p+p'+1.p+p'+2 \ldots p+p'+q+q'+1} \times n^{p+p'+q+q'+1}.$$

Hence, the more the number of the faces of the die is great, the more the probability (for the given order) approaches being

$$\frac{q+1.q+2.q+q'.p+1.p+2 \ldots p+q+1}{p+p'+1.p+p'+2 \ldots p+p'+q+q'+1}.$$

*Fourth remark*. The order, following which the proposed faces must be brought forth not being determined, the greatest probability corresponds to the case where the number of ace & non-ace faces to bring forth are among them (or most approaches being among them) as the number of ace & non-ace faces brought forth. But this has no place for the determined order.

§ 24. When instead of playing with a die of which the number of faces is constant, one draws the tickets from an urn, by not replacing the extracted tickets, the total number of tickets, & the number of tickets of a determined kind, vary according to the known laws by the extractions made. And the factors of the products which enter into the expression of the probabilities, either from the composition of the urn or from the future events, vary also according to known laws.

But, in this case, these variations instead of complicating the results, tend to simplify them. The sequences to sum become those of figurate numbers, or depend on figurate numbers; & their sums are themselves figurate numbers, or depend on figurate numbers.

The results which this case gives, approaches being the same as those which have been obtained in the first case; so much more as the number of faces of the die, & the number of the tickets contained originally in the urn, are greater; & these two results would be the same, if these numbers were supposed infinitely great. This furnishes a new way to determine by the elements the cases treated by Mr. de la Place.

§ 25. In order to facilitate the application of the general results that I just exposed, I join to this memoir a table which presents the probabilities of all the events to obtain in a given number of casts from one to four, corresponding to the events obtained in a number of casts from one to three, by supposing the die of an infinite number of faces.

| | | | |
|---|---|---|---|
| The | first | column of this table contains | the number of coups played. |
| | second | | the faces brought forth of each kind. |
| | third | | the number of coups to play. |
| | fourth | | the faces to bring forth of each kind. |
| | fifth | | the corresponding probabilities of the event, in a determined order. |
| | sixth | | the corresponding probabilities of the event, in any order. |

In order to abridge this table, I have supposed in the second column, the ace faces brought forth, in number greater than, or equal to the one of the non-ace faces brought forth in the contrary case; these probabilities would be the same, by substituting the non-ace faces for the ace faces.

When the number of ace & non-ace faces of the second column were equal, I have exposed in the fourth column & in the following only the cases where the number of ace faces to bring forth, is not smaller than the number of non-ace faces.

In the first & sixth columns, in order to avoid the repetition of the fractional expressions, I have put their denominator outside to the right, one time alone: in a way that it appears in all the following terms in which the denominator is not indicated.

TABLE

| Coups played | Faces brought forth Ace | Non-ace | Coups to play | Faces to bring forth Ace | Non-ace | Probabilities in a deter. order | any order |
|---|---|---|---|---|---|---|---|
| I | 1 | – | I | 1 | – | 2: 3 | 2: 3 |
| | | | | – | 1 | 1 | 1 |
| | | | II | 2 | – | 3: 6 | 3: 6 |
| | | | | 1 | 1 | 1 | 2 |
| | | | | – | 2 | 1 | 1 |
| | | | III | 3 | – | 12: 30 | 12: 30 |
| | | | | 2 | 1 | 3 | 9 |
| | | | | 1 | 2 | 2 | 6 |
| | | | | – | 3 | 3 | 3 |
| | | | IV | 4 | – | 10: 30 | 10: 30 |
| | | | | 3 | 1 | 2 | 8 |
| | | | | 2 | 2 | 1 | 6 |
| | | | | 1 | 3 | 1 | 4 |
| | | | | – | 4 | 2 | 2 |
| II | 2 | – | I | 1 | – | 3: 4 | 3: 4 |
| | | | | – | 1 | 1 | 1 |
| | | | II | 2 | – | 12: 20 | 12: 20 |
| | | | | 1 | 1 | 3 | 6 |
| | | | | – | 2 | 2 | 2 |
| | | | III | 3 | – | 10: 20 | 10: 20 |
| | | | | 2 | 1 | 2 | 6 |
| | | | | 1 | 2 | 1 | 3 |
| | | | | – | 3 | 1 | 1 |
| | | | IV | 4 | – | 60: 140 | 60: 140 |
| | | | | 3 | 1 | 10 | 40 |
| | | | | 2 | 2 | 4 | 24 |
| | | | | 1 | 3 | 3 | 12 |
| | | | | – | 4 | 4 | 4 |
| | 1 | 1 | I | 1 | – | 1: 2 | 1: 2 |
| | | | II | 2 | – | 3: 10 | 3: 10 |
| | | | | 1 | 1 | 2 | 4 |
| | | | III | 3 | – | 2: 10 | 2: 10 |
| | | | | 2 | 1 | 1 | 3 |
| | | | IV | 4 | – | 10: 70 | 10: 70 |
| | | | | 3 | 1 | 4 | 16 |
| | | | | 2 | 2 | 3 | 18 |
| III | 3 | – | I | 1 | – | 4: 5 | 4: 5 |
| | | | | – | 1 | 1 | 1 |
| | | | II | 2 | – | 10: 15 | 10: 15 |
| | | | | 1 | 1 | 2 | 4 |
| | | | | – | 2 | 1 | 1 |
| | | | III | 3 | – | 60: 105 | 60: 105 |
| | | | | 2 | 1 | 10 | 30 |
| | | | | 1 | 2 | 4 | 12 |
| | | | | – | 3 | 3 | 3 |
| | | | IV | 4 | – | 105: 210 | 105: 210 |
| | | | | 3 | 1 | 15 | 60 |
| | | | | 2 | 2 | 5 | 30 |
| | | | | 1 | 3 | 3 | 12 |
| | | | | – | 4 | 3 | 3 |
| | 2 | 1 | I | 1 | – | 3: 5 | 3: 5 |
| | | | | – | 1 | 2 | 2 |
| | | | II | 2 | – | 2: 5 | 2: 5 |
| | | | | 1 | 1 | 1 | 2 |
| | | | | 1 | 2 | 1 | 1 |
| | | | III | 3 | – | 10: 35 | 10: 35 |
| | | | | 2 | 1 | 4 | 12 |
| | | | | 1 | 2 | 3 | 9 |
| | | | | – | 3 | 4 | 4 |
| | | | IV | 4 | – | 15: 70 | 15: 70 |
| | | | | 3 | 1 | 5 | 20 |
| | | | | 2 | 2 | 3 | 18 |
| | | | | 1 | 3 | 3 | 12 |
| | | | | – | 4 | 5 | 5 |