Game-theoretic probability in continuous time

Vladimir Vovk

Department of Computer Science
Royal Holloway, University of London

Workshop on game-theoretic probability and related topics
February 28, 2008

Typical discrete-time GTP protocol

$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
  Forecaster makes his move.
  Skeptic makes his move.
  Reality makes her move.
  $\mathcal{K}_n := \mathcal{K}_{n-1} +$ function of the 3 moves.
END FOR.

$\mathcal{K}_n$: Skeptic's capital.

Two books

- Glenn Shafer and Vladimir Vovk.
  Probability and Finance: It's only a Game!
  New York: Wiley, 2001.
  Abridged Japanese translation:
  ゲームとしての確率とファイナンス.
  Tokyo: Iwanami Shoten, 2006.
- Kei Takeuchi.
  賭けの数理と金融工学
  (Mathematics of Betting and Financial Engineering).
  Tokyo: Saiensusha, 2004.

Moving to continuous time

The protocol is predictive and sequential (APD: prequential).

We could try to emulate this in continuous time. Easiest: use non-standard analysis (Shafer & Vovk, 2001; several working papers at http://www.probabilityandfinance.com).

Disadvantages:

- NSA is not familiar to many people.
- Even more important: dependence on the choice of the infinitesimal atom of time.

Recent breakthrough

Kei Takeuchi, Masayuki Kumon, and Akimichi Takemura [TKT].
A new formulation of asset trading games in continuous time with essential forcing of variation exponent.
August 2007, `arXiv:0708.0275` [math.PR]

Non-predictive. Non-sequential.

The key technique: “high-frequency limit order strategies”.

Non-sequential game-theoretic probability

This talk: two main frameworks.

1. The non-predictive (market) framework: what can be said about the continuous price process of a traded security (one, in this talk)? Assumption: the market is efficient (perhaps in a very weak sense).
2. A free agent (Forecaster) predicts a continuous process. What is the relation between the predictions and the realized trajectory? Assumption: you cannot become infinitely rich gambling against the predictions.

Both frameworks: non-sequential (two-step games); worst-case results.

Outline

1. Non-predictive game-theoretic probability
   - Emergence of randomness
   - Emergence of volatility
   - Emergence of probability
2. Predictive game-theoretic probability
   - Game-theoretic Brownian motion
   - Predicting continuous processes
   - Predicting point processes
3. Discussion

Outline

1. Non-predictive game-theoretic probability
   - Emergence of randomness
   - Emergence of volatility
   - Emergence of probability
2. Predictive game-theoretic probability
   - Game-theoretic Brownian motion
   - Predicting continuous processes
   - Predicting point processes
3. Discussion

Basic game

Players: Reality (market) and Skeptic (speculator).

Time: $[0, \infty)$.

Two steps of the game:

- Skeptic chooses his trading strategy.
- Reality chooses a continuous function $\omega : [0, \infty) \to \mathbb{R}$ (the price process).

Ω: all continuous functions $\omega : [0, \infty) \to \mathbb{R}$.

Processes etc.

$\mathcal{F}_t$, $t \in [0,\infty)$: the smallest σ-algebra on Ω that makes all functions $\omega \mapsto \omega(s)$, $s \in [0,t]$, measurable.

A process $\mathfrak{S}$: a family of functions $\mathfrak{S}_t : \Omega \to \mathbb{R}$, $t \in [0,\infty)$, each $\mathfrak{S}_t$ being $\mathcal{F}_t$-measurable; its trajectories: $t \mapsto \mathfrak{S}_t(\omega)$.

Event: any subset of Ω; not necessarily an element of $\mathcal{F} := \mathcal{F}_\infty := \vee_t \mathcal{F}_t$.

Stopping times $\tau : \Omega \to [0,\infty]$ w.r.t. to $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual. Convenient characterization of $\mathcal{F}_\tau$ measurability: Galmarino's test.

Allowed strategies I

An elementary trading strategy G consists of:

- an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ such that, for any $\omega \in \Omega$, $\lim_{n\to\infty} \tau_n(\omega) = \infty$
- for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable M_n

To such G and initial capital $c \in \mathbb{R}$ corresponds the elementary capital process

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} M_n(\omega)\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\big)$$

(interpretation: the interest rate is zero).
$M_n(\omega)$: Skeptic's bet (or stake) at time τ_n
$\mathcal{K}_t^{G,c}(\omega)$: Skeptic's capital at time t

Allowed strategies II

A positive capital process is a process $\mathfrak{S}$ with values in $[0, \infty]$ that can be represented as

$$\mathfrak{S}_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n, c_n}(\omega),$$

where

- the elementary capital processes $\mathcal{K}_t^{G_n, c_n}(\omega)$ are required to be positive, for all t and ω
- the positive series $\sum_{n=1}^{\infty} c_n$ is required to converge

Compare: the standard definition of expectation for positive random variables.

Upper and lower probability

The upper probability of $E \subseteq \Omega$:

$$\overline{\mathbb{P}}(E) := \inf\left\{\mathfrak{S}_0 \;\middle|\; \forall\omega\in\Omega : \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge \mathbf{1}_E(\omega)\right\},$$

$\mathfrak{S}$ ranging over the positive capital processes.

$E \subseteq \Omega$ is null if $\overline{\mathbb{P}}(E) = 0$. Phrases applied to its complement: almost certain, almost surely (a.s.), for almost all ω.

Lower probability:

$$\underline{\mathbb{P}}(E) := 1 - \overline{\mathbb{P}}(E^c).$$

Robustness

Lemma

The definition of upper probability will not change when $\liminf_{t\to\infty}$ is replaced by $\sup_{t\in[0,\infty)}$ (and, therefore, by $\limsup_{t\to\infty}$).

Proof.

Stop when the capital process hits 1. $\square$

σ-subadditivity

Lemma

For any sequence of subsets $E_1, E_2, \ldots$ of Ω,

$$\overline{\mathbb{P}}\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \overline{\mathbb{P}}(E_n).$$

In particular, a countable union of null sets is null.

Proof.

Countable union of countable sets of elementary capital processes is still countable. $\square$

Contains finite subadditivity as a special case.

Another definition of null events

Lemma

$E \subseteq \Omega$ is null if and only if there exists a positive capital process $\mathfrak{S}$ with $\mathfrak{S}_0 = 1$ such that $\lim_{t\to\infty} \mathfrak{S}_t(\omega) = \infty$ for all $\omega \in E$.

Proof.

Suppose $\overline{\mathbb{P}}(E) = 0$. For each $n = 1, 2, \ldots$, let $\mathfrak{S}^n$ be a positive capital process with $\mathfrak{S}^n_0 = 2^{-n}$ and $\liminf_{t\to\infty} \mathfrak{S}^n_t \geq 1$. It suffices to set $\mathfrak{S} := \sum_{n=1}^{\infty} \mathfrak{S}^n$. □

Coherence

Somewhat informal definition:
A protocol is coherent if $\overline{\mathbb{P}}(\Omega) = 1$. [Usually equivalent to $\overline{\mathbb{P}}(\Omega) > 0$.]

Our current protocol is obviously coherent: consider constant ωs.

Lemma

In a coherent protocol, $\underline{\mathbb{P}}(E) \leq \overline{\mathbb{P}}(E)$, for all $E \subseteq \Omega$.

Proof.

$\underline{\mathbb{P}}(E) > \overline{\mathbb{P}}(E)$ means $\overline{\mathbb{P}}(E) + \overline{\mathbb{P}}(E^c) < 1$, and so some positive capital process makes 1 out of $1 - \epsilon < 1$. $\square$

Comparison with the TKT definition

TKT do not make the second step (positive capital processes via positive elementary capital processes).

Advantage: Simpler definition, with coherence obvious (for all natural protocols I know) and not requiring any assumptions

Disadvantage: Because of the lack of σ-subadditivity, complicated statements of theorems (in terms of ϵ & δ)

Outline

1. Non-predictive game-theoretic probability
   - Emergence of randomness
   - Emergence of volatility
   - Emergence of probability
2. Predictive game-theoretic probability
   - Game-theoretic Brownian motion
   - Predicting continuous processes
   - Predicting point processes
3. Discussion

Qualitative properties of typical trajectories

Qualitative = in terms of order (or order topology)

The textbook properties of typical trajectories of Brownian motion:

- no points of increase [Dvoretzky, Erdős, Kakutani, 1960]; in particular, not monotonic in any interval
- no isolated zeros; the set of zeros is unbounded and has Lebesgue measure zero
- the set of points of local maximum is dense and countable; each local maximum is strict

Which of these manifestations of randomness continue to hold for continuous price processes?

Level sets of the price process

Theorem

Let $b \in \mathbb{R}$. Almost surely, the level set

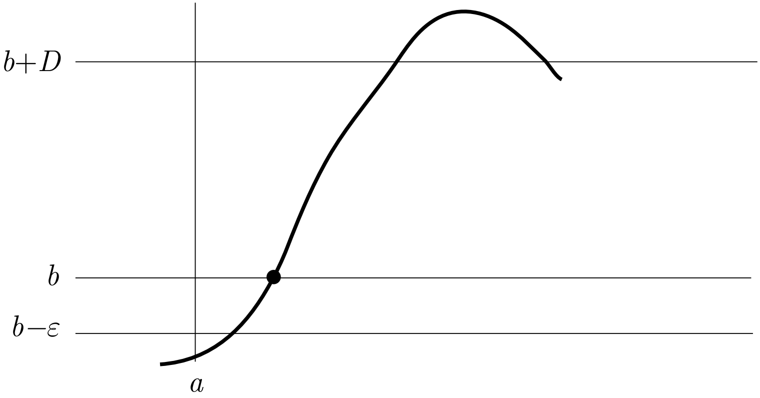
$$\mathcal{L}_\omega(b) := \{t \in [0,\infty) : \omega(t) = b\}$$

has no isolated points in $[0,\infty)$.

Proof.

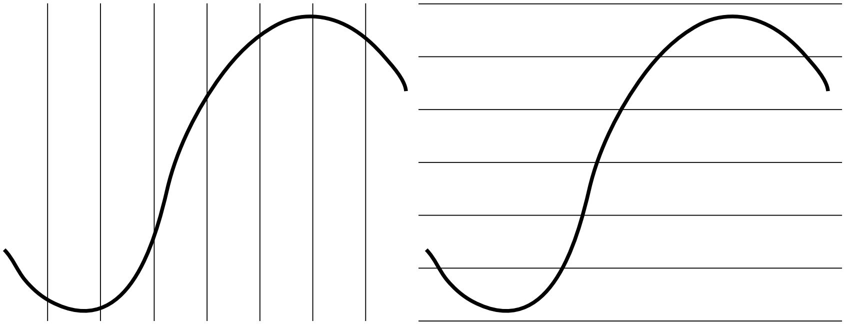
If $\mathcal{L}_\omega(b)$ has an isolated point, there are rational numbers $a \geq 0$ and $D \neq 0$ (let $D > 0$) such that strictly after the time $\inf\{t : t \geq a, \omega(t) = b\}$ ω does not take value b before hitting the value $b + D$. This event, $E_{a,D}$, is null. Apply subadditivity. $\square$

Proof of $\overline{\mathbb{P}}(E_{a,D}) \leq \frac{\epsilon}{\epsilon+D}$



Key technique introduced in GTP by TKT

Riemann vs. Lebesgue trading



Informal discussion: market efficiency

Interpretation: almost certain events are expected to happen in markets that are efficient to some degree. But does becoming rich at ∞ really contradict market efficiency?

The almost certain properties in this talk: if they fail to happen, they will do so “before ∞”, and Skeptic can become infinitely rich at that time. For example: Skeptic can become arbitrarily rich immediately after an isolated point in $\mathcal{L}_\omega(b)$ is observed.

The situation with, e.g., convergence is completely different: Doob’s upcrossings argument only shows that Skeptic will become rich at infinity (but not earlier) if the price process does not converge to a point in $[-\infty, \infty]$.

Corollary

Corollary

For each $b \in \mathbb{R}$, it is almost certain that the set $\mathcal{L}_\omega(b)$ is perfect, and so either is empty or has the cardinality of continuum.

Proof.

Since ω is continuous, the set $\mathcal{L}_\omega(b)$ is closed and so, by the previous theorem perfect. Non-empty perfect sets in $\mathbb{R}$ always have the cardinality of continuum. □

Two completely uncertain events

$E \subseteq \Omega$ is completely uncertain if $\overline{\mathbb{P}}(E) = 1$ and $\underline{\mathbb{P}}(E) = 0$.

Two standard properties of typical trajectories of Brownian motion become completely uncertain for continuous price processes.

Proposition

Let $b \in \mathbb{R}$. The following events are completely uncertain:

1. *the Lebesgue measure of $\mathcal{L}_\omega(b)$ is zero;*
2. *the set $\mathcal{L}_\omega(b)$ is unbounded.*

Proof.

Consider constant ω. $\square$

Non-increase

$t \in [0, \infty)$ is a point of semi-strict increase for ω if there exists $\delta > 0$ such that $\omega(s) \leq \omega(t) < \omega(u)$ for all $s \in ((t - \delta)^+, t)$ and $u \in (t, t + \delta)$.

Theorem

Almost surely, ω has no points of semi-strict increase [or decrease].

Proved in measure-theoretic probability: Dvoretzky, Erdős, and Kakutani 1960 (Brownian motion), Dubins and Schwarz 1965 (continuous martingales). Proof of the game-theoretic result (non-trivial) can be extracted from Burdzy's proof of the DEK 1960 result.

Corollary

Corollary

Almost surely, ω is monotone in no open interval, unless it is constant in that interval.

Proof.

Direct demonstration (for the case of increase): each interval of monotonicity where ω is not constant contains a rational time point a after which ω increases by a rational amount $D > 0$ before hitting the level $\omega(a)$ again; this event, denoted $E_{a,D}$, is null. It remains to apply subadditivity. $\square$

Maxima: definitions

Interval $[t_1, t_2] \subseteq [0, \infty)$ is an interval of local maximum for ω if:

- ω is constant on $[t_1, t_2]$ but not constant on any larger interval containing $[t_1, t_2]$;
- there exists $\delta > 0$ such that $\omega(s) \leq \omega(t)$ for all $s \in ((t_1 - \delta)^+, t_1) \cup (t_2, t_2 + \delta)$ and all $t \in [t_1, t_2]$.

Where $t_1 = t_2$: "interval" $\mapsto$ "point".

Ray $[t, \infty)$ is a ray of local maximum for ω if

- ω is constant on $[t, \infty)$ but not constant on any larger ray $[s, \infty)$, $s \in (0, t)$;
- there exists $\delta > 0$ such that $\omega(s) \leq \omega(t)$ for all $s \in ((t - \delta)^+, t)$.

Cont.

Strict: “$\leq$” $\mapsto$ “$<$.

$t \in [0, \infty)$ is a point of constancy for ω if there exists $\delta > 0$ such that $\omega(s) = \omega(t)$ for all $s \in ((t-\delta)^+, t+\delta)$; all other points $t \in [0, \infty)$ points of non-constancy.

Maxima: statement

Corollary

Almost surely, every interval of local maximum is a point, all points and the ray (if it exists) of local maximum are strict, the set of points of local maximum is countable, and any neighborhood of any point of non-constancy contains a point of local maximum.

Proof.

I will only prove the first statement (the rest are as easy). If ω had an interval of local maximum $[t_1, t_2]$ with $t_1 \neq t_2$, t_2 would be a point of semi-strict decrease. Alternatively: use a direct argument. Therefore: no such $[t_1, t_2]$ can even be an interval of local maximum "on the right". $\square$

Nowhere differentiability

A simple game-theoretic version of the classical result about Brownian motion (Paley, Wiener, Zygmund 1933):

Corollary

Almost surely, ω does not have a non-zero derivative anywhere.

Proof.

A point where a non-zero derivative exists is a point of semi-strict increase or decrease. $\square$

Mixed game/measure-theoretic lemma

Lemma

Suppose $P(E) = 1$, where $E \in \mathcal{F}$ and P is a probability measure on $(\Omega, \mathcal{F})$ which makes the process $X_t(\omega) := \omega(t)$ a martingale w.r. to the filtration $(\mathcal{F}_t)$. Then $\overline{\mathbb{P}}(E) = 1$.

Proof.

Any elementary capital process is a local martingale under P, and so $\overline{\mathbb{P}}(E) < 1$ in conjunction with Ville's inequality for positive supermartingales would contradict the assumption that $P(E) = 1$. $\square$

Necessity of various conditions

Proposition

The following events are completely uncertain:

1. ω *is constant on* $[0, \infty)$*;*
2. *for some* $t \in (0, \infty)$, $[t, \infty)$ *is the ray of local maximum for* ω*;*
3. $\omega'(t)$ *exists for no* $t \in [0, \infty)$*.*

Proof.

Consider the following random processes: constant; Brownian motion; stopped Brownian motion. □

Outline

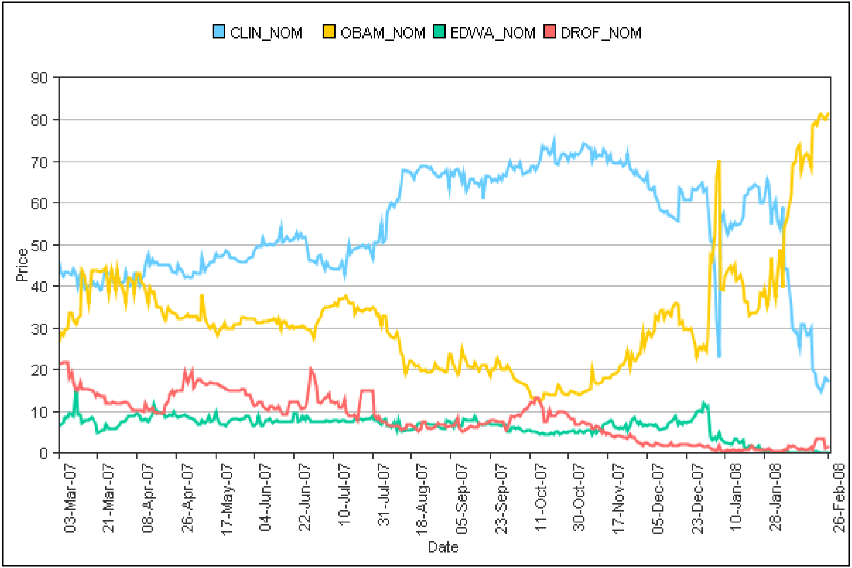
1. Non-predictive game-theoretic probability
   - Emergence of randomness
   - Emergence of volatility
   - Emergence of probability
2. Predictive game-theoretic probability
   - Game-theoretic Brownian motion
   - Predicting continuous processes
   - Predicting point processes
3. Discussion

Sources of volatility

What creates volatility?

- News?
- Merely the process of trading?

Our model: 2 is true [perhaps 1 is also true].


CLIN_NOM
OBAM_NOM
EDWA_NOM
DROF_NOM
Price
0
10
20
30
40
50
60
70
80
90
03-Mar-07
21-Mar-07
08-Apr-07
26-Apr-07
17-May-07
04-Jun-07
22-Jun-07
10-Jul-07
31-Jul-07
18-Aug-07
05-Sep-07
23-Sep-07
11-Oct-07
30-Oct-07
17-Nov-07
05-Dec-07
23-Dec-07
10-Jan-08
28-Jan-08
26-Feb-08
Date

Strong variation

For each $p \in (0, \infty)$ and $[u, v]$ with $0 \le u < v < \infty$, the strong p-variation of $\omega \in \Omega$ over $[u, v]$ is

$$\mathrm{var}_p^{[u,v]}(\omega) := \sup_{\kappa} \sum_{i=1}^{n} |\omega(t_i) - \omega(t_{i-1})|^p ,$$

where n ranges over $\mathbb{N}$ and κ over all partitions $u = t_0 < t_1 < \cdots < t_n = v$ of $[u, v]$.

There exists a unique number $\mathrm{vex}^{[u,v]}(\omega) \in [0, \infty]$ (strong variation exponent of ω over $[u, v]$) such that:

$$\mathrm{var}_p^{[u,v]}(\omega) \begin{cases} < \infty & \text{when } p > \mathrm{vex}^{[u,v]}(\omega) \\ = \infty & \text{when } p < \mathrm{vex}^{[u,v]}(\omega) \end{cases}$$

Notice: $\mathrm{vex}^{[u,v]}(\omega) \notin (0, 1)$.

Emergence of volatility

Theorem

For almost all $\omega \in \Omega$,

$$\forall [u, v] : \mathrm{vex}^{[u,v]}(\omega) = 2 \text{ or } \omega|_{[u,v]} \text{ is constant.}$$

Alternatively: $\mathrm{vex}^{[u,v]}(\omega) \in \{0, 2\}$.

Related results I

Measure-theoretic probability:

- Lévy (1940): for Brownian motion, vex $= 2$ a.s.
- Lepingle (1976): continuous semimartingales
- Bruneau (1979): simple (almost game-theoretic) proof of the measure-theoretic result; can be adapted to prove the game-theoretic result

Game-theoretic probability:

- Shafer and Vovk (2003): uses NSA
- Takeuchi, Kumon, Takemura (2007): more complicated statement

Related results II

"Semi-game-theoretic" probability:

- Rogers (1997): fractional Brownian motion B_h admits arbitrage unless $h = 1/2$.
- Kawada and Kôno (1973) $+$ Marcus and Rosen (2006) [perhaps earlier]: vex $B_h = 1/h$ a.s.

Outline

1. Non-predictive game-theoretic probability
   - Emergence of randomness
   - Emergence of volatility
   - Emergence of probability
2. Predictive game-theoretic probability
   - Game-theoretic Brownian motion
   - Predicting continuous processes
   - Predicting point processes
3. Discussion

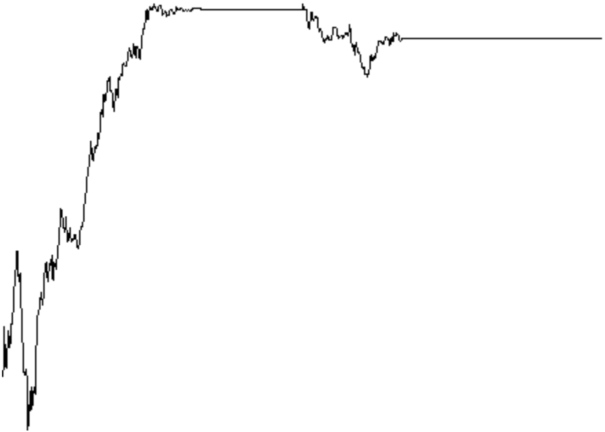
Brownian motion via time change

The previous results suggest: a continuous price process is Brownian motion, up to a time change.

Similar to the Dubins–Schwarz (but not Dambis's) result. Except: now probability is not postulated but emerges.

Problem: the intrinsic life span of the price process can be finite.

Price process with a finite intrinsic life span



Restricted probability

For any $B \subseteq \Omega$, restricted version of upper probability:

$$\overline{\mathbb{P}}(E; B) := \inf\left\{\mathfrak{S}_0 \;\middle|\; \forall\omega \in B : \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge \mathbf{1}_E(\omega)\right\} = \overline{\mathbb{P}}(E \cap B),$$

with $\mathfrak{S}$ ranging over the positive capital processes. (Used only when $\overline{\mathbb{P}}(B) = 1$.)

Restricted lower probability:

$$\underline{\mathbb{P}}(E; B) := 1 - \overline{\mathbb{P}}(E^c; B) = \underline{\mathbb{P}}(E \cup B^c).$$

The analogue in measure-theoretic probability: $E \mapsto \mathbb{P}(E \cap B)$. (Not related to conditional probability $\mathbb{P}(E \mid B)$.)

Quadratic variation I

For each $n \in \mathbb{N}$, let $\mathbb{D}_n := \{k2^{-n} : k \in \mathbb{Z}\}$ and define stopping times T_k^n inductively by

$$T_0^n(\omega) := \inf\{t \geq 0 : \omega(t) \in \mathbb{D}_n\},$$
$$T_k^n(\omega) := \inf\left\{t \geq T_{k-1}^n : \omega(t) \in \mathbb{D}_n \ \& \ \omega(t) \neq \omega(T_{k-1}^n)\right\}, \quad k = 1, 2, \ldots$$

For each $t \in [0, \infty)$ and $\omega \in \Omega$, define

$$A_t^n(\omega) := \sum_{k=0}^{\infty} \left(\omega(T_k^n \wedge t) - \omega(T_{k-1}^n \wedge t)\right)^2$$

(with $T_{-1}^n := 0$) and set

$$\overline{A}_t(\omega) := \limsup_{n\to\infty} A_t^n(\omega), \quad \underline{A}_t(\omega) := \liminf_{n\to\infty} A_t^n(\omega).$$

Quadratic variation II

Lemma

For almost all $\omega \in \Omega$*:*

- $\forall t \in [0, \infty) : \overline{A}_t(\omega) = \underline{A}_t(\omega)$;
- *the function* $t \in [0, \infty) \mapsto A_t(\omega) := \overline{A}_t(\omega) = \underline{A}_t(\omega)$ *is almost surely an element of* Ω.

$$\tau_s := \inf \left\{ t \geq 0 : \overline{A}|_{[0,t)} = \underline{A}|_{[0,t)} \in C[0,t) \ \ \& \ \ \sup_{u<t} \overline{A}_u = \sup_{u<t} \underline{A}_u \geq s \right\}$$

Lemma

τ_s *is a stopping time.*

Convention

An event stated in terms of A_∞, such as $A_\infty = \infty$, happens if and only if $\overline{A} = \underline{A}$ and $A_\infty := \overline{A}_\infty = \underline{A}_\infty$ satisfies the given condition.

We will later restrict $\overline{\mathbb{P}}$ and $\underline{\mathbb{P}}$ to events such as $A_\infty = \infty$.

Images I

Let $P : 2^\Omega \to [0,1]$ (such as $\overline{\mathbb{P}}$ or $\underline{\mathbb{P}}$) and $f : \Omega \to \Psi$. The image (or pushforward) $Pf^{-1} : 2^\Psi \to [0,1]$ is

$$Pf^{-1}(E) := P(f^{-1}(E)).$$

Time change $\mathrm{tc} : \Omega \to \mathbb{R}^{[0,\infty)}$: for each $\omega \in \Omega$, $\mathrm{tc}(\omega)$ is defined to be $s \mapsto \omega(\tau_s)$, $s \in [0,\infty)$ (with $\omega(\infty) := 0$). For each $c \in \mathbb{R}$, $\overline{Q}_c$ and $\underline{Q}_c$ are the images of

$$E \subseteq \Omega \mapsto \overline{\mathbb{P}}(E; \omega(0) = c, A_\infty = \infty),$$
$$E \subseteq \Omega \mapsto \underline{\mathbb{P}}(E; \omega(0) = c, A_\infty = \infty),$$

respectively, under tc.

Images II

For any $S > 0$, restricted time change tc_S is defined by

$$\forall \omega \in \Omega : \mathrm{tc}_S(\omega) := \mathrm{tc}(\omega)|_{[0,S]}.$$

The corresponding images are

$$\overline{Q}_{c,S} := \overline{\mathbb{P}}(\,\cdot\,; \omega(0) = c, \tau_S < \infty)\,\mathrm{tc}_S^{-1},$$
$$\underline{Q}_{c,S} := \underline{\mathbb{P}}(\,\cdot\,; \omega(0) = c, \tau_S < \infty)\,\mathrm{tc}_S^{-1}.$$

Lemma

$\overline{\mathbb{P}}(\omega(0) = c, A_\infty = \infty) = 1$ *(and so* $\overline{\mathbb{P}}(\omega(0) = c, A_\infty \geq S) = 1$ *for each* $S > 0$*).*

Notation for Brownian motion

For $c \in \mathbb{R}$, $\mathcal{W}_c$ is the probability measure on $(\Omega, \mathcal{F})$ corresponding to Brownian motion over time period $[0, \infty)$ started from c.

For all $c \in \mathbb{R}$ and $S > 0$, $\mathcal{W}_{c,S}$ is the probability measure on $(C[0, S], \mathcal{F}_S)$ ($\mathcal{F}_S$ is now a σ-algebra on $C[0, S]$) corresponding to Brownian motion over time period $[0, S]$ started from c.

Theorem: emergence of BM

Theorem

1. *For almost all ω, the function*

$$A(\omega) : t \in [0, \infty) \mapsto A_t(\omega) := \overline{A}_t(\omega) = \underline{A}_t(\omega)$$

exists, is an increasing element of Ω with $A_0(\omega) = 0$, and has the same intervals of constancy as ω.

2. *For all $c \in \mathbb{R}$,*

$$\overline{Q}_c|_{\mathcal{F}} = \underline{Q}_c|_{\mathcal{F}} = \mathcal{W}_c.$$

3. *For any $c \in \mathbb{R}$ and $S > 0$,*

$$\overline{Q}_{c,S}|_{\mathcal{F}_S} = \underline{Q}_{c,S}|_{\mathcal{F}_S} = \mathcal{W}_{c,S}.$$

Remark

The theorem depends on the arbitrary choice of the sequence of partitions $(\mathbb{D}_n)$ to define the quadratic variation process A.

To make this less arbitrary, we could consider all partitions whose mesh tends to zero fast enough and which are definable in the standard language of set theory ($\approx$ Wald's suggested requirement for von Mises's collectives).

Corollary I: points of increase

Corollary

Almost surely, ω has no points of semi-strict increase.

Proof.

By Part 3 of Theorem and the Dvoretzky–Erdős–Kakutani result, the upper probability is zero that there is a point t of semi-strict increase such that $A_t < S < A_\infty$ for $S \in \mathbb{Q}$. Such S will exist: $A_t(\omega) < A_\infty(\omega)$ (except for ω in a null set) for any point t of semi-strict increase of ω follows from Part 1 of Theorem. □

Corollary II: volatility exponent

Corollary

For almost all $\omega \in \Omega$*: for all* $0 \le u < v < \infty$, $\mathrm{vex}^{[u,v]}(\omega) \in \{0, 2\}$.

Proof.

Similar but slightly messier. $\square$

Full proofs and further details

- Vladimir Vovk.
  Continuous-time trading and emergence of randomness.
  http://probabilityandfinance.com, Working Paper 24. Last revised January 2008.
- Vladimir Vovk.
  Continuous-time trading and emergence of volatility.
  http://probabilityandfinance.com, Working Paper 25. Last revised December 2007.
- Vladimir Vovk.
  Continuous-time trading and emergence of probability. To appear on
  http://probabilityandfinance.com.

Outline

1. Non-predictive game-theoretic probability
   - Emergence of randomness
   - Emergence of volatility
   - Emergence of probability
2. Predictive game-theoretic probability
   - Game-theoretic Brownian motion
   - Predicting continuous processes
   - Predicting point processes
3. Discussion

Idea

There is a new player (Forecaster) who tries to predict ω (to be denoted X). Two simple cases:

- Continuous ω (as before). Hope: the Wiener process will emerge.
- "Counting" ω. Hope: the Poisson process will emerge.

But first: the even simpler Lévy game.

Outline

1. Non-predictive game-theoretic probability
   - Emergence of randomness
   - Emergence of volatility
   - Emergence of probability
2. Predictive game-theoretic probability
   - Game-theoretic Brownian motion
   - Predicting continuous processes
   - Predicting point processes
3. Discussion

Lévy game I

Still a game between 2 players. (Forecaster is not an active player as yet.)

An elementary trading strategy G: stopping times $\tau_1 \le \tau_2 \le \cdots$, as before, and, for each $n \in \mathbb{N}$, a pair of bounded $\mathcal{F}_{\tau_n}$-measurable functions, M_n and V_n. Elementary capital process:

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} \Big(M_n(\omega)\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\big)$$

$$+ V_n(\omega)\Big(\big(\omega^2(\tau_{n+1} \wedge t) - (\tau_{n+1} \wedge t)\big) - \big(\omega^2(\tau_n \wedge t) - (\tau_n \wedge t)\big)\Big)\Big).$$

Lévy game II

$M_n(\omega)$ and $V_n(\omega)$: Skeptic's stakes or bets (on $\omega(t)$ and $\omega^2(t) - t$, respectively) chosen at time τ_n.

$$\left.\begin{array}{r}\text{positive capital process}\\ \text{upper and lower probability}\end{array}\right\}\text{ as before}$$

Coherence: follows immediately from the existence of measure-theoretic Brownian motion.

Emergence of the Wiener measure II

Countable subadditivity $\therefore$ $\overline{\mathbb{P}}$ is an outer measure in Carathéodory's sense. Recall: $A \subseteq \Omega$ is $\overline{\mathbb{P}}$-measurable if, for each $E \subseteq \Omega$,

$$\overline{\mathbb{P}}(E) = \overline{\mathbb{P}}(E \cap A) + \overline{\mathbb{P}}(E \cap A^c).$$

Standard result (Carathéodory, 1914): the family $\mathcal{A}$ of all $\overline{\mathbb{P}}$-measurable sets is a σ-algebra and the restriction of $\overline{\mathbb{P}}$ to $\mathcal{A}$ is a probability measure on $(\Omega, \mathcal{A})$.

Theorem

Each event $A \in \mathcal{F}$ is $\overline{\mathbb{P}}$-measurable, and the restriction of $\overline{\mathbb{P}}$ to $\mathcal{F}$ coincides with the Wiener measure $W = W_0$ on $(\Omega, \mathcal{F})$. In particular, $\overline{\mathbb{P}}(A) = \underline{\mathbb{P}}(A) = W(A)$ for each $A \in \mathcal{F}$.

Open problem

What is the class $\mathcal{A}$ of all $\overline{\mathbb{P}}$-measurable subsets of Ω?

Can be easily shown:

Proposition

Each set $A \in \mathcal{F}^W$ in the completion of $\mathcal{F}$ w.r.t. to W is $\overline{\mathbb{P}}$-measurable.

Specific question: $\mathcal{A} = \mathcal{F}^W$?

Outline

1 Non-predictive game-theoretic probability
- Emergence of randomness
- Emergence of volatility
- Emergence of probability

2 Predictive game-theoretic probability
- Game-theoretic Brownian motion
- Predicting continuous processes
- Predicting point processes

3 Discussion

Another definition of trading strategies

Let

- $(\mathcal{F}_t, t \in [0, \infty))$: filtration on some set Ω
- $\Gamma = \mathbb{R}^d$ for some $d \in \{1, 2\}$
- μ: adapted càdlàg process (basic martingale)

An elementary trading strategy G consists of:

- stopping times $\tau_1 \leq \tau_2 \leq \cdots$, as before
- for each $n \in \mathbb{N}$, a bounded $\mathcal{F}_{\tau_n}$-measurable Γ-valued M_n

Another definition of probability

Elementary capital process:

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} M_n(\omega) \cdot \left(\mu_{\tau_{n+1}\wedge t} - \mu_{\tau_n \wedge t}\right).$$

Positive capital process, upper and lower probability: as before.

Dambis game: informal picture

- Skeptic chooses a trading strategy.
- Forecaster chooses continuous $B : [0, \infty) \to \mathbb{R}$ and continuous increasing $A : [0, \infty) \to [0, \infty)$ with $A(0) = 0$; for simplicity $A(\infty) = \infty$.
- Reality chooses continuous $X : [0, \infty) \to \mathbb{R}$.

The interaction between Forecaster and Reality: not formalized.

Example of Forecaster's strategy

$$B(t) := \int_0^t b(s, X(s))\mathrm{d}s, \quad A(t) := \int_0^t a^2(s, X(s))\mathrm{d}s$$

for some functions b and a.

In the language of measure-theoretic probability, Forecaster models Reality by the SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + a(t, X_t)\mathrm{d}W_t.$$

In general: B is the "trend process" and A is the "volatility process" for X ($X - B$ and $(X - B)^2 - A$ are [local] martingales).

Definition of the Dambis game

Ω is the set of all triples (B, A, X) (as above).

$\mathcal{F}_t$, $t \in [0, \infty)$: the smallest σ-algebra that makes all functions $(B, A, X) \mapsto (B(s), A(s), X(s))$, $s \in [0, t]$, measurable.

The basic martingale:

$$\mu_t(\omega) := \left(X(t) - B(t), (X(t) - B(t))^2 - A(t)\right),$$

where $\omega := (B, A, X)$.

Game-theoretic Dambis result

Theorem

Let $s > 0$,

$$\tau_s := \inf\{t : A(t) = s\}.$$

Then $X(\tau_s) - B(\tau_s) \sim \mathcal{N}_{0,s}$ in the sense that

$$\overline{\mathbb{P}}(X(\tau_s) - B(\tau_s) \in E) = \underline{\mathbb{P}}(X(\tau_s) - B(\tau_s) \in E) = \mathcal{N}_{0,s}(E)$$

for all $E \in \mathcal{B}(\mathbb{R})$.

Vovk (1993): time-changed $X - B$ is Wiener process. (But the statement in that paper is more awkward than the statement for the Lévy game, since only elementary capital processes are considered.)

Measure-theoretic results

Dubins–Schwarz 1965 and Dambis 1965 are usually mentioned together but in fact they are very different: the former is non-predictive and the latter is predictive. The difference is much less pronounced in measure-theoretic probability.

The case $B = 0$ and $A(t) = t$: Lévy's characterization of Brownian motion.

Outline

1 Non-predictive game-theoretic probability
- Emergence of randomness
- Emergence of volatility
- Emergence of probability

2 Predictive game-theoretic probability
- Game-theoretic Brownian motion
- Predicting continuous processes
- Predicting point processes

3 Discussion

Definitions for the Meyer–Papangelou game

Ω is the set of all pairs (A, X), where A is as before and $X : [0, \infty) \to \{0, 1, 2, \dots\}$ is a càdlàg function with $X(0) = 0$ and all jumps of size 1 (“counting function”).

$\mathcal{F}_t$, $t \in [0, \infty)$: the smallest σ-algebra that makes all functions $(A, X) \mapsto (A(s), X(s))$, $s \in [0, t]$, measurable.

The basic martingale:

$$\mu_t := X(t) - A(t).$$

Game-theoretic Meyer–Papangelou result

Theorem

Let $s > 0$,

$$\tau_s := \inf \{t : A(t) = s\} .$$

Then $X(\tau_s) \sim \mathcal{P}_s$, where $\mathcal{P}_s$ is the Poisson distribution with parameter s, in the sense that

$$\overline{\mathbb{P}}(X(\tau_s) \in E) = \underline{\mathbb{P}}(X(\tau_s) \in E) = \mathcal{P}_s(E), \quad \forall E \subseteq \{0, 1, \ldots\}.$$

Vovk (1993): time-changed X is Poisson process (in a weak sense).

Measure-theoretic results

Time-changed point processes become Poisson process: Meyer (1971) and Papangelou (1972).

When $A(t) = t$: cf. Watanabe's (1964) characterization of Poisson process.

Proofs

- Vladimir Vovk.
  Game-theoretic Brownian motion.
  http://probabilityandfinance.com, Working Paper 26. Last revised January 2008.
- Vladimir Vovk.
  Forecasting point and continuous processes: prequential analysis.
  Test, **2**:189–217, 1993.

Outline

1. Non-predictive game-theoretic probability
   - Emergence of randomness
   - Emergence of volatility
   - Emergence of probability
2. Predictive game-theoretic probability
   - Game-theoretic Brownian motion
   - Predicting continuous processes
   - Predicting point processes
3. Discussion

Big Question

Question

Will the Lévy game remain coherent if the measurability restrictions on stopping times and stakes are dropped?

- Positive answer $\rightarrow$ simpler and more intuitive definitions ($\approx$ as in discrete-time GTP)
- Negative answer: a counter-intuitive phenomenon akin to the Banach–Tarski paradox

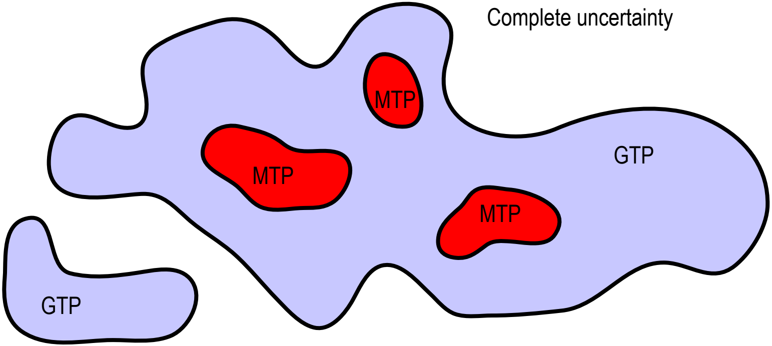
Related question: will dropping the requirement that M and V should be bounded lead to loss of coherence?

Philosophy: relation between MTP and GTP

Spectrum of views:

- MTP is enough
- GTP will eventually supplant MTP
- middle way: MTP and GTP have always co-existed and will always co-exist

Perhaps: the right philosophy will eventually be determined by the mathematical results we eventually get. This talk's mathematics: philosophically leans to the middle way.


Complete uncertainty
MTP
MTP
GTP
MTP
GTP

Our goal: push out both areas, red (MTP) and blue (GTP).

Will the gap shrink or expand?

Thank you for your attention!