The efficient index hypothesis and its implications in the BSM model

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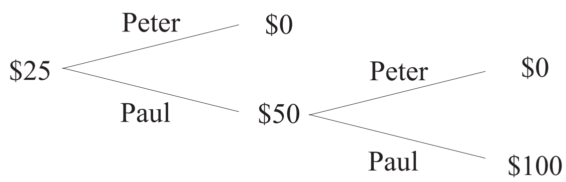

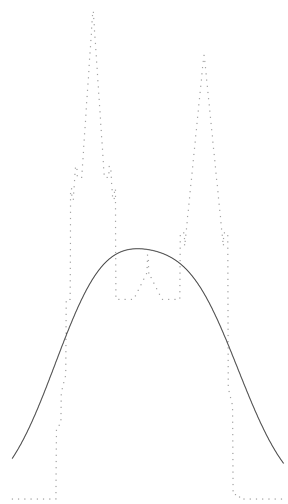

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Abstract

This article studies the behavior of an index I_t which is assumed to be a tradable security, to satisfy the BSM model $\mathrm{d}I_t/I_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$, and to be *efficient* in the following sense: we do not expect a prespecified trading strategy whose value is almost surely always nonnegative to outperform the index greatly. The efficiency of the index imposes severe restrictions on its appreciation rate; in particular, for a long investment horizon we should have $\mu \approx r + \sigma^2$, where r is the interest rate. This provides another partial solution to the equity premium puzzle. All our mathematical results are extremely simple.

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1 Introduction

The *efficient index hypothesis* (*EIH*) is a version of the random walk hypothesis and the efficient market hypothesis. It is a statement about a specific index, such as S&P 500, and says that we do not expect a prespecified trading strategy to beat the index by a factor of $1/\delta$ or more, for a given threshold δ (such as $\delta = 0.1$). The trading strategy is assumed to be *prudent*, in the sense of its value being nonnegative a.s. at all times. By saying that it beats the index by a factor of $1/\delta$ or more we mean that its initial value is $\mathcal{K}_0 > 0$ and its final value $\mathcal{K}_T$ satisfies $\mathcal{K}_T/I_T \geq (1/\delta)(\mathcal{K}_0/I_0)$. (By the value of a trading strategy we always mean the undiscounted dollar value of its current portfolio.) We will see that the EIH has several interesting implications, such as $\mu \approx r + \sigma^2$ for the appreciation rate μ of the index.

We use the EIH in the interpretation of our results, but their mathematical statements do not involve this hypothesis. For example, in Section 2 we prove that there is a prudent trading strategy that, almost surely, beats the index by a factor of at least 100 unless

$$\frac{I_T}{\mathrm{e}^{rT}} \in \left(\mathrm{e}^{\sigma^2 T/2 - 2.58\sigma\sqrt{T}}, \mathrm{e}^{\sigma^2 T/2 + 2.58\sigma\sqrt{T}}\right) \tag{1.1}$$

(see Proposition 2.1). If we believe in the EIH (for $\delta = 0.01$), we should believe in (1.1). But even if we do not believe in the EIH, the proposition gives us a way of beating the index when (1.1) is violated.

As used in this article, the EIH is a weaker assumption than it appears to be. There might be sophisticated prudent trading strategies that do beat the index (by a large factor), but we are not interested in such strategies. It is sufficient that the primitive strategies considered in this article be not expected to beat the index.

Our EIH is obviously related, and has a similar motivation, to the standard efficient markets hypothesis [3]. There are, however, important differences. For example, the EIH does not assume that the security prices are "correct" in any sense, or that investors' expectations are rational (individually or *en masse*). The EIH controls for risk only by insisting that our trading strategies be prudent. Admittedly, this is a weak requirement, and so the threshold value of δ should be a small number; in our examples, we use $\delta = 0.1$ or $\delta = 0.01$. If a trader is worried about losing all money, nothing prevents her from investing only part of her capital in prudent strategies that can lose everything. In Section 6, for example, we consider a trading strategy that is a weighted average of the index, with weight 90%, and a prudent trading strategy that, almost surely, beats the index by a factor of at least 100 unless (1.1) holds, with weight 10%. This strategy either underperforms the index by 10% (if (1.1) holds) or beats it by 990% (if not).

Remark. In [13, 17], the EIH was referred to as the "efficient market hypothesis", whereas the standard hypothesis of market efficiency as the "efficient markets hypothesis", with "markets" in plural. However, nowadays the standard hypothesis is more often called the "efficient market hypothesis" than the "efficient markets hypothesis", and so it is safer to use a different term for our hypothesis. The results of this article agree with the results of [17] (see, e.g., (1) of [17] as applied to $s_n := r$, $\forall n$), which were obtained using very different methods.

We start the main part of the article with results about the final value of the index under the EIH (Section 2). The main insight here is that the index outperforms the bond approximately by a factor of $e^{\sigma^2 T/2}$ (cf. (1.1)). In the following section, Section 3, we show that, under the EIH, $\mu \approx r + \sigma^2$. Section 4 applies this result to the equity premium puzzle; the equity premium of σ^2 is closer to the observed levels of the equity premium than the predictions [9, 7, 10] of some standard theories. Section 5 discusses our findings from the point of view of game-theoretic probability (see, e.g., [13]), and Section 6 discusses them from the point of view of equilibrium asset pricing (see, e.g., [11]).

2 The final value of the index

The time interval in this article is $[0, T]$, $T > 0$; in the interpretation of our results the horizon T will be assumed to be a large number. The value of the index at time t is denoted I_t. We assume that it satisfies the BSM (Black–Scholes–Merton) model

$$\frac{\mathrm{d}I_t}{I_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t \tag{2.1}$$

and that $I_0 = 1$. The interest rate r is assumed constant. We will sometimes interpret e^{rt} as the price at time t of a zero-coupon bond whose initial price is 1.

Let z_p be the upper p-quantile of the standard Gaussian distribution $N_{0,1}$; i.e., z_p is defined by the requirement that $\mathbb{P}(\xi \geq z_p) = p$, where $\xi \sim N_{0,1}$.

Proposition 2.1. *Let $\delta > 0$. There is a prudent trading strategy (depending on σ, r, T, δ) that, almost surely, beats the index by a factor of $1/\delta$ unless*

$$\frac{I_T}{e^{rT}} \in \left(e^{\sigma^2 T/2 - z_{\delta/2}\sigma\sqrt{T}}, e^{\sigma^2 T/2 + z_{\delta/2}\sigma\sqrt{T}} \right). \tag{2.2}$$

Equation (2.2) says that for large T the efficient index can be expected to outperform the bond $e^{\sigma^2 T/2}$-fold. The case $\delta \geq 1$ in Proposition 2.1 is trivial, but we do not exclude it to simplify the statement of the proposition; the upper quantile z_p is understood to be $-\infty$ when $p \geq 1$.

If we are only interested in a lower or upper bound on I_T, we can use the following proposition.

Proposition 2.2. *Let $\delta > 0$. There is a prudent trading strategy that, almost surely, beats the index by a factor of $1/\delta$ unless*

$$\frac{I_T}{e^{rT}} > e^{\sigma^2 T/2 - z_\delta \sigma\sqrt{T}}.$$

There is another prudent trading strategy that, almost surely, beats the index by a factor of $1/\delta$ unless

$$\frac{I_T}{e^{rT}} < e^{\sigma^2 T/2 + z_\delta \sigma\sqrt{T}}. \tag{2.3}$$

In the rest of this section we will prove Proposition 2.1; Proposition 2.2 can be proved in the same way. It will be clear from the proof that Propositions 2.1 and 2.2 are tight in the sense that the factor $1/\delta$ cannot be improved. The main idea of the proof is reminiscent of the argument in [1].

Let $\mathbf{1}\{\ldots\}$ be 1 if the condition in the curly braces is satisfied and 0 otherwise. We start from the following simple analytic lemma.

Lemma 2.3. *Let $u \in \mathbb{R}$, $E \subseteq \mathbb{R}$ be a Borel set, and $\xi \sim N_{0,1}$. Then*

$$\mathbb{E}\left(e^{u\xi}\, \mathbf{1}\{\xi \in E\} \right) = e^{u^2/2}\, \mathbb{P}\left(\xi + u \in E\right).$$

Proof. This follows from

$$\begin{aligned}
\mathbb{E}\left(e^{u\xi}\, \mathbf{1}\{\xi \in E\} \right) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{uz}\, \mathbf{1}\{z \in E\}\, e^{-z^2/2}\, dz \\
&= \frac{1}{\sqrt{2\pi}} e^{u^2/2} \int_{\mathbb{R}} \mathbf{1}\{z \in E\}\, e^{-(z-u)^2/2}\, dz \\
&= \frac{1}{\sqrt{2\pi}} e^{u^2/2} \int_{\mathbb{R}} \mathbf{1}\left\{w + u \in E\right\} e^{-w^2/2}\, dw \\
&= e^{u^2/2}\, \mathbb{P}\left(\xi + u \in E\right).
\end{aligned}$$

□

The risk-neutral version of (2.1) is

$$\frac{\mathrm{d}I_t}{I_t} = r\mathrm{d}t + \sigma\mathrm{d}W_t, \tag{2.4}$$

and the explicit strong solution to this SDE is

$$I_t = \mathrm{e}^{(r-\sigma^2/2)t+\sigma W_t}\,. \tag{2.5}$$

Proof of Proposition 2.1. Let A be the event that (2.2) is violated and $\mathbf{1}_A$ be its indicator function. The BSM price at time 0 of the European contingent claim whose payoff at time T is $I_T\,\mathbf{1}_A$ can be computed as the discounted expected value

$$\begin{aligned}
&\mathrm{e}^{-rT}\,\mathbb{E}(I_T\,\mathbf{1}_A)\\
&\quad= \mathrm{e}^{-rT}\,\mathbb{E}\left(\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}\xi}\,\mathbf{1}\left\{\xi \le \sigma\sqrt{T} - z_{\delta/2} \text{ or } \xi \ge \sigma\sqrt{T} + z_{\delta/2}\right\}\right)\\
&\quad= \mathbb{P}\left(\xi \le -z_{\delta/2} \text{ or } \xi \ge z_{\delta/2}\right) = \delta,
\end{aligned}$$

where $\xi \sim N_{0,1}$; we have used (2.5) and Lemma 2.3. Since the BSM price can be hedged perfectly (see, e.g., [4], Theorem 5.8.12), there is a prudent trading strategy Σ with initial value δ and final value $I_T\,\mathbf{1}_A$ a.s. We can see that Σ beats the index by a factor of $1/\delta$ if A happens. ☐

3 Implications for the appreciation rate of the index

The following corollary of Proposition 2.1 shows that the EIH and the BSM model (2.1) imply $\mu \approx r + \sigma^2$.

Proposition 3.1. *For each $\delta > 0$ there exists a prudent trading strategy $\Sigma = \Sigma(\sigma, r, T, \delta)$ that satisfies the following condition. For each $\epsilon > 0$, either*

$$\left|r + \sigma^2 - \mu\right| < \frac{(z_{\delta/2} + z_\epsilon)\sigma}{\sqrt{T}} \tag{3.1}$$

or Σ beats the index by a factor of at least $1/\delta$ with probability at least $1-\epsilon$.

Intuitively, $\mu \approx r + \sigma^2$ unless we can beat the index or a rare event happens (assuming that δ and ϵ are small and T is large).

Proof of Proposition 3.1. Without loss of generality, assume $\delta, \epsilon \in (0,1)$. As Σ we take a prudent trading strategy that beats the index by a factor of $1/\delta$ unless (2.2) holds. Therefore, we are only required to prove that the event that (2.2) holds but (3.1) does not has probability at most ϵ. We can rewrite (2.2) as

$$\left| \ln I_T - rT - \frac{\sigma^2}{2} T \right| < z_{\delta/2} \sigma \sqrt{T}. \tag{3.2}$$

Remembering that (2.1) has explicit solution $I_t = \mathrm{e}^{(\mu - \sigma^2/2)t + \sigma W_t}$, we can rewrite (3.2) as

$$\left| \sigma \sqrt{T} \xi - (r + \sigma^2 - \mu) T \right| < z_{\delta/2} \sigma \sqrt{T},$$

where $\xi \sim N_{0,1}$, i.e., as

$$\left| \xi - \frac{(r + \sigma^2 - \mu)\sqrt{T}}{\sigma} \right| < z_{\delta/2}. \tag{3.3}$$

If (3.1) is violated, we have either $r + \sigma^2 - \mu < -(z_{\delta/2} + z_\epsilon)\sigma/\sqrt{T}$ or $r + \sigma^2 - \mu > (z_{\delta/2} + z_\epsilon)\sigma/\sqrt{T}$. The two cases are analogous, and we consider only the first. In this case, (3.3) implies $\xi < -z_\epsilon$, the probability of which is ϵ. $\square$

Proposition 3.1 shows that the arbitrariness of μ in the BSM model (2.1) for the index is to a large degree illusory if we accept the EIH.

The strategy Σ of Proposition 3.1 depends only on σ, r, T, and δ. If we allow, additionally, dependence on μ and ϵ, we can use Proposition 2.2 instead of Proposition 2.1 and strengthen (3.1) by replacing $\delta/2$ with δ.

Proposition 3.2. *Let $\delta > 0$ and $\epsilon > 0$. Unless*

$$\left| r + \sigma^2 - \mu \right| < \frac{(z_\delta + z_\epsilon)\sigma}{\sqrt{T}}, \tag{3.4}$$

there exists a prudent trading strategy $\Sigma = \Sigma(\mu, \sigma, r, T, \delta, \epsilon)$ that beats the index by a factor of at least $1/\delta$ with probability at least $1 - \epsilon$.

Proof. Suppose (3.4) is violated. Since the cases $r + \sigma^2 - \mu < -(z_\delta + z_\epsilon)\sigma/\sqrt{T}$ and $r + \sigma^2 - \mu > (z_\delta + z_\epsilon)\sigma/\sqrt{T}$ are analogous, we will assume

$$r + \sigma^2 - \mu < -\frac{(z_\delta + z_\epsilon)\sigma}{\sqrt{T}}. \tag{3.5}$$

(Our trading strategy depends on which of the two cases holds, and so depends on μ and ϵ.) As Σ we take a prudent trading strategy that beats the index by a factor of $1/\delta$ unless (2.3) holds. We are required to prove that the probability of (2.3) is at most ϵ. We can rewrite (2.3) as

$$\ln I_T - rT - \frac{\sigma^2}{2}T < z_\delta \sigma \sqrt{T},$$

i.e.,

$$\xi - \frac{(r + \sigma^2 - \mu)\sqrt{T}}{\sigma} < z_\delta,$$

where $\xi \sim N_{0,1}$. The last inequality and (3.5) imply $\xi < -z_\epsilon$, whose probability is ϵ. □

4 Equity premium puzzle

The equity premium is the excess of stock returns over bond returns, and it appears to be higher in the real world than suggested by some standard economic theories. Mehra and Prescott dubbed this phenomenon the equity premium puzzle [9]. There is no consensus as to the explanation, or even to the existence, of the equity premium puzzle; for recent reviews see, e.g., [7, 10]. In this section we will see that our results can be interpreted as providing a partial solution to the puzzle.

According to Proposition 3.1, under the EIH we can expect $\mu \approx r + \sigma^2$. This gives the equity premium σ^2. The annual volatility of S&P 500 is approximately 20% (see, e.g., [8], p. 3, or [7], p. 8), which translates into an expected 4% equity premium. The standard theory, as applied by Mehra and Prescott, predicts an equity premium of at most 1% ([9], p. 146, [7], p. 11).

The empirical study by Mehra and Prescott reported in [8], Table 2, estimates the equity premium over the period 1889–2005 as 6.36%. Taking into account the later years 2006–2010 reduces it, but not much, to 6.05%. (The recent news about bonds outperforming stocks over the past 30 years [5] were about 30-year Treasury bonds, whereas Mehra and Prescott use short-term Treasury bills for this period.) Our figure of 4% is below 6.05%, but the difference is much less significant than for Mehra and Prescott's predictions. If the years 1802–1888 are also taken into account (as done by Siegel [14], updated until 2004 by Mehra and Prescott [8], Table 2, and until 2010 by myself), the equity premium goes down to 5.17%.

Equation (2.2) allows us to estimate the accuracy of our estimate σ^2 of the equity premium. Namely, we have, almost surely,

$$\frac{1}{T}\int_0^T \frac{\mathrm{d}I_t}{I_t} - r - \sigma^2 = \frac{\ln I_T + \sigma^2 T/2 - rT - \sigma^2 T}{T} \in \left(-\frac{z_{\delta/2}\sigma}{\sqrt{T}}, \frac{z_{\delta/2}\sigma}{\sqrt{T}}\right) \quad (4.1)$$

unless a prespecified prudent trading strategy beats the index by a factor of $1/\delta$. Plugging $\delta := 0.1$ (to obtain a reasonable accuracy), $\sigma := 0.2$, and $T := 2010 - 1888$, we evaluate $z_{\delta/2}\sigma/\sqrt{T}$ in (4.1) to 2.98% for the period 1889–2010, and changing T to $2010 - 1801$, we evaluate it to 2.28% for the period 1802–2010. For both periods, the observed equity premium falls well within the prediction interval.

5 Three kinds of probabilities for the index

In this section we will take a broader view of the simple results of the previous sections. We started from the "physical" probability measure (2.1), used the risk-neutral probability measure (2.4), and saw the importance of the probability measure

$$\frac{\mathrm{d}I_t}{I_t} = (r + \sigma^2)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad (5.1)$$

which will be called the *efficient-index measure*. We will see that the last two are essentially special cases of game-theoretic probability, as defined in [13] (and extended to continuous time in [15, 16]). If E is a Borel subset of the Banach space $\Omega := C([0,T])$ of all continuous functions on $[0,T]$, we define its *upper probability with bond as numéraire* by

$$\overline{\mathbb{P}}_{\mathrm{b}}(E) := \inf\left\{\mathcal{K}_0 \;\middle|\; \frac{\mathcal{K}_T}{\mathrm{e}^{rT}} \ge \mathbf{1}_E \text{ a.s.}\right\},$$

where $\mathbf{1}_E$ is the indicator function of E, $\mathcal{K}$ ranges over the value processes of prudent trading strategies, and "a.s." means with probability one under the physical measure (2.1) (equivalently, under (2.4) or under (5.1)). In other words, $\overline{\mathbb{P}}_{\mathrm{b}}(E)$ is the infimum of $\delta > 0$ such that a prudent trading strategy can beat the bond by a factor of $1/\delta$ or more on the event E (except for its subset of zero probability). We define the *upper probability* of E *with index as numéraire* by

$$\overline{\mathbb{P}}_I(E) := \inf\left\{\mathcal{K}_0 \;\middle|\; \frac{\mathcal{K}_T}{I_T} \ge \mathbf{1}_E \text{ a.s.}\right\}.$$

In other words, $\overline{\mathbb{P}}_I(E)$ is the infimum of $\delta > 0$ such that a prudent trading strategy can beat the index by a factor of $1/\delta$ or more on the event E.

For each Borel E, $\overline{\mathbb{P}}_{\mathrm{b}}(E)$ is the risk-neutral measure of E and $\overline{\mathbb{P}}_I(E)$ is its efficient-index measure. It is standard in game-theoretic probability to define the corresponding *lower probabilities*

$$\underline{\mathbb{P}}_{\mathrm{b}}(E) := 1 - \overline{\mathbb{P}}_{\mathrm{b}}(E^c) \text{ and } \underline{\mathbb{P}}_I(E) := 1 - \overline{\mathbb{P}}_I(E^c),$$

where $E^c := \Omega \setminus E$. Since our market is complete, upper and lower probabilities always coincide. A major difference of the definitions of $\overline{\mathbb{P}}_{\mathrm{b}}$ and $\overline{\mathbb{P}}_I$ from the usual definitions of upper probabilities in game-theoretic probability is the presence of "a.s."; in game-theoretic probability "a.s." is absent as there is no probability measure to begin with.

The processes (2.4) and (5.1) are in some sense reciprocal. By Itô's formula, if I_t satisfies (2.4), then $I_t^* := \mathrm{e}^{2rt}/I_t$ will satisfy (5.1) with I^* in place of I and $-W$ in place of W, and vice versa. (The definition of I_t^* makes the bond's price e^{rt} the geometric mean of I_t and I_t^*.) In particular, the appreciation rate of typical trajectories of (2.4) is approximately $\mathrm{e}^{(r-\sigma^2/2)t}$, and the appreciation rate of typical trajectories of (5.1) is approximately $\mathrm{e}^{(r+\sigma^2/2)t}$.

6 Connections with optimal portfolio selection

A surprising feature of the BSM theory of option pricing is that the BSM prices of contingent claims do not depend on the investors' attitudes toward risk. Similarly, our results show that the appreciation rate μ of the index should be close to $r + \sigma^2$ regardless of the investors' attitudes toward risk (but relying on the EIH). In this section we will discuss these results from the point of view of Merton's theory of optimal portfolio selection; of course, which portfolio is optimal very much depends on the chosen utility function.

Let us consider an investor whose utility of a final wealth x is $F_\alpha(x)$, where

$$F_\alpha(x) := \begin{cases} x^\alpha/\alpha & \text{if } \alpha \neq 0 \\ \ln x & \text{if } \alpha = 0 \end{cases} \tag{6.1}$$

and $\alpha < 1$ is a constant. The isoelastic utility function $F_\alpha(x)$ has a constant relative risk aversion equal to $1-\alpha$: for all $x > 0$, $-xF_\alpha''(x)/F_\alpha'(x) = 1-\alpha$. Notice that $F_0(x) = \lim_{\alpha\to 0}(F_\alpha(x) - 1/\alpha)$ (and adding a constant does not change a utility function in an essential way).

The investor's goal is to maximize the expectation of the utility of her final wealth by investing into the index and the bond starting from an initial capital of 1. As shown in [11] ((4.25); see also [2], (9.22)), the optimal fraction w of the investor's wealth invested in the index is

$$w = \frac{\mu - r}{(1-\alpha)\sigma^2}. \tag{6.2}$$

We start from an example that I learned from Robert Merton, who believes that it was used already by Paul Samuelson (in the case of discrete time). The role of this example in this article is heuristic as it replaces the constant interest rate r by an equilibrium interest rate; it also uses less familiar economic notions than the rest of the article.

Example 6.1 (heuristic)**.** Consider a primitive economy with one risky asset I satisfying (2.1). Since there is only one risky asset, we regard I_t as both the index and the market portfolio. The role of the risk-free asset is played by a financial asset (sometimes called "inside money") traded inside the economy and serving the purpose of borrowing and lending. The net amount of inside money in the economy is always zero, and the interest rate r adjusts to clear the market. Each investor in the economy has the same utility function (6.1) (so the constant α is the same for all investors), and her goal is to maximize the expectation of the utility of her final wealth. Since all investors have identical utility functions, w defined by (6.2) is the fraction of aggregate wealth invested in the market portfolio; the fraction invested in the risk-free asset is $1 - w$. As the net amount of the risk-free asset is 0, the equilibrium interest rate r must be such that $1 - w = 0$, i.e., $w = 1$. This gives the equilibrium relation

$$\mu = r + (1-\alpha)\sigma^2 \tag{6.3}$$

between μ, r, and σ. Our formula $\mu \approx r + \sigma^2$ is compatible with (6.3) only when $\alpha = 0$ (the case of logarithmic utility). Setting $\mu := r$ is compatible with (6.3) only when $\alpha = 1$. This corresponds to the risk-neutral probability measure (2.4) used in BSM option pricing.

It might seem that there is a contradiction between our results and the argument leading to (6.3). We do not longer need the assumption of Example 6.1 that the risk-free asset is inside money, and so we return to our framework in which r is constant. Suppose that (6.3) happens to be satisfied. Then the optimal fraction of the investor's wealth invested in the index is 1. How can this be optimal for $\alpha \neq 0$ and large T if, by our results, the investor can beat the index with high probability instead of investing in it?

Let us see that our strategy that beats the index with high probability does not in fact lead to a higher expected utility than the index. Until the end of this section we assume that (6.3) is satisfied. If $\alpha < 0$ (this is believed to be the practically interesting case), the explanation is trivial: since our strategy (one satisfying Proposition 2.1) loses all wealth with positive probability, its expected utility is $-\infty$. Therefore, we will consider the trading strategy Σ that is a weighted average of the index, with weight $1-\gamma$, and a strategy satisfying Proposition 2.1, with weight γ, where $\gamma \in (0,1)$. The specific values that we will discuss later are $\delta := 0.01$ and $\gamma := 0.1$; in this case Σ either underperforms the index by 10% (with a very low probability, for large T) or beats it by 990% (with probability close to 1).

For simplicity, we will assume zero interest rate, $r = 0$. Remember that

$$I_T = \mathrm{e}^{(\mu-\sigma^2/2)T+\sigma\sqrt{T}\xi} = \mathrm{e}^{(1/2-\alpha)\sigma^2 T+\sigma\sqrt{T}\xi},$$

where $\xi \sim N_{0,1}$.

We will calculate the expected utility of the final value I_T of the index and of the final wealth of Σ separately over the three regions implicit in (2.2); for now, we only assume $\alpha \in (-\infty, 1)$ and $\alpha \neq 0$.

1. Over the region $I_T \geq \mathrm{e}^{\sigma^2 T/2+z_{\delta/2}\sigma\sqrt{T}}$ we have:

   (a) by Lemma 2.3, the expected utility of the index is

$$\begin{aligned}
&\mathbb{E}\left(F_\alpha(I_T)\,\mathbf{1}\left\{I_T \geq \mathrm{e}^{\sigma^2 T/2+z_{\delta/2}\sigma\sqrt{T}}\right\}\right)\\
&= \frac{1}{\alpha}\,\mathrm{e}^{\alpha(1/2-\alpha)\sigma^2 T}\,\mathbb{E}\left(\mathrm{e}^{\alpha\sigma\sqrt{T}\xi}\,\mathbf{1}\left\{\xi \geq \alpha\sigma\sqrt{T}+z_{\delta/2}\right\}\right)\\
&= \frac{1}{\alpha}\,\mathrm{e}^{\alpha(1/2-\alpha)\sigma^2 T}\,\mathrm{e}^{\alpha^2\sigma^2 T/2}\,\mathbb{P}\left(\xi \geq z_{\delta/2}\right)\\
&= \frac{\delta}{2\alpha}\,\mathrm{e}^{\alpha(1-\alpha)\sigma^2 T/2};
\end{aligned}$$

   (b) replacing $F_\alpha(I_T)$ by $F_\alpha((1-\gamma+\gamma/\delta)I_T)$, we find the expected utility of the final wealth of Σ as

$$(1-\gamma+\gamma/\delta)^\alpha\,\frac{\delta}{2\alpha}\,\mathrm{e}^{\alpha(1-\alpha)\sigma^2 T/2}\,.$$

2. Over the region

$$\mathrm{e}^{\sigma^2 T/2-z_{\delta/2}\sigma\sqrt{T}} < I_T < \mathrm{e}^{\sigma^2 T/2+z_{\delta/2}\sigma\sqrt{T}} \qquad (6.4)$$

   (cf. (2.2)) we have:

(a) by Lemma 2.3, the expected utility of the index is

$$\begin{aligned}
&\mathbb{E}\left(F_\alpha(I_T)\,\mathbf{1}\left\{\mathrm{e}^{\sigma^2T/2-z_{\delta/2}\sigma\sqrt{T}}<I_T<\mathrm{e}^{\sigma^2T/2+z_{\delta/2}\sigma\sqrt{T}}\right\}\right)\\
&=\frac{1}{\alpha}\,\mathrm{e}^{\alpha(1/2-\alpha)\sigma^2T}\,\mathbb{E}\left(\mathrm{e}^{\alpha\sigma\sqrt{T}\xi}\,\mathbf{1}\left\{\alpha\sigma\sqrt{T}-z_{\delta/2}<\xi<\alpha\sigma\sqrt{T}+z_{\delta/2}\right\}\right)\\
&=\frac{1}{\alpha}\,\mathrm{e}^{\alpha(1/2-\alpha)\sigma^2T}\,\mathrm{e}^{\alpha^2\sigma^2T/2}\,\mathbb{P}\left(-z_{\delta/2}<\xi<z_{\delta/2}\right)\\
&=\frac{1-\delta}{\alpha}\,\mathrm{e}^{\alpha(1-\alpha)\sigma^2T/2};
\end{aligned}$$

(b) replacing $F_\alpha(I_T)$ by $F_\alpha((1-\gamma)I_T)$, we find the expected utility of the final wealth of Σ as

$$(1-\gamma)^\alpha\frac{1-\delta}{\alpha}\,\mathrm{e}^{\alpha(1-\alpha)\sigma^2T/2}.$$

3. Over the region $I_T\le\mathrm{e}^{\sigma^2T/2-z_{\delta/2}\sigma\sqrt{T}}$ we have:

(a) by Lemma 2.3, the expected utility of the index is

$$\begin{aligned}
&\mathbb{E}\left(F_\alpha(I_T)\,\mathbf{1}\left\{I_T\le\mathrm{e}^{\sigma^2T/2-z_{\delta/2}\sigma\sqrt{T}}\right\}\right)\\
&=\frac{1}{\alpha}\,\mathrm{e}^{\alpha(1/2-\alpha)\sigma^2T}\,\mathbb{E}\left(\mathrm{e}^{\alpha\sigma\sqrt{T}\xi}\,\mathbf{1}\left\{\xi\le\alpha\sigma\sqrt{T}-z_{\delta/2}\right\}\right)\\
&=\frac{1}{\alpha}\,\mathrm{e}^{\alpha(1/2-\alpha)\sigma^2T}\,\mathrm{e}^{\alpha^2\sigma^2T/2}\,\mathbb{P}\left(\xi\le-z_{\delta/2}\right)\\
&=\frac{\delta}{2\alpha}\,\mathrm{e}^{\alpha(1-\alpha)\sigma^2T/2};
\end{aligned}$$

(b) replacing $F_\alpha(I_T)$ by $F_\alpha((1-\gamma+\gamma/\delta)I_T)$, we find the expected utility of the final wealth of Σ as

$$(1-\gamma+\gamma/\delta)^\alpha\frac{\delta}{2\alpha}\,\mathrm{e}^{\alpha(1-\alpha)\sigma^2T/2}.$$

For regions 1 and 3 we obtained identical results.

If we divide the total expected utility of Σ over the three regions by the total expected utility of the index, the factor $\frac{1}{\alpha}\,\mathrm{e}^{\alpha(1-\alpha)\sigma^2T/2}$ will cancel out and we will obtain

$$\delta(1-\gamma+\gamma/\delta)^\alpha+(1-\delta)(1-\gamma)^\alpha. \tag{6.5}$$

Figure 1 plots this ratio for a range of αs and for $\delta=0.01$ and $\gamma=0.1$. The ratio is more than 1 when $\alpha<0$ and less than 1 when $\alpha\in(0,1)$. As the utility is negative when $\alpha<0$ and positive when $\alpha>0$, this means that the expected utility of Σ is lower than that of the index. The following lemma states this formally.

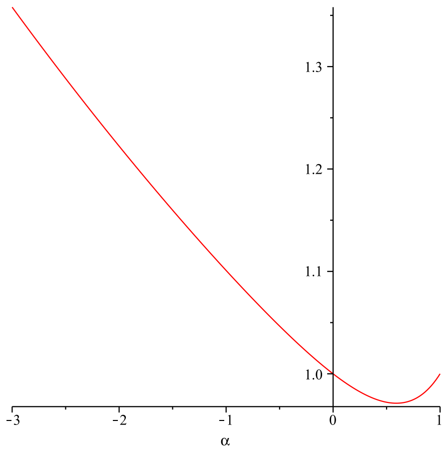

Figure 1: The ratio (6.5) of the expected utilities of Σ and of the index for values of α ranging from 1 (the risk-neutral case) via 0 (logarithmic utility) to -3 (relative risk aversion 4) for $\delta = 0.01$ and $\gamma = 0.1$.

Lemma 6.2. *For all $\gamma, \delta \in (0,1)$ and all $\alpha \in (-\infty, 1)$,*

$$\delta(1-\gamma+\gamma/\delta)^\alpha + (1-\delta)(1-\gamma)^\alpha \begin{cases} > 1 & \text{if } \alpha < 0 \\ < 1 & \text{if } \alpha > 0. \end{cases}$$

Proof. When $\gamma = 0$,

$$f(\gamma) := \delta(1-\gamma+\gamma/\delta)^\alpha + (1-\delta)(1-\gamma)^\alpha = 1.$$

It remains to check that the derivative $f'(\gamma)$, $\gamma > 0$, is positive when $\alpha < 0$ and negative when $\alpha \in (0,1)$. □

Let us first discuss the case $\alpha < 0$; it is believed that typical values of relative risk aversion for real-world investors are between 2 and 10, which corresponds to α between -1 and -9. Utility in this case is negative, and we will couch our discussion in terms of *disutility*, which we define as minus utility. For concreteness, let $\alpha := -2$ (so the relative risk aversion is 3). According to Figure 1, the disutility of Σ is approximately 1.2 times greater than the disutility of the index for $\delta = 0.01$, $\gamma = 0.1$ (a more accurate value is 1.222). This looks counterintuitive, but is explained by the property of F_{-2} (and generally F_α for $\alpha < 0$) to become "saturated". Let us assume that T is large. The utility function F_{-2} is bounded above, and even large differences in wealth (on absolute, relative, or any other scale) cease to matter as wealth increases. In the context of this article, beating the index by a factor of, say, 10 quickly becomes less and less impressive. For example, the utility of increasing wealth 10-fold is 100 more valuable when the current wealth is 1 than when it is 10. This appears to contradict the way institutional investors are usually evaluated: beating the index by a factor of 10 is as big an achievement today as it was 20 years ago.

In the context of the calculations of this section (for $\alpha = -2$), region 1 ($I_T \geq \mathrm{e}^{\sigma^2 T/2 + z_{\delta/2}\sigma\sqrt{T}}$) has a probability very close to 1 and in this case Σ beats the index by a factor of more than 10. But these typical values for the index are regarded as too large by the utility function, the difference between Σ and the index is essentially disregarded, and F_{-2} concentrates on region 2, (6.4). This is a very unlikely region and Σ loses only 10% as compared to the index; this is, however, sufficient for F_{-2} to punish Σ harshly. In region 3, Σ again beats the index by a factor of more than 10, but its probability is too small to change the outcome of the comparison.

In the case $\alpha \in (0, 1)$, the expected utility of the index is still greater than that of Σ. The utility function F_α, $\alpha \in (0, 1)$, prefers the index to Σ because Σ loses 10% as compared to the index in the same low-probability region (6.4); the 10-fold outperformance over the index outside this region is not sufficient to counterbalance the potential modest and low-probability underperformance.

It is interesting that in both cases, $\alpha < 0$ and $\alpha > 0$, the utility function F_α focuses on the same region, (6.4). A very bad performance outside this region is tolerated. For $\alpha = 0$ this region at least has a high probability; for other values of α, this is a surprising phenomenon.

The logarithmic utility function is often regarded to be the most fundamental utility function; e.g., maximizing expected logarithmic utility often leads to the best asymptotic growth rate: see, e.g., [6] (and see [12] for the analysis of Kelly's rule from the point of view of maximizing expected utility). This section (and article in general) can be regarded as another manifestation of the fundamental character of the logarithmic utility function. Other utility functions in the F_α family lead to decisions based on potential events (far from catastrophic) that have very low probability.

As a final informal remark, it is not clear that important players in stock markets, such as mutual and hedge funds, have goals that can be easily translated into utility functions. An important goal for them is to beat the market, and our results in the bulk of the article address this goal directly rather than via a conventional utility function.

Acknowledgments

The idea of this article originated in my attempts to understand Bodie's paradox [1], saying that it is expensive to insure against a shortfall of stock returns as compared to bond returns. I am grateful to Prof. Robert Merton for drawing my attention to Bodie's paper back in 2000 and for his recent comments resulting in Section 6. Thanks to Wouter Koolen for illuminating discussions. The data for the empirical studies in Section 4 have been provided by Yahoo! Finance and processed using R. Figure 1 was plotted in Maple™. This research has been supported in part by NWO Rubicon grant 680-50-1010.

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