CORRESPONDENCE
OF NICOLAS BERNOULLI WITH MONTMORT

NICOLAS BERNOULLI
1710–1713

Remarks of Mr. Nicolas Bernoulli
Appendix to the letter of Mr. Jean Bernoulli to Mr. Montmort
From Basel this 17 March 1710

Pg. 23, 24, 25.[1] In general, if $q > 2$, the prerogative of the one holding the cards is Pharaon

$$= \frac{1}{4} \times \frac{q}{p-q+1} - \frac{1}{8} \times \frac{q.q-1}{p-q+1.p-q+2} + \frac{1}{16} \times \frac{q.q-1.q-2}{p-q+1.p-q+2.p-q+3} - \dots$$

$$\text{all the way to } \pm \frac{1}{2^q} \times \frac{q.q-1.q-2\dots 2}{p-q+1.p-q+2\dots p-1}$$

in A.

Pag. 34[2] *seqq. Maintenant, &c.* This author supposes the game to be broken off with Peter ruined, which is against the laws of this game, for the game is continued, & the hand or privilege of distributing the cards is transfered to Paul, who is to the right of Peter; whence it follows, that gain not only must be added with the found gain, which comes forth out of the expectation of retaining the hand, but from that also the loss must be subtracted, which Peter has to be feared, if he will have lost the hand. Toward finding therefore the lot of Peter, let that be put $= x$ (N. B. by lot here I understand what by which out of the money of his adversary he is expecting) the lot of Jacob $= y$, & the lot of Paul $= z$ by which deposited there will be $x = \frac{3}{17}A + \frac{2351x+4024y}{6375}$, for because 2351 cases of retaining the hand, & 4024 cases that of losing, Peter will have besides the lot $\frac{3}{17}A$ previously found, thus far 2351 cases to be remaining in that state, in which it was of the game from the beginning, & 4024 cases to be acquiring y or the harmful lot of the gamester Jacob; further because the lots of Jacob & of Paul for one game only are $\frac{-106}{2125}A$ & $\frac{-269}{2125}A$, there will be $y = \frac{-106}{2125}A + \frac{2351y+4024z}{6375}$, & $z = \frac{-269}{2125}A + \frac{2351z+4024x}{6375}$, by which reduced equations & in addition with $x + y + z = 0$ put, there will be found $x = \frac{161}{1006}A$, $y = \frac{-481}{4024}A$, $z = \frac{-163}{4024}A$. Lansquenet

Note. In all games, which certainly must consist of the number of games l, & in which any whatsoever, who holds the hand, that if he loses to his neighbor is held to cede to the gamester at right, if the number of players A, B, C, D, &c. be $= p$, & the first A holds the hand, and B is to the left of A, C to the left of B, D to the left of C, &c. & has the cases of preserving the hand to the cases of losing that the ratio as m to n, the lots of the players A, B, C, D, &c. will be expressed z, y, x, u, &c. respectively by means of the following

Date: October 25, 2009.
[1]See page 97.
[2]See page 110.

series, certainly

$$
\begin{aligned}
z =& a + \frac{ma+nb}{s} + \frac{mma+2mnb+nnc}{ss} + \frac{m^3a+3mmnb+3mnnc+n^3d}{s^3} + \text{ \&c.}\\
y =& b + \frac{mb+nc}{s} + \frac{mmb+2mnc+nnd}{ss} + \frac{m^3b+3mmnc+3mnnd+n^3e}{s^3} + \text{ \&c.}\\
x =& c + \frac{mc+nd}{s} + \frac{mmc+2mnd+nne}{ss} + \frac{m^3c+3mmnd+3mnne+n^3f}{s^3} + \text{ \&c.}\\
u =& d + \frac{md+ne}{s} + \frac{mmd+2mne+nnf}{ss} + \frac{m^3d+3mmne+3mnnf+n^3g}{s^3} + \text{ \&c.}
\end{aligned}
$$

or again & thus again

$$
\begin{aligned}
z = aq \times \overline{1-r} &+ bq \times \overline{1-r\times\overline{1+\frac{t.l}{1}}} + cq \times \overline{1-r\times\overline{1+\frac{t.l}{1}+\frac{tt\cdot l.l-1}{1.2}}}\\
&+ dq \times \overline{1-r\times\overline{1+\times t.l1+\frac{tt\cdot l.l-1}{1.2}+\frac{t^3l.l-1\cdot l-1}{1.2.3}}} + \text{\&c.}\\
y = bq \times \overline{1-r} &+ cq \times \overline{1-r\times\overline{1+\frac{t.l}{1}}} + dq \times \overline{1-r\times\overline{1+\frac{t.l}{1}+\frac{tt\cdot l.l-1}{1.2}}}\\
&+ eq \times \overline{1-r\times\overline{1+\times t.l1+\frac{tt\cdot l.l-1}{1.2}+\frac{t^3l.l-1\cdot l-1}{1.2.3}}} + \text{\&c.}\\
x = cq \times \overline{1-r} &+ dq \times \overline{1-r\times\overline{1+\frac{t.l}{1}}} + eq \times \overline{1-r\times\overline{1+\frac{t.l}{1}+\frac{tt\cdot l.l-1}{1.2}}}\\
&+ fq \times \overline{1-r\times\overline{1+\times t.l1+\frac{tt\cdot l.l-1}{1.2}+\frac{t^3l.l-1\cdot l-1}{1.2.3}}} + \text{\&c.}\\
u = dq \times \overline{1-r} &+ eq \times \overline{1-r\times\overline{1+\frac{t.l}{1}}} + fq \times \overline{1-r\times\overline{1+\frac{t.l}{1}+\frac{tt\cdot l.l-1}{1.2}}}\\
&+ gq \times \overline{1-r\times\overline{1+\times t.l1+\frac{tt\cdot l.l-1}{1.2}+\frac{t^3l.l-1\cdot l-1}{1.2.3}}} + \text{\&c.}
\end{aligned}
$$

where in whichever series as many terms are summed, as are units in l; but there is $s = m+n$, $q = \frac{s}{n}$, $r = \frac{m^l}{s}$, $t = \frac{n}{m}$, $z + y + x + u +$ &c. $= 0$. If $p = 2$, now among the lots a, b, c, d, e, &c. as z, y, x, u, &c. always every other & every other are equals, as $a = c = e =$ &c. $b = d = d =$ &c. $z = x =$ &c. $y = u =$ &c. If $p = 3$, every third are equal, if $p = 4$ every four, & thus in succession.

$$
\left.\begin{array}{c} z=a\\ y=b\\ x=c\\ u=d\\ \text{\&c.}\end{array}\right\}\text{if } l=1,\quad
\left.\begin{array}{l}\text{If } l=\text{inf.}\\ \text{or the number}\\ \text{is great enough}\end{array}\right\}\text{will be }
\begin{array}{l}
z{=}y+aq{=}\frac{q}{p}\times p-1\times a+p-2\times b+p-3\times c+\cdots 0\\
y{=}x+bq{=}\frac{q}{p}\times p-1\times b+p-2\times c+p-3\times d+\cdots 0\\
x{=}u+cq{=}\frac{q}{p}\times p-1\times c+p-2\times d+p-3\times e+\cdots 0
\end{array}
$$

Treize *Page 58 On the game of Treize*.[3] Let the cards which Peter holds be designated by the letters $a, b, c, d, e,$ &c. of which the number is n, the number of all possible cases will be

[3]See page 134.

$= 1.2.3\ldots n$, the number of cases when a is in the first place

$$= 1.2.3\ldots n-1;$$

the number of cases when b is in the second, but a not in the first

$$= 1.2.3\ldots n-1 - 1.2.3\ldots n-2;$$

the number of cases when c is in the third place, yet neither a in the first nor b in the second

$$= 1.2.3\ldots n-1 - 2\times 1.2.3\ldots n-2 + 1.2.3\ldots n-3;$$

the number of cases when d is in the fourth, none indeed of the preceding in its place

$$= 1.2.3\ldots n-1 - 3\times 1.2.3\ldots n-2 + 3\times 1.2.3\ldots n-3 - 1.2.3\times n-4;$$

and generally, the number of cases, in which it is able to happen when the letter which is at rank m, but none of the preceding is in its place,

$$= 1.2.3\ldots n-1 - \frac{m-1}{1}\times 1.2.3\ldots n-2 + \frac{m-1.m-2}{1.2}\times 1.2.3\ldots n-3 - \frac{m-1.m-2.m-3}{1.2.3}\times 1.2.3\times n-4 + \ldots \text{up to } \pm\frac{m-1.m-2\ldots m-m+1}{1.2.3\ldots m-1}\times 1.2.3\ldots n-m$$

hence the risk of the player who in this letter finally, which is at rank m, wishes to win, is

$$\frac{1}{n} - \frac{m-1}{1}\times\frac{1}{n.n-1} + \frac{m-1.m-2}{1.2}\times\frac{1}{n.n-1.n-2} - \frac{m-1.m-2.m-3}{1.2.3}\times\frac{1}{n.n-1.n-2.n-3} + \ldots \text{up to } \pm\frac{m-1.m-2\ldots m-m+1}{1.2.\ldots m-1}\times\frac{1}{n.n-1\ldots n-m+1},$$

& the risk of the player who at least in the case of some m of the letters wishes to win $=$ the sum of all the possible preceding values of the series being put for m successively 1.2.3 &c. that is

$$\frac{m}{n} - \frac{m.m-1}{1.2}\times\frac{1}{n.n-1} + \frac{m.m-1.m-2}{1.2.3}\times\frac{1}{n.n-1.n-2} - \frac{m.m-1.m-2.m-3}{1.2.3.4}\times\frac{1}{n.n-1.n-2.n-3} + \ldots \text{up to } \pm\frac{m.m-1.m-2\ldots m-m+1}{1.2.\ldots m-1}\times\frac{1}{n.n-1\ldots n-m+1},$$

I put $m = n$ the risk of the player is

$$= 1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \ldots \text{ up to } \pm\frac{1}{1.2.3\ldots n}.$$

In another way. Either a is in first place, or it is not; if a is in first place, thereupon the risk is $= 1$, if it is not, thereupon he has as many chances to obtain 1, which were held if the number of letters were $n-1$, with this excepted case, in which it happens, when this letter, of which a entered the position, again is in first place, for these do not surrender 1 to him, but merely that expectation, which he had if the number of letters were $n-2$; however there are as many cases when this happens, as they admit variations of $n-2$ letters, certainly $1.2.3\ldots n-2$; hence putting the strength of him when the number of letters is $n-2 = d$, & g for the strength when the number of letters is $n-1$, there will be with the existing number of letters $= n-1$, out of the entire cases $1.2.3\ldots n-1$, $1.2.3\ldots n-1\times g$ winning cases (for he has the whole deposit or 1 to the value of the expectation the same ratio as the number of all cases to the number of winning cases) hence the expectation which he has if a not be in its place is

$$= \frac{\overline{1.2.3\dots n-1\times g - 1.2.3\times n-2} + 1.2.3\dots n-2d}{1.2.3\dots n-1} = \frac{\overline{n-1}\times g - 1 + d}{n-1},$$

since therefore out of n cases precisely one is when a is in first place, & $n-1$ cases when it is not, the obtained risk will be

$$= \frac{1\times 1 + n - 1\frac{\overline{n-1}\times g-1+d}{n-1}}{n} = \frac{n-1\times g+d}{n}.$$

Hence it appears the difference between the sought strength & the one which he has, if the number of letters is $n-1$, to be $= \frac{-g+d}{n}$ = difference between this same strength & the one, which he has if the number of letters is $n-2$, but supposing negative & dividing by the number of letters n, whence with the existing number of letters 0 & 1, furthermore the risk is 0 & 1, will be the difference between the strength if the number of letters is 2, & between the preceding strength, when certainly the number of letters is less by unity, $= -\frac{1}{2}$; if the number of letters be 3, $= +\frac{1}{2.3}$; if 4, $= -\frac{1}{2.3.4}$; if 5, $= +\frac{1}{2.3.4.5}$, & generally if the number of letters be $n = \pm\frac{1}{2.3.4\dots n}$, and even the total risk

$$= 1 - \frac{1}{2} + \frac{1}{2.3} - \frac{1}{2.3.4} + \dots \text{ up to } \pm\frac{1}{2.3.4\dots n}.$$

Bassette

Pag. 73[4] Another formula. If $q < 1$, the gain of the one holding the cards is

$$= -\frac{1}{3}\frac{q.p-q}{p.p-1} + \frac{1}{2}\times\frac{q}{p} - \frac{1}{4}\times\frac{q.q-1}{p.p-q+1} + \frac{1}{8}\times\frac{q.q-1.q-2}{p.p-q+1.p-q+2}$$
$$- \frac{1}{16}\times\frac{q.q-1.q-2.q-3}{p.p-q+1.p-q+2.p-q+3} + \cdots$$

always to $\pm\frac{1}{2}q - 1 \times \frac{q.q-1\dots 2}{p.p-q+1.p-q+2\dots p-2}$ in A. If $q=1$, you must add $\frac{1}{p}\times A$.

Pag. 74. In the Table, in the last case there is an error of calculation, for instance the gain of the one holding the cards when all 4 suits are hidden of all 52 inverted cards is not $\frac{2453842}{175592235}a$, but $\frac{454}{32487}a = \frac{2453870}{175592235}a$.

Letter of Nicolas Bernoulli to M. de Montmort
From Basel this 26 February 1711 (pages 308–314)

This is to thank you, Sir, for your very accommodating Letter, by which you have wished to assure me of your friendship & of your esteem, of which I infinitely indebted to you. My uncle, to whom his affairs hitherto have not permitted all the good things of which you have filled the Letter which you have taken the pain to write to him, has charged me to make it & to respond to you; by attending therefore of the leisure of it, I have hitherto the response that I owe you.

Treize

I have not yet attempted the general solution of the problem on the game of Treize, because it seems to me almost impossible; this is also why I was greatly astonished by that which you say, that you have found $\frac{690568237871898897}{241347817621535625}A$ for the advantage of the one who holds the cards; but in examining the thing a little more closely, I had the thought, that you perhaps have resolved generally this problem only under the supposition, that the one who holds the cards having won or lost, the game would conclude; that which confirms

[4]See page 153.

to me in this thought, is that I have found for this hypothesis a general formula, which applied to the particular case of 52 cards, gives for the advantage of the one who holds the cards this fraction $\frac{99177450342464537}{336245122781568000}A$ which is a little greater than yours, but which has for denominator a number composed of nearly the same factors as the one of yours, this which makes me believe that you have made an error of calculation in the application of your formula: here is mine of which I just spoke.

$$S = \frac{1}{1} - \frac{n-p}{1.2 \times n-1} + \frac{n-2p}{1.2.3 \times n-2} - \frac{n-3p}{1.2.3.4 \times n-3} + \&\text{c.}$$

up to the a term which is $= 0$; by p I intend the number of times that each different card is repeated, & by n the number of all cards. I have also calculated the case for 4 cards, of which you speak, & I have found $\frac{130225}{172279} = \frac{56908325}{75285923}$ as you; but it is apropos to observe here, that according to the rules of this game there, it is not necessary to suppose that the game is complete, when the one who has the hand just loses, because then he is obligated to cede the hand to another, & the game continues; this is why the advantage of the one who holds the cards being diminished by the disadvantage that he had in losing the hand, will be in the aforesaid case only $\frac{130225}{344558}$ the half of that which had been found. If one assumes that there were many players against the one who has the hand, & that their number is $= n$, his advantage will be $\frac{130225}{344558} \times n$, & the one of the other players $\frac{130225}{344558} \times$either $n-2$, or $n-4$, or $n-6$, &c. according to the rank that each occupies by relation to the right of the one who holds the card. This remark extends itself on all of the players in which the hand passes from one to the other; also in your first case of Lansquenet I have found that the advantage of Pierre is only $\frac{161}{1006}A$, the disadvantage of Paul — $\frac{163}{4024}A$, & the one of Jacques $\frac{481}{4024}A$.

The formula which you have found for proposition 31[5] is quite correct & very useful for the usage. I have found the same although under another expression by the method of combinations.

Duration of play

The Problem that you propose on the game which is played in many games by reducing is quite difficult; nonetheless seeing that you wished that I find a solution of it, I have applied myself, & I have found a general rule in order to express the lot of the one who would wager that one of the Players will have won in such number of trials as we will wish, be that they play in one equal or unequal game, be that one has already won some games or none: here it is in words. Let the two Players be Pierre & Paul, the number of parts which are lacking to Pierre$= m$, the number of parts which are lacking to Paul$= n$, their sum$= m + n = s$, the number of cases favorable to Pierre$= p$, the number of cases favorable to Paul$= q$, their sum $= p + q = r$, the number of trials $= h = m + 2k$, the number of times that s is contained in $k = t$; this put, I say that the difference between the sum of all the possible values (that is to say by putting for t all the values which it can have from 0 to the greatest) of this series

$$\begin{aligned}
&1 \times (p^{2k-2ts} + q^{2k-2ts}) + h \times (p^{2k-2ts-1}q + q^{2k-2ts-1}p) \\
&\quad + \frac{h.h-1}{1.2} \times (p^{2k-2ts-2}qq + q^{2k-2ts-2}pp) \\
&\quad + \frac{h.h-1.h-2}{1.2.3} \times (p^{2k-2ts-3}q^3 + q^{2k-2ts-3}p^3) \\
&\quad + \&\text{c. to } \frac{h.h-1.h-2\cdots h-k+ts+1}{1.2.3.4\cdots k-ts} \times pq^{k-ts}
\end{aligned}$$

[5]See page 46.

the whole multiplied by $\frac{p^{ts+m}q^{ts}}{r^h}$, & the sum of all these values of this here

$$\begin{aligned} &1 \times (p^{2k-2ts-2n} + q^{2k-2ts-2n}) \\ &\quad + h \times (p^{2k-2ts-2n-1}q + q^{2k-2ts-2n-1}p) \\ &\quad + \frac{h.h-1}{1.2} \times (p^{2k-2ts-2n-2}qq + q^{2k-2ts-2n-2}pp) \\ &\quad + \text{\&c. to } \frac{h.h-1.h-2 \cdots h-k+ts++n+1}{1.2.3.4 \cdots k-ts-n} \times pq^{k-ts-n}, \end{aligned}$$

the whole multiplied by $\frac{p^{ts+s}q^{ts+n}}{r^h}$, will express the lot of the one who would wager that Pierre will win the game in at least h trials. If k is smaller than $ts+n$; that is to say, if after having divided k by s, the rest of the division is smaller than n, it is not necessary in the last series to put for t all the values from 0 to t, but only to $t-1$. In order to have the lot of the one who would wager that Paul will win it in h trials, it will be necessary only to substitute into this formula the letters $q,\ p,\ n,\ m,$ in place of p, q, m, n. The sum of these two lots together will be the lot of the one who would wager that the game will be decided in h trials. The application of this formula to some particular cases, when $p = q = 1$, is quite easy; I have found not more than you, since without calculation, that for six games the lot of the one who would wager that the game will be ended in 26 trials will be $\frac{16607955}{33554432}$, & in 28 trials $\frac{35485125}{67108864}$, or $\frac{7090250}{134217738}$, & not $\frac{70970250}{133432831}$, as you have written in error; but for twelve games I have found that we can already wager with advantage that the game will be ended in 110, & it would be disadvantageous to wager that it will end in 108 trials; because the lot for these two numbers of trials will be $\frac{329\cdots}{649\cdots}$ & $\frac{810\cdots}{1622\cdots}$,[6] it must be therefore that you yourself are mistaken, since you say that we can wager with advantage when the game will be decided only in 124 trials. It must be however to confess that it is necessary by groping in order to find when the strength will be $\frac{1}{2}$; this is why if you have a better method than that here, I pray you to communicate it to me, & I will be much obliged to you. It is clear that this formula, which I just gave, will serve also to find the lot of the same Players; because for this end it will be necessary only to suppose that the number of trials is infinite, by putting therefore $h,\ k,$ & $l = $*inf.* we will find that the lot of Pierre will be

$$= \frac{(p+q)^h \times p^s - p^{s-n}q^n}{r^h \times p^s - q^s} = \frac{p^s - p^m q^n}{p^s - q^s};$$

& consequently that of Paul $\frac{p^m q^n - q^s}{p^s - q^s}$, which I have found formerly by a different way from that which I have followed in the research on this Problem. If $m = n$, & $s = 2m$, their lots will be as $p^{2m} - p^m q^m$ & $p^m q^m - q^{2m}$, or as p^m & q^m; & by supposing $m = 12$, $p = 9$, $q = 5$, we will have 9^{12} & 5^{12} for the strengths of Pierre & Paul, which is the case of the fifth Problem of Huygens. If $p = q$, the lots of the two Players are as n & m, which is found easily by dividing $p^s - p^m q^n$, & $p^m q^n - q^s$ by $p - q$; because we will have by this division two geometric progressions, of which the number of terms of the 1st will be $= n$, & that of the 2nd $= m$, & of which the terms, by supposing $p = q$, will become all equal. If $p = q$, & $s = m + m = 12$, we have the case of page 178[7] This refers to the first

[6]*Translator's note*. These values are correct. Indeed, the exact probability that the game terminate in 110 trials is $\frac{329756296122611431546168042626736}{649037107316853453566312041152512}$ and the exact probability that the game terminate in 108 trials is $\frac{81057262276448668848223046732461}{162259276829213363391578010288128}$. The first quotient is approximately 0.5080700200 and the second is 0.4995539476.

[7]See page 277.

edition. of your Book. Your formula in order to find how many trials there are in order to bring forth precisely a certain number of points with a certain number of dice is quite correct; as also the method which you give in order to find the sum of the figurate numbers raised to any powers; my late uncle has given the same rule in his Treatise, not only for the figurate numbers to any exponent; but generally for all the numbers which are similar to the figurate numbers, that is to say, which have the first, or the second, or the third, & equal differences; beyond this method, there are again others, of which here is one which has been found some time ago by my living uncle; it consists in the assumption of a series of terms affected of indeterminate coefficients; for example, if one would wish to have the sum of the triangular numbers squared, that is to say the sum of all the $\frac{p.p+1}{1.2} \times \frac{p.p+1}{1.2}$ or of all the $\frac{1}{4}p^4 + \frac{1}{2}p^3 + \frac{1}{4}pp$, I suppose it equal to $ap^5 + bp^4 + cp^3 + dpp + ep + f$. In order to determine the unknown coefficients, I put in these two expressions $p + 1$ instead of p, & I will have

Powers of figurate numbers

$$\left.\begin{array}{l} ap^5 + 5ap^4 + 10ap^3 + 10app + 5ap + a \\ + bp^4 + 4bp^3 + 6bpp + 4bp + b \\ + cp^3 + 3cpp + 3cp + c \\ + dpp + 2dp + d \\ + ep + e \\ + f \end{array}\right\} = \text{to}$$

$\frac{1}{4}\overline{p+1}^4 + \frac{1}{2}\overline{p+1}^3 + \frac{1}{4}\overline{p+1}^2 +$ the sum of all the

$$\begin{aligned} \frac{1}{4}p^4 + \frac{1}{2}p^3 + \frac{1}{4}pp = \frac{1}{4}p^4 + \frac{3}{2}p^3 + \frac{13}{4}pp + 3p + 1 \\ + ap^5 + bp^4 + cp^3 + dpp + ep + f, \end{aligned}$$

by subtracting on both sides

$$ap^5 + bp^4 + cp^3 + dpp + ep + f;$$

& by comparing next the homogeneous terms, we will find

$$a = \frac{1}{20},\ b = \frac{1}{4},\ c = \frac{5}{12},\ d = \frac{1}{4},\ e = \frac{1}{30},\ f = 0;$$

therefore the formula for the sum of the triangular numbers squared will be

$$\frac{1}{20}p^5 + \frac{1}{4}p^4 + \frac{5}{12}p^3 + \frac{1}{4}pp + \frac{1}{30}p = \frac{3p^5 + 15p^4 + 25p^3 + 15pp + 2p}{3.4.5},$$

as you have found. One can also find the sum of such numbers by reducing them to the figurate numbers; for example,

$$\begin{aligned} \frac{p.p+1}{1.2} \times \frac{p.p+1}{1.2} &= \frac{1}{4}p^4 + \frac{1}{2}p^3 + \frac{1}{4}pp \\ &= 6 \times \frac{p.p+1.p+2.p+3}{1.2.3.4} - 6 \times \frac{p.p+1.p+2}{1.2.3} + \frac{p.p+1}{1.2}; \end{aligned}$$

therefore the sum of all the $\frac{\overline{p.p+1}^2}{1.2}$ will be

$$= 6\int \frac{p.p+1.p+2.p+3}{1.2.3.4} - 6\int \frac{p.p+1.p+2}{1.2.3} + \int \frac{p.p+1}{1.2}$$
$$= 6 \times \frac{p.p+1.p+2.p+3.p+4}{1.2.3.4.5} - 6 \times \frac{p.p+1.p+2.p+3}{1.2.3.4} + \frac{p.p+1.p+2}{1.2.3}$$
$$= \frac{3p^5 + 15p^4 + 25p^3 + 15pp + 2p}{3.4.5},$$

as above.

Lottery of Lorraine

The problem which you have had plan to propose to the Geometers has no difficulty: here is how I have concluded the thing. The question is to find how often the condition of rendering their 25 livres to those who having taken 50 tickets would have won no lot in their 50 tickets, gives advantage or disadvantage to the one who holds the Lottery, that which is the same thing as if we wished to seek the lot of the one who would undertake to bring forth with 20000 dice with 1000000 faces, of which 50 alone are marked with some points, in a single trial at least one of the marked faces; now the number of cases that this will not happen is 999950^{20000} & the number of all the cases is 1000000^{20000}; whence it follows that this condition to render the silver to each of those who win no lot in their 50 tickets, is worth $\left(\frac{999950}{1000000}\right)^{20000} \times 25$ livres that which makes in all $\left(\frac{999950}{1000000}\right)^{20000} \times 500000$ livres=(that which is found by logarithms) 184064 livres. & about ten sols. Therefore the disadvantage of the Banker, who retains only 75000 livres will be $109064\frac{1}{2}$ livres so that he must not be amazed at all if the one who has held one such Lottery has been bankrupted. One can by this same method & by two words resolve proposition 44 of your Book.[8]

Printing of the *Ars Conjectandi*

Here is, Sir, that which I have found necessary to write to you on these matters, one other time when I will have more leisure, I will take the pleasure to examine some other curious things of your work. For that which regards the Treatise of my late Uncle, I have proposed to offer it as you have made to publish this manuscript to my Cousin the brother of the deceased, who is the master of it. I have also written over there to Mr. Herman, & I have prayed him to take care that this manuscript is soon printed; but I have not at all yet received a response. It is a great pity that the fourth part of this Treatise, which must be the principal, was not at all achieved; it is but scarcely begun, & contains only five chapters, in which there are only some general things: that which is the most remarkable of it is the last chapter, where he gives the solution of a quite curious Problem, which he has preferred even to the quadrature of the circle, this is to find how many observations it is necessary to make in order to attain to such degree of probability as we would wish, & where he demonstrates at the same time that by the observations often reiterated we can discover strongly to the correct the ratio that there is among the number of cases where a certain event will happen, & the number of cases where it will not happen. It would be wished that some one would wish to undertake to finish this last part, & to treat at foundation the things of politics & of morals; & as I know no person who is more capable to succeed to it as you, Sir, who have givne some proofs so excellent in your Work, I pay you to motivate the views that you have on this matter, you oblige much the Public, & particularly me who is with much respect & esteem,

Sir,

Your very humble & very obedient Servant,
N. Bernoulli.

[8]See page 228.

Letter of Mr. de Montmort to Mr. Nicolas Bernoulli (pag. 315–323)

At Montmort 10 April 1711.

I cannot express to you, Sir, how much I am obliged to you with the complacence that you have had to work on the matters which are contained in the letter that you have given me the honor to write me. I will have well also to congratulate you on so many beauties of which it is filled; but I know that the Philosophers do not love the praises, & especially those which merit them as much as you.

It is necessary, Sir, that you have badly copied your general formula on Treize, for I am not able to find my count: here is mine. Let n be the number of cards, p the number of times that each different card is repeated; it is also $\frac{n}{p} = m$ & $n - m = q$, one will have the sought lot $= p^m - mp^{m-1} \times q + 1 + \frac{m \cdot m-1}{1.2} p^{m-2} \times q + 1.q + 2 - \frac{m \cdot m-1 \cdot m-2}{1.2.3} p^{m-3} \times q + 1.q + 2.q + 3 + \frac{m \cdot m-1 \cdot m-2 \cdot m-3}{1.2.3.4} \times q + 1.q + 2.q + 3.q + 4 -$ &c the whole divided by as many products of the numbers $n - 1.n - 2.n - 3.n - 4$ &c as there are units in $\frac{n}{p}$. Treize

Note, 1°. That it is necessary to take as many terms of this sequence as m expresses units. 2°. That it is necessary to change all the signs of this sequence when m is an even number.

Thus I find that the lot of the one who holds the cards at the beginning of the game is $\frac{310404641408725522}{241347817621535625}$, & his advantage $\frac{69056823787189897}{241347817621535625}$. I do not believe that there is an error in calculation; but surely there is none in the method.

I admire your formula for the duration of the games that we play by reducing; I sense that it is quite correct, but I am forced to say to you that I do not understand it. You have given me great pleasure, for me to facilitate the understanding of it, by making application of it in an example: for example, in the one where we play to six games, & where we find that there is advantage to wager that it will endure less than 28 trials. It is true that I deceived myself in the denominator, you have also deceived yourself inadvertently, it must be 134217728, & not 13421738: these kinds of errors slip in quite easily, when we are tired from a long calculation. I begin to doubt in any case as you that we can wager with advantage that playing to eleven games the game will end in 124 games, & not in 122. I have made that which I have been able to recall my ideas on this Problem which is assuredly quite difficult & quite abstract. I have not been able to find the papers where the demonstrations of these Problems are figured, & I believe that they are in Paris: immediately as I will be there I will do for you part of that which I have found on this matter. I will say to you only that we both have followed a quite different path, which you will understand quite easily, Sir, when you know that this number 70970250 is the sum of these six 34597290, 20030010, 10015005, 4292145, 1560780, 475020, which are the 7, the 8, the 9, the 10, the 11 & the 12th terms of the 30th perpendicular band. I find in a Book where I have put formerly some remarks that the odds are $\frac{35103333817}{2 \times 24359738368}$ that playing to 7 games the game will be ended in 37 games at least,[9] & $\frac{8338160273}{2 \times 8589934592}$ that it will be ended in less than 35,[10] which shows that there is advantage in 37, & disadvantage in 35. You can verify by your formula if this calculation is correct: besides your formula amazes me for its generality; I see that you draw from it the best as can be the fifth Problem of M. Huygens, & that of page 178.[11] It is nearly two months that I have sent to Paris my solution Duration of play

[9]*Translator's note*. The text says "au moins" but the probability given is that the contest terminate in at most 37 games. The value is also incorrect and should be $\frac{35102333827}{68719476736}$ which is approximately 0.5108.

[10]*Translator's note*. The text says "en moins de." However, this makes no sense in light of the fact that the probability the contest terminates in 35 games is actually $\frac{8338160273}{17179869184}$, which is approximately 0.4853.

[11]See page 277.

of the Problem in order to find the sum of the figurate numbers raised to any exponent, it has been sent in the Journal de France 23 March of this year.

Lottery of Lorraine

The Anagram which I give for the solution of the Problem which I propose on the Lottery of Loraine, contains these words *20000 moins un divisé par 20000 élevé à l'exposant 20000*, that which gives a solution conformed to yours. This problem has always appeared to me more curious than difficult; nonetheless its difficulty is such to my opinion, that it can stop some persons who could be not at all as you & Mr. your Uncle some Geometers of the first order, & capables of the greatest things: Many Geometers of my friends have worked uselessly. Besides the solution of this Problem is only a particular case of the formula which I have sent to Mr. your Uncle in my last Letter

$$\overline{m-1}^{p-q} \times p \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \frac{p-3}{4} \text{ \&c. divided by } m^p,$$

but beyond that one has not yet much thought to these sorts of Problems of combinations, it was necessary to be advised to reduce the Problem of the Lottery to a question of dice.

Treize

You say, Sir, that you have calculated the case for four cards, *page 64*,[12] & that you have found as I $\frac{130225}{172279}$; but you have added that according to the rules of this game it is not necessary to suppose that the game is finite, when the one who has the hand comes to lose; for then, say you, he is obligated to cede the hand to the other. This is why the advantage of the one who holds the cards being diminished by the disadvantage that he has in losing the hand, will be only $\frac{130225}{344558} \times n$, & the one of the other Players $\frac{130225}{344558} \times n-2$, $\frac{130225}{344558} \times n-4$ &c. according as the rank that each occupies. You extend next this remark onto Lanquenet, & it seems that you arrange in series of opinion of applying it to all sorts of games. For me I believe to have some reasons to think otherwise: I am going to expose them to you. Firstly, in regard to Treize, it is certain that the one who quits the hand is not at all obliged to continue to play, & besides he is not obliged to put the same sum into the game; on the contrary it happens that in this game those who have themselves noticed, how easy it is perceived by practice, that the advantage is for the one who holds the cards, they hold all when they have the hand, & they put little silver into the game when they do not have the hand. There is yet to remark that in this game the stakes increase or diminish without ceasing as well as the number of the Players. So that in my opinion one is able to say nothing useful & certain on these games, that by taking the part to determine at each coup the advantage or the disadvantage of the one who holds the hand with respect to a determined number of stakes of the Players. If I have made enter into Lansquenet the consideration of expectation that the one who holds the cards has to make the hand, this has been only by elegance, for in the fund it is just only by supposing that the number of Players will always be the same as much as Pierre will have the hand, this which is uncertain. It suffices it seems to me in order to be instructed, as perfectly as it is possible, of the hazards of these games, for example of Lansquenet, to know that with respect to such numbers of Players & of stakes there is so much advantage & disadvantage for each of the Players, according as the different places that they occupy.

Here is, Sir, that which I believe must oppose to your Remarks & to those of Mr. your Uncle has already made on this subject. If you find that they permit some reply, you will give me pleasure to caution myself of it. Dealing with Lansquenet, one one my friends has made me observe that it would be quite possible to have some cases in Lansquenet where the one who is to the left of Pierre would have the advantage. This suspicion would appear well-founded, & I would have wished to study it more thoroughly for the case of

[12]See page 143

Waldegrave

five or six players; but the length of the calculation has turned me away until now. This same Geometer[13] who is a Gentleman of much intellect, has proposed to me lately & has resolved a quite pleasing Problem which is here. *Pierre, Paul & Jacques play a pool at Trictrac or at Piquet. After one has deduced whom will play it is found that Pierre & Paul begin. We demand, 1°. What is the advantage of Jacques. 2°. How great are the odds that Pierre or Paul will win rather than Jacques. 3°. How many games must the pool naturally endure.*

Pool

As you do not know perhaps what it is to play a pool, I am going to explain it to you, nothing is more simple. If Pierre wins, Jacques will enter in the place of Paul & will put an écu into the pool; then if Pierre wins, the pool is ended, & Pierre wins two écus. If Jacques wins, Paul enters in the place of Pierre. In a word the one who enters always puts an écu into the game, & the one who wins two games in sequence takes away all that which is in the pool. If there were four Players, it would be necessary to win three games in sequence; & four if there were five Players. I have found that to three Players the advantage of Jacques, naming a the stake of each Player, was contained in this series

$$\frac{3}{2^2}a + \frac{5a}{2^5} + \frac{7a}{2^8} + \frac{9a}{2^{11}} + \frac{11a}{2^{14}} + \&c.$$
$$-\frac{a}{2} - \frac{a}{2^3} - \frac{2a}{2^4} - \frac{2a}{2^6} - \frac{3a}{2^7} - \frac{3a}{2^9} - \frac{4a}{2^{10}} - \frac{4a}{2^{12}} - \frac{5a}{2^{13}} - \&c.$$

that which is reduced to this simpler series;

$$\frac{a}{2^3} + \frac{\text{zero}}{2^6} - \frac{a}{2^9} - \frac{2a}{2^{12}} - \frac{3a}{2^{15}} - \frac{4a}{2^{18}} - \&c.$$
$$= \frac{a}{8} - \frac{1}{8^2} \times \frac{a}{8} + \frac{2a}{8^2} + \frac{3a}{8^3} + \frac{4a}{8^4} + \&c.$$
$$= \frac{a}{8} - \frac{a}{8^2} \times \overline{m + 2mm + 3m^3 + 4m^4 + 5m^5},$$

supposing $m = \frac{1}{8}$ in it. Now in order to find the sum of this series $m + 2mm + 3m^3 + 4m^4$+&c. where the coefficients & the exponents are in arithmetic progression, I observe that

$$\begin{aligned} \tfrac{m}{1-m} &= m + mm + m^3 + m^4 + m^5 \\ \tfrac{mm}{1-m} &= \quad mm + m^3 + m^4 + m^5 \\ \tfrac{m^3}{1-m} &= \quad m^3 + m^4 + m^5 \\ \tfrac{m^4}{1-m} &= \quad m^4 + m^5 \end{aligned}$$

Whence I conclude that the sought sum is equal to this one

$$= \frac{m}{1-m} + \frac{mm}{1-m} + \frac{m^3}{1-m} + \frac{m^4}{1-m}\&c. = \frac{m}{\overline{1-m}^2},$$

and consequently the advantage of Jacques $\frac{6}{49}$. I have further found that although there is the advantage for Jacques, there are odds five against 4 that Pierre will win the pool rather than Jacques.

If we wish to know how long the pool will endure among three Players, we will find that there are odds three against one that it will endure no more than three games, 7 against 1, 15 against 1, 31 against 1, that it will not endure more than 5, 7, 9, games; I have similarly sought how long the pool would endure among four Players, & I have found this sequence

$$\frac{1}{4}, \frac{3}{8}, \frac{8}{16}, \frac{19}{32}, \frac{43}{64}, \frac{94}{128}, \frac{201}{256}, \frac{423}{512}, \frac{880}{1024}, \frac{1815}{2048}, \frac{3719}{4096}, \&c.$$

[13] Mr. Waldegrave.

of which the sequence was not easy to notice. I have wished to seek the lot of the Players when there are four, & also how long the pool will endure when there are five or six Players; but this has appeared to me too difficult, or rather I have lacked courage, because I would be sure to succeed to it.

I have worked over some days to resolve this Problem, drawing from a deck of cards a certain number of cards at will; namely in how many ways one is able to bring forth a certain point. This Problem has much relation with the one of the dice on page 141,[14] of which I have sent you the solution; but it has some particular difficulties, & I have been able to come to the end of it only be supposing that there are neither jacks, nor queens, nor kings.

Traité du Jeu

One has sent me recently from Paris a Book which has for title, *Traité du Jeu*,[15] it is a Book on morals. The Author appears judicious & writes well; but in the places where he speaks of the usage of Geometry in order to determine the hazards of Games, it appears to me that he is deceived: Here is an example of it. The Author cites as an evident thing, that a Player who plays two coups against another one, must set into the game two against one; however it is certain that this is false. If one plays with one die of which the number of faces is p, I have found & you will find very easily that the advantage of a Player, who playing two coups against one, wagers only 2 against one, is

$$\frac{\overline{p-1}\times pp-\frac{1}{2}\times\overline{2\times\overline{p-1}^{3}+3\times\overline{p-1}^{2}+p-1}}{p^{3}}=\frac{p-1}{2pp},$$

this which shows that the advantage diminishes according as the number of faces is greater; but that there is always advantage. Would one be able to say that in *petit palet* or in *franc du carreau*, this advantage would be null because of the divisibility of matter to infinity?

Her

Tas

I have undertaken since some time to achieve the solution of the Problems that I propose at the end of my Book; I find that in Her, when there remains no more than two Players Pierre & Paul, the advantage of Paul is greater than $\frac{1}{85}$, & less than $\frac{1}{84}$. This Problem has some difficulties of a singular nature. I have begun also the Problem of Tas, & I have found that when the Tas are only two cards, & when the cards are only two aces, two deuce, two threes, two fours, &c the loss of the one who wagers to make the Tas is expressed by the formula $\frac{p-1}{2p-1}$, I call p the number of the Tas. The difficulty will be much greater under the ordinary assumption of four aces, four deuces, four threes, &c. & of the Tas composed of four cards. It is time to end this Letter. The pleasure that I find in undertaking with you on these matters carries me too far, & I must fear to annoy you. I pray to you, Sir, to assure Mr. your Uncle of the perfect veneration that I have for him, & to believe me, Sir, with an infinite esteem,

Your, &c.

Lottery of Lorraine

Postscript. I send you the Memoir that I have given in the Journal of France on the manner to find the sum of the numbers which are a constant difference. The method of Mr. your Uncle in order to find the figured numbers of which he has pleased to make me part is very beautiful & very different from mine. This manner to employ the undetermined coefficients of which Mr. Descartes is the inventor, has been worth to us nearly all the great discoveries which have been made in Geometry; but the application of it is often difficult, & it has yet been employed only by the great Masters. I propose to the Geometers the solution of the Problem on the Lottery of Lorraine. I invite you, Sir, to render public that

[14] See page 46.

[15] Jean Barbeyrac, published 1709.

which you have found. As there remains nearly no more copies of my Book, I believe that I will have to give soon a new edition of it. When I will be determined, I will demand permission from you, & to Mr. your Uncle, to insert your good Letters which will be the principle ornament of it. One counsels me to change the order & form of it, & to reassemble in the first Part all the Theory of Combinations. I have similar design to give the demonstrations of quantities of propositions & of difficult solutions that I have omitted by design in the first edition. You obligate me much, Sir, to give me your opinion on this subject.

It is not it seems to me of these demonstrations as of the demonstrations of Geometry, those touching the numbers & combinations are infinitely more embarrassing, & one is able to have them very sharply in the mind with being able to set them on paper. You arrange in series content, for example, of the demonstration which is for proposition 14, page 97.[16] You give me too much honor, Sir, to believe me capable of fulfilling the views that the late Mr. your Uncle had, to treat by Geometry the things of politics & morals. For me the more I touch & the more I recognize my insufficiency in this regard: I have some ideas & some materials, but it is yet mere trifles. The concern is to discover the truths of practice & in the usage of civil society. It is necessary to be based on some exact & well established hypotheses, to conserve especially this exactitude of which the Geometers are stung more than the rest of men, all that demands a strong head & a very great work. I have read lately a quite beautiful morsel of Mr. your Uncle in the Memoires de l'Academie de Berlin. I am astonished to see the Journals of Leipzig so stripped of morsels of Mathematics: they owe their reputation in part to the excellent Memoirs that Messers your Uncles sent often: the Geometers no longer find five or six years the same riches as othertimes, make some reproaches to Mr. your Uncle, & permit me to make of your also, *Luceat lux vestra coram hominibus.*[17] I am, &c.

Letter of Mr. Nicolas Bernoulli to Mr. de Montmort (pag. 323–337)

At Basel this 10 November 1711.

Sir,

I am totally confused to have such a long time kept silence, & I know not nearly how you can excuse me; I will say to you only that I am not able rather to satisfy to the desire as I will have to respond to all the points of your last Letter, & to resolve the Problems which you propose to me, because of other studies & affairs, which often interrupt my calculations, not leaving any time which was necessary to me in order to apply myself to our matters. But wishing finally to acquit myself of my debt, I have resolved to give resignation for a little to other studies, & to break at this hour this annoying silence, which I pray you to pardon me, by promising you what I will try in the future to be more exact & more regular. Here is therefore, Sir, my response that I will make also short as it will be possible.

You have reason to say that you have not found your count in my formula for Treize, because an error is slipped there; it is necessary to put Treize

$$S = \frac{1}{1} - \frac{n-p}{1.2.n-1} + \frac{n-p.n-2p}{1.2.3.n-1.n-2} - \frac{n-p.n-2p.n-3p}{1.2.3.4.n-1.n-2.n-3} + \&c.$$

[16]See page 44.

[17]Let your light so shine among men. Matthew, 5:16.

instead of
$$S = \frac{1}{1} - \frac{n-p}{1.2.n-1} + \frac{n-2p}{1.2.3.n-2} - \frac{n-3p}{1.2.3.4.n-3} + \text{\&c.}$$
This error, to that which I myself can remember, comes from that which by making the calculation I have put on the table on these last factors of the terms of each fraction, in order to indicate the law of the progression which there is among the terms of this series; whence it happens that next no more remembering the true solution, I have allowed to escape the other factors. You will see that this formula thus corrected will agree exactly with yours. The number
$$\frac{69056823787189897}{241347817621535625}$$
which you give for the case $n = 52$ & $p = 4$ is not yet correct, it is necessary according to your formula & mine
$$\frac{69056823706869897}{241347817621535625} = \frac{7672980411874433}{26816424180170625}.$$
The method of which I am being served in order to find this formula is the same as that of which I was being served once in my Latin Remarks for the resolution of the particular case of $p = 1$.

Duration of Play

I am displeased that the series
$$\begin{aligned} &1 \times (p^{2k-2ts} + q^{2k-2ts}) \\ &\quad + h \times (p^{2k-2ts-1}q + q^{2k-2ts-1}p) \\ &\quad + \frac{h.h-1}{1.2} \times (p^{2k-2ts-2}qq + q^{2k-2ts-2}pp) \\ &\quad + \frac{h.h-1.h-2}{1.2.3} \times (p^{2k-2ts-3}q^3 + q^{2k-2ts-3}p^3) + \text{\&c.} \\ &\quad \text{to } \frac{h.h-1.h-2\cdots h-k+ts+1}{1.2.3.4\cdots k-ts} \times pq^{k-ts} \times \frac{p^{ts+m}q^{ts}}{r^h}, \end{aligned}$$
&
$$\begin{aligned} &1 \times (p^{2k-2ts-2n} + q^{2k-2ts-2n}) \\ &\quad + h \times (p^{2k-2ts-2n-1}q + q^{2k-2ts-2n-1}p) \\ &\quad + \frac{h.h-1}{1.2} \times (p^{2k-2ts-2n-2}qq + q^{2k-2ts-2n-2}pp) + \text{\&c.} \\ &\quad \text{to } \frac{h.h-1.h-2\cdots h-k+ts++n+1}{1.2.3.4\cdots k-ts-n} \times pq^{k-ts-n} \times \frac{p^{ts+s}q^{ts+n}}{r^h} \end{aligned}$$
that I have given in order to determine the duration of the games that we play by reducing, has not been sufficiently intelligible to you. It is in these sorts of matters sometimes difficult to make them well understood, especially when we do not take care to avoid all the ambiguities that can be encountered, as I believe that it has happened to me; because it seems to me that the cause for what you have not understood by me, consists only in that which I have said, that it is necessary to put for t all the values that it can have from 0 to the greatest, in which there is a little ambiguity that I could have avoided by putting in the formula for t one letter, for example v, & by saying that in the application it is necessary to put for v successively $0,\ 1,\ 2,\ 3,\ 4,$ &c. to t, which expresses the number of times that s is contained in k. In order to give it to you in greater clarity, I am going to demonstrate how I have deduced from these series a general rule for the games which are played in an

equal game, to which I will apply next in some particular cases of seven games. It is clear that when $p = q = 1$, & $r = p + q = 2$, these two series are changed into this one here,

$$1 \times 2 + h \times 2 + \frac{h.h-1}{1.2} \times 2 + \frac{h.h-1.h-2}{1.2.3} \times 2 + \text{ \&c. to,}$$

$\frac{h.h-1.h-2 \cdots h-k+vs+1}{1.2.3 \cdots k-vs} \times 1$, the whole divided by 2^h, &

$$1 \times 2 + h \times 2 + \frac{h.h-1}{1.2} \times 2 + \frac{h.h-1.h-2}{1.2.3} \times 2 +$$

&c. to $\frac{h.h-1.h-2 \cdots h-k+vs+n+1}{1.2.3 \cdots k-vs-n} \times 1$. The whole divided by 2^h (I put here v in place of t, for the reason that I just said.) Now the terms of these series are nothing but those of the perpendicular band of the arithmetic triangle of M. Pascal, of which the heading is expressed by $h+1$, each multiplied by 2, except the last; whence it follows that their sums are correctly the sums of as many of the terms of the following band, of which the heading is $h+2$; since therefore the number of these terms of the first series is $k - vs + 1$, & that of the second $k - vs - n + 1$, the sums of all the possible values of these two series, by taking for v successively $0, 1, 2, 3, 4,$ &c. to t, will be $\boxed{k+1} + \boxed{k-s+1} + \boxed{k-2s+1} + \boxed{k-3s+1} +$&c. And $\boxed{k-n+1} + \boxed{k-s-n+1} + \boxed{k-2s-n+1} + \boxed{k-3s-n+1} +$&c. By this arbitrary mark $\boxed{k+1}$ I intend the sum of as many of the first terms of the perpendicular band which correspond to the heading $h+2$, as there are units in $k+1$. The difference of these two sums $\boxed{k+1} - \boxed{k-n+1} + \boxed{k-s+1} - \boxed{k-s-n+1} + \boxed{k-2s+1} - \boxed{k-2s-n+1} +$&c. divided by 2^h will express the lot of the one who would wager that Pierre will win the game in less than h trials. In order to apply this to some particular cases; we suppose, for example, that we play for seven games, & that we wish to know how much we could wager at the beginning of the game that one of the Players, for example Pierre, will win the game in less than 35 trials. We will have $m = n = 7$, $s = m + n = 14$, $h = 35 = 7 + 2k$: therefore k will be $= 14$, & $t = 1$; & the formula $\boxed{k+1} - \boxed{k-n+1} + \boxed{k-s+1} -$&c. divided by 2^h will be changed into this one here $\boxed{15} - \boxed{8} + \boxed{1}$ divided by 2^{35}, which indicates that it is necessary to divide the sum of the $15, 14, 13, 12, 11, 10,$ 9th & 1st term of the 37th perpendicular band, that is to say 8338160273, by 2^{35} in order to have the lot of the one who would wager that Pierre will win in less than 35 games, & that it is necessary to divide it by 2^{34} in order to have the lot of the one who would wager that the game will be ended in 35 games, conforming to our calculation; but for 37 games I find that it must be $\frac{35102333827}{2 \times 34359738368}$, & not $\frac{35103333817}{2 \times 34359738368}$, as you have written by error. If we suppose that $m = 5$, $n = 9$, $s = 14$, & $h = 35$, that is to say, that Pierre has already won two games, & that we wish to know the probability that Pierre or Paul will win the game in 35 trials, we will find $\boxed{16} - \boxed{7} + \boxed{2}$ divided by 2^{35} or $\frac{13914410549}{34359738368}$ for the lot of the one who would wager that Pierre will win the game in 35 trials, & $\boxed{14} - \boxed{9}$ divided by 2^{35} or $\frac{4511602732}{34359738368}$ for the lot of the one who would wager that Paul will win it in 35 trials. The sum of these two lots $\frac{18426013281}{34359738368}$ will express the lot of the one who would wager that the game will be decided in 35 trials, which shows that it would be to the advantage. I believe that this will suffice for you to make understood the sense of my formula: we pass to other things.

As you have invited me to render public my solution of your Problem on the Lottery of Loraine, I have sent it to Mr. Varignon four months ago to insert it into the Lottery of Lorraine

Journal des Sçavans, where it appeared the thirteenth of July, that which you know perhaps already. For that which is your solution, I have remarked that beyond that your Anagram $4a, 5e, 5i, 13o, 3u, 2l, 2n, 2p, 4s, 3, 2, c, d, m, r,$ does not contain exactly these words: *20000 moins un divisé par 20000 élevé à l'exposant 20000*; because the Anagram will have no correct sense, & will not give at all the sought value 184064. Moreover that which you say that the solution of this Problem is only a particular case of the formula $m - 1^{p-q} \times p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times$&c. divided by m^p, is only true when the number which expresses how many of the tickets one must draw, finally that having won no lot in all these tickets, one can redraw his silver, is correctly a fractional part of the number of all the tickets; because in order to reduce the cases where this is not found in the Problem for the dice, in order to bring forth a certain number of points, it would be necessary to suppose that each die has many faces marked with a point which one proposes to bring forth, this is to what your formula does not extend at all. But to what serves it to go seeking so far the manner of solving this Problem, doesn't one see first & most easily that it is only a particular case of proposition 44[18] of your Book?

Treize

I am surprised, Sir, to see your objections against my remarks on the games in which the hand turns from one to another; it seems to me that you are much wrong to oppose me some things which are also as much against you as against me; because if you are in a state to suppose, for example, at Lansquenet, that the number of players & of the wagers are always the same, & that the game continues as long as Pierre will have the hand, why would there not be permitted to me to suppose again the same thing, the same after Pierre will have lost the hand? You say that one is able to say nothing useful & certain on these games, because the number of wagers & of the players are always able to vary there: this is true, & this is also the reason why one must make a certain hypothesis to which one can take oneself in the calculation. I have therefore made this hypothesis, namely that one continues to play when one just loses the hand, because it is more natural & more conforming to that which happens ordinarily, than yours which supposes that the game continues as long as Pierre will have the hand, this which is a condition which is being scarcely practiced among the players, especially when they know that there is advantage to have the hand. But you oppose me still when, by example at Treize, the one who quits the hand is no longer obligated to continue the game, to which I respond that an honest man must be held obligated to it, although one is not expressly agreed to that; because it is certain that ordinarily one begins the game with the plan to make a great enough number of games, & not to end immediately after the first move, this which engages the players tacitly to continue the game during a certain time. It will not be permissible to quit the game after having had the advantage of the hand, at least one does not wish to pass in order for a man who thinks rather of grabbing the money of the others than to amuse them. You see by this, Sir, that you would not have done badly to take into consideration, not only the advantage that one has in conserving the hand; but also the disadvantage that one has in losing it.

Waldegrave
Pool

As I do not understand well the rules of Lansquenet, nor that which one calls rejoicing, to pay all around, cards of resumption, &c. I am not able to examine if it could have the cases where the one who is at the left of Pierre would have the advantage, as you say that it has been proposed to you by one of your friends. But for this other Problem that this Geometer has proposed to you, I have resolved it generally in all its three points. Let be named n the number of trials which it is necessary to win in sequence, or the number of Players less one; Pierre & Paul two Players who follow immediately in the

[18] See page 228.

order of play; so that Paul, for example, enters into the game immediately after Pierre; a the probability that Pierre will win the pool; b the probability that Paul will win it; A the advantage or the disadvantage of Pierre; B the advantage or disadvantage of Paul, I find generally $b = \frac{a\times 2^n}{1+2^n}$, & $B = \frac{\overline{A+a}\times 2^n - nb}{1+2^n}$. The first of these two equations demonstrate that there are odds of $1+2^n$ against 2^n that Pierre will win the pool rather than Paul, that which gives in the particular case of $n = 2$, five against four, thus as you have found. It is easy to find by these two equations or Theorems, the advantage or the disadvantage of each Player, & the probability that each has to win the pool, because the sum of the advantages & of the disadvantages of all the Players together must be equal to zero; as also the sum of the probabilities which they have to win the pool, must make an entire certitude or 1. For example in the case of the three Players or of $n = 2$, we find first that the probability which the first & second have to win, that is to say those who play first, is $= \frac{5}{14}$, & that what the third or the one who enters into the game last is $= \frac{2}{7}$, having substituted these values into the second equation, & having named x the advantage or the disadvantage of the first, the one of the second will also be $= x$, & the one of the third $= \frac{\overline{x+\frac{5}{14}}\times 4 - 2\times\frac{2}{7}}{5} = \frac{4x+\frac{6}{7}}{5}$, to which, if we add $2x$, we will have $\frac{14x+\frac{6}{7}}{5} = 0$; whence we deduce $x = -\frac{3}{49}$, that which shows that the first two players have disadvantage, & that the advantage of the third is $= \frac{6}{49}$, thus as you have found it by a very different way from that. I have also made application for the case of four & five Players, & I have found that in a pool of 4 Players the disadvantage of the first two is $= -\frac{2700}{22201}$, the advantage of the third $= \frac{1176}{22201}$, & the advantage of the fourth $= \frac{4224}{22201}$; but to five Players the disadvantage of the first two is $= -\frac{24059828}{131079601}$, the advantage of the 3rd $= \frac{-2402712}{131079601}$, the advantage of the fourth $= \frac{16789760}{131079601}$, & the one of the fifth $= \frac{33732608}{131079601}$. For the last point of this Problem, namely how many games must the pool naturally endure, I have found a general formula which expresses the probability that that it will be decided in at least p games, here it is

$$\frac{p+1}{1.2^n} - \frac{p-n.p-n+3}{1.2.2^{2n}} + \frac{p-2n.p-2n+1.p-2n+5}{1.2.3.2^{3n}} - \frac{p-3n.p-3n+1.p-3n+2.p-3n+7}{1.2.3.4.2^{4n}} + \frac{p-4n.p-4n+1.p-4n+2.p-4n+3.p-4n+9}{1.2.3.4.5.2^{5n}} - \&c.$$ [19]

It is necessary to take as many terms of this series as there are units in $\frac{p+n}{n}$. Now if you like more the series such as you have given for three & four Players, here is a general method to find them. It is necessary to construct a series of fractions, of which the denominators increase in double ratio, & in which the first term is $\frac{1}{2}$ raised to $n-1$, that is to say, to the exponent expressed by the number of Players less 2, & the numerator of each other term the sum of the numerators of as many preceding terms as there are units in $n-1$. This being made, the sums of the terms of this series will give the terms of the sought series; namely, the first term will be also the first of the sought series, the sum of the first two will be the second term, the sum of the first three will be the third term, that of the first four the fourth, & thus in sequence. By this manner we will find for five Players this sequence $\frac{1}{8}, \frac{3}{16}, \frac{8}{32}, \frac{20}{64}, \frac{47}{128}, \frac{107}{256}, \frac{238}{512}, \frac{520}{1024}, \frac{1121}{2048}, \frac{2391}{4096}$,&c. of which the terms are the sums of these $\frac{1}{8}, \frac{1}{16}, \frac{2}{32}, \frac{4}{64}, \frac{7}{128}, \frac{13}{256}, \frac{24}{512}, \frac{44}{1024}, \frac{81}{2048}, \frac{149}{4096}$, &c. in which each numerator is the sum of the three preceding. I am astonished that by giving you two series for the case of three & of four Players you have not observed the progression which there is between these series, & which on the contrary it has appeared too difficult or too painful to you to continue them for a great number of Players; but I am not astonished that you have found the same difficulty in wishing to seek the lot of the Players, when there are more than three, because it is extremely difficult to find it by the say of infinite series, as you have done for the case of three Players; & if I had not found a method to resolve this Problem by analysis, it would have been absolutely impossible to me to succeed to it. I would have well wished to make you part of this method; but as it would be too lengthy to put it here, & as I would have the pain to make myself well understood, I will leave to you the pleasure of finding it for yourself.

I believe well, Sir, that you have again been able to resolve this Problem, *drawing from a deck of cards a certain number of cards, knowing in how many ways one can bring forth a certain point at will*, only by supposing that it is neither Jacks, nor Queens, nor Kings; because these three kinds of cards being counted for a tine or for some other point, are that it is not an equal number of ace, of two, of three, of four, &c. Now it is quite discomforting to find for this suppostion a general formula; & one would not know to take a middle route in order to resolve this Problem in general, that to seek firstly all the different dispositions of the numbers which can form the proposed point, & next how many cases there are where each of these dispositions in particular can happen: For example, if we wish to know how many cases there are to bring forth the point 12 in three cards, it is necessary to seek first all the different ways by which we can partition the number 12 into three parts, of which each is a whole number, we will find twelve, namely 1.1.10; 1.2.9; 1.3.8; 1.4.7; 1.5.6; 2.2.8; 2.3.7; 2.4.6; 2.5.5; 3.3.6; 3.4.5; & 4.4.4. If we wish now to suppose that generally the number of aces is $= a$, that of two $= b$, that of three $= c$, that of four $= d$, &c. we will have $\frac{a.a-1}{1.2} \times k$ for the number which expresses in how many ways the first of these dispositions 1.1.10 can happen; $a \times b \times i$ for the number of cases of the second disposition 1.2.9; $a \times c \times h$ for that of the third 1.3.8; $a \times d \times g$ for that of the fourth 1.4.7; $a \times e \times f$ for that of the fifth 1.5.6; $\frac{b.b-1}{1.2} \times h$ for that of the sixth 2.2.8; & thus in sequence.

Traité du Jeu

I find quite correct the sentiment that you have for the Book intitled *Traité du Jeu*, of which the author is Mr. Barbeyrac, who is at present Professor of Law at Lausanne in Switzerland; but your critique for the particular example that you allege does not seem to me well founded. For me I am rather of the sentiment of the Author than of your; because it is evident that between two Players the one who plays two trials against the other one, & puts also two écus against one, can be considered as two persons, of whom each play one trial, & put into the game one écu; now as in this last case there is no inequality at all, it will not be any more in the first. That which you have found that if one plays with a die, of which the number of faces is p, to whom will bring forth the highest point, the one who playing two games against one, wagers only two against one, has the advantage, & that this advantage is $\frac{p-1}{2pp}$, comes from that which you have supposed that each withdraw his money, when the one who plays two games, who I will name Pierre, brings forth a point equal to the one of his antagonist, who is called Paul, is who makes it twice in sequence, or only one time, instead in this last case you must suppose that the two Players Pierre & Paul divide the money equally; because Paul bringing forth the same point as Pierre, acquires the same right as him, & Pierre loses the prerogative which he had before. If we wish nevertheless to make another supposition as that, which seems to me to be the most natural, it will be according to this supposition that will change also the lot of the Players. In order to find generally, let the advantage of Pierre be named x, y his advantage when he brings forth only one time the same point as Paul, & z his advantage when he brings forth twice the same point as Paul; we will find $x = \frac{2x+2py-2y+p-1}{2pp}$. If we put $x = y = 0$, which is your hypothesis, we will have $x = \frac{p-1}{2pp}$, as you have found. If $z = y = -\frac{1}{2}$, that is to say, if we suppose that the Players divide the money of the game equally, when they bring forth both an equal point, either that Pierre makes it one or two times, Pierre will have disadvantage, & this disadvantage will be $= \frac{1}{2pp}$. If $z = y = x$, that is to say if we suppose that the Players, after having brought forth the same point, restart the game, the equality found will be changed to this $x = \frac{2px+p-1}{2pp}$, that which will give $x = \frac{1}{2p}$. Finally if $z = 0$, & $y = -\frac{1}{2}$, which is the supposition of which I just spoke above, we will find $x = 0$, that which demonstrates that then Pierre will have neither advantage nor disadvantage, conforming to that which has been said by our Author, who consequently can support with reason that a Player who plays two trials during which the other plays only one, must put double against simple. There is however place which shows that this Author does not intend well enough these things; this is at page 122, where he cites quite badly à propos, to that which seems to me, these words of Mr. de Fontenelle in the eulogy of my late Uncle: *It is to remark that often the advantages or the forces are incommensurables, such that the two Players can never be perfectly equal*, which does not prove that which the Author has intention to prove; because in order to understand the sense well, & that which Mr. de Fontenelle has wished to say, it is necessary to know what my late Uncle has left, beyond the Latin Treatise *De Arte Conjectandi*, another Manuscript written in French, in which he treats on the Game of Tennis in particular, & where he has resolved many questions which one can form on this Game, of which these two are the principals. 1° If we suppose the Players unequal, we demand what advantage the stronger must accord to the other in order that the game be equal or reciprocally. 2°. If we suppose that one has accorded a certain advantage, & that thence their strength is made equal, we demand by how much it is stronger, or what ratio there is between their abilities; & it is in this Problem which he has found that their forces or abiliites will be incommensurables, & that one will not know how to express by any number the ratio which they have between themselves. I am amazed that you have not spoken at all of this Game in your Book. It is true that, the research of the similar Problems is not easy, but it is quite curious, & not failing of usage. Here are some other ones which you could seek the solution in order to see if it will agree with that of my Uncle. 1. *Pierre & Paul play Tennis by linking four games, Pierre is twice as able as Paul, we demand what advantage Pierre must accord to Paul? 2. Pierre accords to Paul half-thirty, we demand by how much he is more able than Paul? 3. Pierre accords to Paul half-30, & to Jean forty-five, we demand how much Paul can accord to Jean? 4. Pierre & Paul play together against Jean, & their respective forces are as 1. 2. 3, we demand how much this last can accord to the two others?* Tennis

I have also resolved the Problem on Her in the most simple case, here is that which I have found. If we suppose that each of the Players observes the conduct which is the most advantageous to him, it is necessary that Paul holds himself only to one card which is higher than a seven, & Pierre to one which is higher than an eight, & we will find that in this supposition the lot of Pierre will be to the lot of Paul as 2697 to 2828. If we suppose that Paul holds himself also at a seven, the Pierre must hold himself to an eight, & their lots will be again as 2697 to 2828. It is nonetheless more advantageous for Paul to not hold himself to a seven as to hold himself to it, that which is an enigma which I leave to you to develop. Her

Your formula $\frac{p-1}{2p-1}$ for the game of Tas, when the Tas has only two cards, & if he has only 2 aces, 2 twos, 2 three, 2 four, &c. is quite correct; but I wish to know how you have found it, I have been able to find it yet otherwise only by induction, by putting for the number of the tas successively 2, 3, 4, 5, &c. Tas

You will make well, Sir, when you will give a new Edition of your Book by changing Figurate numbers the order, & by reassembling in the first part the material on combinations; you might add quantity of propositions, & treat this material more amply than you have done in the first Edition. My Uncle, who made you his compliments, well wishes to permit you, & me I consent also, that you insert in this new Edition our Letters, if you find that they can give some light to your good discoveries. I find quite conforming to the truth that which you say, that the demonstrations touching the numbers & combinations, are much more awkward than are the demonstrations of Geometry, & that we can often more easily have them in the mind than put them on the paper. I am quite content of the demonstration which you give in your Letter on the formula, *page 99, Proposition 14.*[20]

Figurate numbers

But it is not likewise of that which you give, *page 88,*[21] to demonstrate the most principal property of the figurate numbers; this demonstration, & all those which one has given until here to this end, although quite correct, are all seen as one has previewed this property, & have need of induction, at least at the beginning; it is necessary in these sorts of matters, in order to entirely satisfy our mind, not only to demonstrate that having found by chance that a certain property agrees, for example, with the first terms of a sequence, it must hold by necessity in the following terms; but it is necessary further to demonstrate the road by which one can attain to the discovery of this property. Thus in order to demonstrate Corollary 4, page 91,[22] it is necessary for you to demonstrate before the preceding Lemma; but you do not show how you are arrived to the knowledge of this Lemma, which you perhaps would not have found, if you had not known before the formula of the figurate numbers. This is why I make you part here of a method which I have for some years to find the formulas & the sums of the figurate numbers by pure Analysis, without supposing anything known touching the form of their expressions. This method is quite singular & so much more curious as it serves of the differential calculus, here it is. Let be proposed to find, for example, the sum of the triangular numbers, or the formula of the pyramidal numbers, I multiply these numbers 1, 3, 6, 10, 15, 21, &c. by the terms of this geometric progression $1,\, x,\, xx,\, x^3,\, x^4$, &c. This being done I will have this sequence $1,\, 3x,\, 6xx,\, 10x^3,\, 15x^4$, &c. which I decompose into these sequences.

$A\, 1,\, 2x,\, 3xx,\, 4x^3,\, 5x^4$, &c.
$B\, x,\, 2xx,\, 3x^3,\, 4x^4$, &c.
$C\, xx,\, 2x^3,\, 3x^4$, &c.
$D\, x^3,\, 2x^4$, &c.
$E\, x^4$, &c.

Now I find in the same manner that you have made in your Letter in order to find the sum of this sequence $m + 2mm + 2m^3 + 4m^4$, &c. that by putting p for the number of terms the sequence A is

[20]See page 44.
[21]See page 10.
[22]See page 13.

$$= \frac{1 - \overline{p+1}x^{p+1} + px^{p+2}}{1 - 2x + xx},$$
$$B = \frac{x - px^{p+1} + \overline{p-1}x^{p+2}}{1 - 2x + xx},$$
$$C = \frac{xx - \overline{p-1}x^{p+1} + \overline{p-2}x^{p+2}}{1 - 2x + xx},$$
$$D = \frac{x^3 - \overline{p-2}x^{p+1} + \overline{p-3}x^{p+2}}{1 - 2x + xx}, \text{\&c.}$$

& that the sum of all these sequences $A + B + C + D$, &c. will be

$$= \frac{z - \overline{pp + 3p + 2}x^{p+1} + \overline{2pp + 4p}x^{p+2} - \overline{pp + p}x^{p+3}}{1 - 2x + xx}$$

which by putting $x = 1$ will give the sum of the triangular numbers; now in this supposition of $x = 1$ the two terms of this fraction vanish; this is why in serving myself of the rule of my Uncle, which the late Sir the Marquis de l'Hôpital has inferred in his Analysis of the infinitely small, page 145; & by differentiating three times the sequence of the numerator & the denominator, I find for the sought sum

$$\frac{1}{12} \times \overline{p^5 + 3p^4 + p^3 - 3pp - 2p}x^{p-2} + \frac{1}{12} \times \overline{2p^5 + 10p^4 + 16p^6 + 8pp}x^{p-1}$$
$$- \frac{1}{12} \times \overline{p^5 + 7p^4 + 17p^3 + 17pp + 6p}x^p,$$

or by putting 1 for x, & by dividing next the numerator & by -2,

$$\frac{p^3 + 3pp + 2p}{6} = \frac{p.p+1.p+2}{1.2.3},$$

that which it is necessary to find. The formula for the pyramidal numbers being thus found, we will find in the same manner that of the triangulo pyramidal, & thus in sequence all the formulas of the figurate numbers. It would be useless & too long to make here the proof by the calculus. I end by assuring you that I am with all possible consideration,

SIR,

Your very humble & very
obedient Servant
N. BERNOULLY

Letter from Mr. de Montmort to Mr. Nicolas Bernoulli (pages 337–347)

At Paris 1 March 1712

The Letter which you have taken the pain to write to me, Sir, dated 13 November, is filled up with admirable things. The affairs which I have at Paris have not left me at all since I am there, & will not leave me at all so much that I will be left the liberty of mind necessary to examine all the beauties of your Letter. Thus until upon my return to Montmort I have regained this leisure & this tranquility of mind that I esteem so much, & of which I have besides absolutely need in order to follow you in your algebraic meditations. I will limit myself in this here to make you part of the reflections of two of my friends who I have left at Montmort, & who I have strongly invited, in quitting them, to examine the Problems which you propose on Tennis, & that which you say touching Her; I will join those that I have made lightly enough on some points of your Letter.

Her

Now so that you know, Sir, to whom you have to make, & who are these Sirs who full of admiration for your talent, dare however not at all to be submitted to your decisions; you know that one of the two is called Mr. the Abbé de Monsoury. We are neighbors in the country, his Abbey is only at a mile and a half from Montmort. The other is named Mr. Waldegrave. This is an English Gentleman, brother of the late Milord Waldegrave, who had married a natural daughter of King James. When I worked on Her some years ago, I shared with Mr. the Abbé of Monsoury that which I had found; but neither my calculations nor my reasonings could convince him. He upheld to me always that it was impossible to determine the lot of Pierre or the one of Paul, because we could not determine at what card Paul must be held, without knowing to what card Pierre must be held, *& vice versa*, that which made a circle, & rendered in his opinion the solution impossible. I added a quantity of subtle reasonings which made me to doubt that I had caught the truth. I was there when I had proposed to you to examine this Problem, my goal was to assure myself through you of the goodness of my solution, without having the pain of reporting my ideas on the subject which were entirely effaced. I have seen with pleasure that you have found as me that Paul could do better only to be held to an eight whatever choice as Pierre took; & that Pierre, when Paul is held, could make his choice better only by being held solely at the nine, whatever choice that Paul has taken; & that under this assumption that Pierre & Paul both take the choice which is their most advantageous, the lot of Paul was to the lot of Pierre :: $2828 : 2697$. The honor that your decision has made to my solution, has given rise to our Sirs, & on all to Mr. the Abbé of Monsoury, to examine this Problem at the base. Here is that which they have written to me on this subject it is more than a month.

"In order for you to render account, Sir, of the judgment that Mr. the Abbé & me ourselves have dared to pronounce against Mr. Bernoulli on the subject of his solution on Her; it is not true, according to us, that Paul must be held only at the eight, & Pierre at the nine. We claim that it is indifferent to Paul to change or to be held at the seven, & to Pierre to change or to be held at the eight. In order to prove it, I must first expose their lot in all the cases. The one of Paul having a seven, is $\frac{780}{50\times51}$ when he changes, & when he is held his lot is $\frac{720}{50\times51}$ if Pierre is held at an eight, & $\frac{816}{50\times51}$ if Pierre changes at the eight. The lot of Pierre having an eight is $\frac{150}{23\times50}$ if he is held, & $\frac{210}{23\times50}$ if he changes in the case that Paul never is held at the seven; & $\frac{350}{27\times50}$ by being held, & $\frac{314}{27\times50}$ by changing in the case that Paul is held at the seven, here are them all in sequence. The lots of Paul $\frac{780 \text{ or } 720 \text{ or } 816}{50\times51}$, those of Pierre $\frac{150 \text{ or } 210}{23\times50}$ or $\frac{350 \text{ or } 314}{27\times50}$.

"720 being more below 780 than 816 is above, it seems that Paul must deduce a reason to change at the seven. I call this weight which carries Paul to change A, likewise Pierre of his different lots must deduce a reason to change at the eight, I call this weight B. This put, we say that the same weights carry Pierre & Paul equally to the two choices: Therefore, &c. A carries Paul to change his seven, & consequently carries Pierre to change his eight; but that which carries Pierre to change his eight, carries also Paul to be held at his seven. Therefore A carries Paul equally to change his seven, & to be held. Likewise B carries Pierre to change his eight, & consequently Paul has to be held at the seven; but that which carries Paul to guard his seven, carries also Pierre to guard his eight, & consequently carries Paul to change his seven. Therefore B carries Paul equally to change & to guard his seven: it is likewise of Pierre. Therefore the same weights carry equally, &c. Therefore it is false that Paul must be held only at the eight, & Pierre only at the nine. Apparently Mr. Bernoulli is contented to regard the fractions which express the different lots of Pierre & of Paul, without paying attention to the probability of that which the other will make."

They have confirmed in some later Letters that which they advance in this here, & add: *As we do not agree with Mr. Bernoulli that Paul must be held only at the eight & Pierre at the nine; we have not sought the explication of his enigma which we believe founded on a false supposition.*

These Sirs have also sent to me quite long calculations on the first of your Problems on Tennis: these calculations are exact; but as there is much groping in their method, & as besides there is much lacking in them that the Problem is not resolved, I will not put them here. Tennis

For me, Sir, before undertaking the solution of it, I have believed I must demand enlightenment on that which follows.

1° When you say *Pierre is two times more able than Paul*, do you understand that Pierre has two times more facility than Paul to win each fifteen, or more exactly, that the ratios of the facility are as 2.1.

2° By this word, *parts of Tennis*, do you understand of the parts composed of six games? Do you conceive that when Pierre & Paul have each forty-five, this which is called to be at deuce, one returns necessarily into two fifteen, this which is practiced here.

3° When you say: *One demands what advantage Pierre must make to Paul*. Do you demand how many fifteen or fractions of fifteen Pierre must accord to Paul in each game? You know that the strongest gives often in order to be equal to the weakest some bisques, some entire games, to save the first or the second, to play entirely on one side, &c. all that wish to be determined. It would not be the same thing, for example, to give three games in each part of six games, or 30 in each game of the same part.

4° The fourth Problem that you enunciate thus: *Pierre & Paul play together against Jean, & their respective forces are as 1.2.3, one demands how much this last can accord to the first two*. This Problem, I say, seems to include no exactitude. Often two persons less strong in particular than Pierre, can play without disadvantage with him; & to the contrary two persons as strong can play with disadvantage, according as they will know or will not know to accommodate themselves together, this which is a particular talent independent of the one to play well being alone.

5° When you say in the second Problem, *Pierre accords to Paul half-thirty*. Do you intend that Paul will have 30 in the first game, & next 15, & thus in sequence 30 & 15 alternatively, or if Paul will commence by having 15, & next 30, &c. this which would be perhaps quite different.

By you writing this, Sir, I have had the curiosity to make some tries on your four Problems. Here is the path that I have made.

You know, Sir, that naming p the number of the games which are lacking to Pierre, q the number of games which are lacking to Paul, a the degree of facility that Pierre has to win each point, b the degree of facility that Paul has to win each point, & supposing $p+q-1=m$, the formula which expresses the lot of Pierre is

$$a^m b^0 + m\cdot a^{m-1}b^1 + \frac{m.m-1}{1.2}a^{m-2}b^2 + \frac{m.m-1.m-2}{1.2.3}a^{m-3}b^3 + \&\text{c.}$$

& likewise that the formula which expresses the lot of Paul is

$$b^m a^0 + m\cdot b^{m-1}a^1 + \frac{m.m-1}{1.2}b^{m-2}a^2 + \frac{m.m-1.m-2}{1.2.3}b^{m-3}a^3 + \&\text{c.}$$

that it is necessary to continue the first series until the number of terms expressed by q, & the second until the number of terms expressed by p, & to divide both by $\overline{a+b}^m$.

I have found that if one wishes that when there is lacking more than one point to each of the two Players, one returns to deuce by necessity; these same formulas can again serve with the two restrictions which follow, 1° m must be $= p+q-2$, instead as one supposed $m = p+q-1$. 2° It is necessary to multiply the last term of the series which expresses the lot of Pierre, by $\frac{aa}{aa+bb}$; & the last term of the series which expresses the lot of Paul, by $\frac{bb}{aa+bb}$.

Next from these preliminaries, I have sought the solution of some of your Problems, or of others which have relation, here is that which I have found. 1° *Pierre plays against Paul, & he is two times stronger: there is lacking to him four points, one demands how many there must be lacking of them to Paul, that is to say, what must be the value of q, p being* $= 4$.

Under the ordinary assumption that one not return, one has this equation

$$4m^3 - 8mm + 14m + 6 = 3^{m+1},$$

of which one can find the root by the intersection of a logarithm & of a cubic parabola, I find $m = 5 + \frac{57}{230}$, this which teaches me that Pierre must give to Paul one point, & $\frac{263}{320}$ on the 2nd point, this which I explicate in this manner. One will put three hundred twenty tokens into a pouch, of which there will be 263 whites & 57 blacks; & it will be said that if drawing a token at random one draws a white, Pierre will give two points to Paul on the part, & that if one draws a black token, he will give to him only one of them. One can render the lots perfectly equal only by using this skill, & it is only in this manner that it is necessary to explicate the fractions of things which are not shared at all by the coups, the points, &c.

If one wishes to suppose that the Players will return when they are at deuce, that is to say every time that they will have each three points, as it is the rule in the game of Tennis.

One finds that under this assumption Pierre must give to Paul two points, & $\frac{11}{224}$ on the third point, in order that the game be equal, & it is here, it seems to me, the solution of the first of your Problems.

2° *Pierre gives to Paul two points out of four, & beyond this* $\frac{11}{224}$ *on the third point, one demands by how much he must be stronger than Paul in order to give to him this advantage, one finds that the ratio sought of his force to that of Paul is contained in this equality of the sixth degree,*

$$224a^6 + 830a^5b - 1142a^4bb - 1792a^3b^3 - 1568a^2b^4 - 896ab^5 - 224b^6 = 0.$$

Whence one deduces $a = 2b$.

If one supposes that *Pierre has reason to give one point to Paul, & that one demands how much he must be stronger in order to give this advantage to him, b being* $= 1$, I find $a = 1+\sqrt{2}$, that is to say that Pierre must be stronger than Paul in the ratio of 1 to $\sqrt{2}-1$; & generally that if $q = 1$, it is necessary that Pierre be stronger than Paul in the ratio of 1 to $\sqrt[p]{2}-1$.

3° I had commenced to make an attempt on one kind completely parallel to the second of your four Problems, but more simple, here is what it is.

Pierre plays in two linked games against Paul in the "petit palet," each of the games is of two points. They agree that Paul will have one point in the first part, that he will have none at all in the second, that is to say that they will play it to goal; & that in the third, if the game is not finished before, Paul will have one point. This returns to that which one calls half-fifteen in Tennis. One demands how much it is necessary that Pierre be stronger than Paul in order to give him this advantage. I have found that this Problem would depend on the resolution of this equality of the 7th degree

$$a^7 + 7a^6b + 5a^5b^2 - 21a^4b^3 - 29a^3b^4 - 21a^2b^5 - 7ab^6 - b^7 = 0,$$

this which would lead me to some quite long calculations. Your second Problem demands of it yet greater, the equality being more composed; thus I pray you to say to me if you have some other secret than me in order to avoid the resolution of these equalities. My method has been every time to rest myself when I am come to the equation of it, & to leave to the curious to seek the roots of it. I will change only in order to please you in the case that you testified that you required this sacrifice here from me; I say sacrifice, because in truth it is the farthest that I myself remember to have resolved some equalities which exceed the fourth degree, & it seems to me that of all the occupations it is the least agreeable.

4 ° I have made further some reflections on the third of your four Problems, here is the one that I have proposed to myself, which is a little more simple, but which contains the same difficulty.

Pierre playing against Paul in two points can give one of them to him, Paul playing against Jacques can give to him one of them, one demands how much Pierre can give of them to Jacques.

I have found that he must give to him one point & $8\sqrt{2} - 11$ on the second point, this which I explicate in this way: Let be supposed $\sqrt{2} = \frac{1414}{1000}$, I say that putting 125 tokens into a pouch, of which there are 39 blacks & 86 whites; if one draws a token at random, & if it is found white, Pierre must give one point only to Paul, & if black is encountered he must give to him two of them. One sees here the example of a case where it is absolutely impossible to render the parts equal, whatever compensation that one can imagine. Here is, Sir, that which I have found quite in a hurry; if I had had more leisure to meditate on these matters, & to make long calculations, I would have perhaps better success. If I myself am deceived, give me grace in favor of my allegiance. I pass now to the other places of your Letter.

Your formula for Treize is very correct. I myself am quite doubtful that the error of the preceding was able to proceed only by some inadvertence in transcribing. The idea that I myself have made of your infallibility in Geometry does not permit me to suspect that you had been able to deceive yourself in the foundation of a method. Treize

I understand perfectly your formula for the games by reducing; it assuredly needed explication in order to be understood; I have seen with surprise & admiration that it was not much different from mine. You yourself will have without doubt noticed in my last Letter, when I had sent you that this number 70970250 is the sum of the six $34597290, 20030010, 10015005, 4292145, 1560780, 475020,$ which are the 7, the 8, the 9, the 10, the 11 & the 12th terms of the 30th perpendicular band. I do not know if it there must be a 3 in place of a 4 in this numerator 13914410549, I find 13914410539: this number is formed from the sum of these nine, & again from these two 36. 1. Duration of Play

556790260
3796297200
2310789600
1251677700
600805296
254186856
34143280
30260340
8347670

It would not be useful that I report to you here my method all at length, it is slightly different from yours only in the manner of statement, with the exception that I have had in view only the supposition of equal chances for one & the other Player, instead as you suppose them in any ratio, here it is briefly. Let p be the number of trials, m the number of games which are lacking to Pierre, n the number of games which are lacking to Paul, it is necessary to choose a perpendicular column of which the heading is $p+2$, & in this column to choose the heading which corresponds to the quantity $\frac{p+2-m}{2}$; to add to this number the superiors to the quantity n. Take the nine first terms, for example, if there is lacking nine games to Paul, then by omitting the quantity m, to add the quantity n, to omit the quantity m, & thus in sequence alternately, to divide these numbers by 2^p; we will have a fraction which expresses how much the odds are that Pierre will win the game at least by as many trials as p expresses units. If we wish to have for Paul that which we have for Pierre, it is necessary to put everywhere m in place of n, & n in place of m.

Pool

Your method to find the advantages or disadvantages of those who play a pool at Tric-trac, at the rate of the order according to which they enter the game, can be only perfectly beautiful. This Problem is assuredly quite difficult. I have wished to discover how we can apply your method for three Players to the case of four or five Players, but uselessly. The route that you have followed is apparently very remote. I will work seriously as soon as I will have the leisure. Your series to determine how many matches the pool must naturally endure is quite correct.

Lottery of Lorraine

In the explication which I have sent to you on my Anagram, it is necessary to read *20000 moins 1, divisé par 20000 élevé à la puissance 20000*. I had put through distraction *exposant* in the place of puissance: this is that which has prevented you from understanding, because besides it is clear that it is necessary to multiply this number by 25×20000, & this goes without saying.

I will replicate nothing to that which you say to me on the games in which the hand turns, I do not believe to have a thing to add to that which I have already said; besides I agree that your speculation is beautiful & good.

Traité du Jeu

It suffices, Sir, for the justification of Mr. Barbeyrac, that you approve the place that I have criticized; & it is rather that he has on this subject diversity of opinions for which he has no wrong. For me, Sir, I would believe, & all those to which I have spoken believe it also, that Pierre having brought forth in one or the other of this two coups the same point as Paul, one must suppose either that they each will take their stake, or that they will recommence immediately. I know well that this must not be so, but it is necessary from the calculus & the reasoning in order to find that that must not be; & as simple as these reasonings are, I am quite led to believe that each of those who have held this part have not made them; it is necessary to suppose for rule of the game, not that which must be done, but that which is practiced ordinarily among the Players.

Here is a wager quite similar to the kind of Mr. de B. that I have seen made sometime. Pierre wagers one écu against Paul to make one thing in two coups, for example, to pass a ball through a hole; in the case of this wager here is that which happens. If Pierre puts into the hole on the first coup, he does not recommence the second, because there is no longer anything to win, the game is ended, Pierre has won the écu; if he plays his second coup, it will be for amusement, without fear of losing anything by not setting a second time, & without expectation of winning anything in setting a second time.

Figurate numbers

Your analytic demonstration of the formula of figured numbers is of an extreme beauty. I do not know…[23]

You know without doubt the death of My Lord the Dauphin, it is a great loss for France, & in particular for the Sciences, he loved them & would have protected them. I am with an infinite esteem,

[23]Here begins remarks on recent scientific works.

SIR,

Your very humble & very
obedient Servant R. de M. . .

Letter from Mr. Nicolas Bernoulli to Mr. de Montmort (pages 348–352)

At Basel 2 June 1712

SIR,

I am writing you this Letter quite with haste, being on the point to depart tomorrow or after tomorrow for Holland; I am quite displeased that my affairs have not permitted me to respond rather at the honor of your last Letter, & I am so much more that I see myself constrained to abreviate the response which I must for the ample material on which you have furnished the occasion to speak on your good Letter.

I am very sensible to the honor which your two Friends, Mr. the Abbé d'Orbais & Mr. de Waldegrave, have made me in examining that which I have written you on Her, & I am quite obliged to you of their sentiments on that subject which you have communicated to me. The lots which they have found for Pierre & Paul are quite correct; but the reasoning by which they wish to prove that it is indifferent to Pierre to change at the seven or to hold to himself, & to Paul to change or to hold to himself at the eight, cannot convince me; because in examining it more closely we will find that it is a sophism, & that we cannot reason thus: *The weight A carries Paul to change his seven* (when it is uncertain to what Pierre will determine,) *& consequently carry Pierre to change his eight,* (supposing that Pierre knows that Paul changes at a seven;) *but that which carries Pierre to change his eight, carries also Paul to hold to himself at his seven: Therefore A carries Paul equally to change his seven & to hold to himself;* because we suppose two contradictory choices at the time; namely, that Paul knows & that he is ignorant at the same time what choice Pierre will take & Pierre what choice Paul will take. It is quite true that the weight A carries Paul to change at the seven. Having therefore made this hypothesis that Paul has the maxim to change at the seven, it follows that Pierre will do better to change at the eight; but we must stop there, & not pass beyond, because it is not permitted to return to Paul & to conclude; therefore Paul must guard his seven, because according to this hypothesis we have already fixed that Paul has the maxim to change at the seven, & that Pierre changes at the eight only on condition that Paul changes at the seven: Therefore Paul is not able to change from maxim & hold to himself at the seven, without that Pierre change also at the seven; so that following the reasoning of these Sirs we would go always in a circle, that which is a demonstration that we can prove nothing from it. Moreover it is clear by the calculus that it is not indifferent to the Players to change at the seven or at the eight, or to hold to himself; because if this were, we would find also the same lots for all these cases there; now we find by the calculus that their lots are different according as they hold to themselves at such or such card. Therefore it is false that the same weights carry Pierre & Paul equally to the two choices. If these Sirs are not content in this response, I will give myself the honor of writing to you at the first occasion more amply on this, & to make you part of the method of which I have served myself to resolve this Problem, & I hope that these Sirs will find nothing to retell. I am quite pressed presently to enter into detail on all these things. I pray you to assure these two Sirs of my very humble respects, & to thank them on my part of the particular esteem which they wish well to have for me. Her

I found a general formula for the lots of the Players, when we suppose that the degrees of ability which they have to win changes alternately, as it happens when a Player accords Tennis to the other or half-fifteen or half-thirty, or some other similar point, here it is: Let p be the number of games which lack to Pierre, q the number of games which lack to Paul, a the degree of facility that Pierre has to win the 1st, 3rd, 5th, 7th, &c. game; b the degree of facility which Paul has to win the 1st, 3rd, 5th, 7th, &c. game; c the degree of facility which Pierre has to win the 2nd, 4th, 6th, &c game; d the degree of facility which Paul has to win the 2nd, 4th, 6th, &c game; let moreover $m+n=p+q-1$, & $m=n$ if $m+n$ is an even number, & $m=n+1$ if $m+n$ is an odd number; I say that the lot of Pierre will be the sum of all possible values of this sequence:

$$\begin{aligned}
6^n &\times \frac{m.m-1.m-2\ldots m-s+1}{1.2.3\ldots s}a^{m-s}b^s \\
&+nc^{n-1}d\times\frac{m.m-1.m-2\ldots m-s+2}{1.2.3\ldots s-1}a^{m-s+1}b^{s-1} \\
&+\frac{n.n-1}{1.2}c^{n-2}dd\times\frac{m.m-1.m-2\ldots m-s+3}{1.2.3\ldots s-2}a^{m-s+2}b^{s-2} \\
&+\frac{n.n-1.n-2}{1.2.3}c^{n-3}d^3\times\frac{m.m-1.m-2\ldots m-s+4}{1.2.3\ldots s-3}a^{m-s+3}b^{s-3}+_{c.}
\end{aligned}$$

the whole divided by $\overline{a+b}^m\times\overline{c+d}^n$. This is extended by taking for x successively $0,\ 1, 2\ , 3,$ &c. until $q-1$ inclusively. If one is at two in the game, & if he fails to win two games in sequence in order to win the game, it is necessary to put $m+n-1$, instead of $m+n$; & it is necessary yet to multiply that which results by substituting for s the last value $q-1$ by $\frac{ac}{ac+bd}$. You will see quite easily that this formula in the case of $a=c$, & $b=d$ will become exactly with yours. I have also found throughout the same solutions which you give in your Letter, except only this equation

$$a^7+7a^6b+a^5bb-15a^4b^3-29a^3b^4-21aab^5-7ab^6-b^7=0,$$

instead of which I have found this one

$$a^7+7a^6b+13a^5bb-21a^4b^3-35a^3b^4-21aab^5-7ab^6-b^7=0,$$

in order to determine by how much Pierre must be stronger than Paul, so that by playing to two games, of which each is of two points, he must give to him a point at the first game, & nothing at the second; & if the game is not finished before, next a point in the third game; but I have no more than point to you of another secret to avoid the resolution of these equalities but approximations, & I would take also for a sacrifice if it would be necessary to seek the roots of these sorts of equations, this is a work which I leave quite gladly to the curious.

Pool

For that which is my method to find the advantages or disadvantages of those who play a pool, I have believed to have explained it quite clearly, & I am bothered that you have not been able to apply it to the case of four or of five Players; I am going therefore to clarify it more for you by applying the two Theorems which I have found in the case of the four Players. Let the four Players be Pierre, Paul, Jacques & Jean, who enter into the game according to the order which they are ranked here; so that Pierre & Paul play first together; next the one who will have won will play with Jacques, & the one who will have won of these two there with Jean, & thus in sequence; let be named p the probability that Pierre or Paul has to win the pool, q the probability that Jacques has to win, & r the probability that Jean has to win it, x the advantage of Pierre or Paul, y the advantage of Jacques, & z the advantage of Jean, we will have by supposing $n=3$ which is the number of games which it is necessary to win in sequence; by the first Theorem $q=\frac{p\times 2^3}{1+2^3}$, $r=\frac{q\times 2^3}{1+2^3}$, &

by the second Theorem $y = \frac{\overline{x+p} \times 2^3 - 3q}{1+2^3}$, & $z = \frac{\overline{y+q} \times 2^3 - 3r}{1+2^3}$; now $p+p+q+r$ must be $= 1$, & $x+x+y+z = 0$; we will have therefore these six equations $q = \frac{8p}{9}$, $r = \frac{8q}{9}$, $y = \frac{8x+8p-3q}{9}$, $z = \frac{8y+8q-3r}{9}$, $2p+q+r = 1$, & $2x+y+z = 0$, which being compared together by the ordinary methods, gives $x = -\frac{2700}{22201}$, $y = \frac{1176}{22201}$, & $z = \frac{4224}{22201}$. The route which I have followed in order to find these two Theorems is not at all remote, I would communicate it to you willingly if I were not so pressed, that which is also the cause that I pass under silence the other places of your Letter. I know nothing at all new of the sciences, except that Mr. de Moivre who is member of the Society in England, made imprinted at London a Book on Chances. As I believe that you will be curious to have this Book when it will be imprinted, & that I hope to pass from Holland into England, I will try to procure a copy. Moivre

Besides, if I can on my trip make some other thing for your service, or for one of your friends, I pray you to make known to me. If you wish to give me the honor of writing to me, you can send your Letters to Basel as before, one will make always to keep them for me. I believe that it might happen that I will return by France, in which case I will flatter myself to have the honor to see you, & to demonstrate to you that I am more than I would know how to say,

SIR,

Your very humble & very
obedient Servant
N. BERNOULLY.

Letter of M. de Montmort to M. N. Bernoulli (p. 352–360)

At Montmort this 8 June 1712.

The Letter that you have given me the honor to write to me, Sir, dated 10 November, had put me into a great passion of algebra, & I commenced, it seems to me, to be busy, when I made response the first of May 1712. I myself proposed to work strongly on your scholarly Letter & with assiduity, in the hope of finding something which is able to give pleasure to you, & render me more worthy of the trading of Letters that you wish well to have with me: a weakness of mind of which I am unable to discern the cause has not permitted me to the present. I have been three months without daring to think, & even without being able to taste the pleasure of reading, it is only since some days that I commence to be able to count on my sanity.

In reading your Letter & my response I have perceived in that a fault of which I have believed must caution you. Instead of this equality $a^7 + 7a^6b + a^5bb - 15a^4b^3 - 29a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0$, it is necessary $a^7 + 7a^6b + 13a^5bb - 21a^4b^3 - 35a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0$, of which the root is approximately 1.77. I have again made some attempts on some Problems similar to those you propose to me, but always uselessly. I fall into some equalities which appear to me always to demand immense calculations, of which the difficulty belongs to algebra, & which do not demand invention. I suppose therefore, Sir, that I am not on a good path with respect to these Problems, & I pray you to set me: Here is one of them very simple in appearance which was proposed to me: Tennis

Pierre plays for three points, Paul for two, & Jacques for one, their lots being equal, one demands what must be the ratio of their lots.

In naming their respective forces a, b, c, one has, conformably to the Problem of page 175,[24] the lot of Pierre $= a^3c + ra^b + a^4$, the one of Paul $b^4 + 4a^1b^3 + 4abbc + 6aabb +$

[24]See page 242.

$bbcc + 2cb^3$, the one of Jacques $c^4 + 2cb^3 + 4bc^3 + 5bbcc + 4ac^3 + 8abbc + 12abcc + 6aacc + 12aabc + 3a^3c$, the whole divided by $(a + b + c)^4$.

I see well that in comparing these equalities, of which each $= \frac{1}{3}$, I will come in the end to determine the value of the three unknowns; but this will not be without resolving an equality quite composed which would demand perhaps 30 hours of calculation, with counting the risk of deceiving myself. I will await therefore on all these questions your help & your light. In waiting, & in order to fill the paper, I am going to make you part of two rather curious remarks, it seems to me, that I have made a long time ago on the occasion of this Problem.

Powers of multinomials

Mr. your Uncle has remarked quite subtly that my formula on page 137,[25] or his, $\frac{1.2.3.4...p}{1.2.3...b \times 1.2.3...c \times 1.2.3...d \times 1.2.3...e \times}$ &c. are able to serve for the determination of the coefficient of any term that one will wish of any polynomial raised to any power. It is assuredly quite curious to see to two Problems so different reunited under one same formula; but the principal convenience of the formula $\overset{f}{\underset{B}{\square}} \times \overset{f-B}{\underset{C}{\square}} \times \overset{f-B-C}{\underset{D}{\square}} \times \overset{f-B-C-D}{\underset{E}{\square}} \times \overset{p}{\underset{b}{\square}} \times \overset{p-b}{\underset{c}{\square}} \times \overset{p-b-c}{\underset{d}{\square}} \times \overset{p-b-c-d}{\underset{e}{\square}} \times$ & c. is to make distinguished how many terms there are which have the same coefficient, & where the exponents of the letters are the same. In order to make me understand, let the trinomial $a + b + c$ be what one wishes to raise to the fourth power, & of which one demands all the coefficients: I reduce the Problem to this here; being supposed four dice which have each three faces: one demands how many there are of different coups in order that casting them at random there is found either a quadruple, or a triple & one simple, or two doubles, or a double & two simples. Now if I substitute into the formula above for p, 4, the number of dice, & for f, 3, the number of faces of each die, I find for the first case 3×1, for the 2nd 6×4, for the 3rd 3×6, for the 4th 3×12, of which the sum $= 81$, the fourth power of 3.

Now in order to be assured that these two kinds are the same, it is necessary to observe that by raising the trinomial $a + b + c$ to the 4th power, one makes precisely the same thing as in the Problem of the dice; that is, that one takes the 4th power of each of the three letters a, b, c. 2° That one takes the cube of each with each of the two others as many times as it is possible. 3° That one takes the square of one of the three letters with the square of another in as many ways as it is able; & finally the square of one of the three with the product of the two others, as one knows as this is practiced in the formation of the powers which is only a reiterated multiplication; thus the formula above is able to serve for the formation of any multinomial as well, & is able to be more advantageous than the ordinary formulas for multinomials, which appears to me to be all only an easy extension of the formula $(a + b)^m = a^n + \frac{n-0}{1}a^{n-1}b + \frac{n-0}{1} \times \frac{n-1}{2}a^{n-2}bb + \frac{n-0 \times n-1 \times n-2}{1.2.3}a^{n-3}b^3, +$ &c. since in order to change this formula into that of the trinomial there is concern only to substitute into each term $b + c$ in the place of b, & that in the same fashion the formula of the trinomial is changed into a formula for the quadrinomial, &c. this which leaps to the eye, although the invention of these formulas for the indefinite powers of the multinomials has been quite praised; but in order to return to the usage of which I just spoke, one ses that there are three terms where the letter is to the 4th power. 2° That there are six terms where the coefficient is 4, & where there is found a cube with a simple letter. 3° That there are three where the coefficient is 6, & where there is found two squares. Finally, that there are

[25] See page 242.

three where the coefficient is 12, & where there is found a square with two simple letters, the part of the formula which is in capital letters gives the first numbers 3, 6, 3, 3: the part of the formula which is in small letters gives the others 1, 4, 6, 12: Here is my second remark.

A person who I know has again examined how many terms produces any multinomial according as it is raised to such or such exponent. The rule is that of which I am served myself in order to know how many terms the formula must have, page 63, which gives the sum of any sequence of figured numbers raised to any exponent; this rule, I say, is such. In order to know how many terms the multinomial q will contain raised to the power p, there is only to take the quantity $p + 1$ of the order q of the figured numbers, one will have the sought number; so that if one seeks, for example, how many terms a quadrinomial raised to the 8^{th} power will have, one will find 165, the 9^{th} number of the 4^{th} order for the sought number; & if one wishes to know how many terms a quintinomial raised to the 11th power will have, one will find that it will have 1365 of them, this number 1365 is the 12^{th} of the 5^{th} order.

. . . Sir, Your, &c.

[The remainder of the letter concerns a logarithmic curve]

Letter of M. de Montmort to M. (Nicolas) Bernoulli (p. 361–370)

At Montmort this 5 September 1712.

Sir,

When I received the Letter which you have made the honor to write me, dated 2 June, I was ill in bed of a gross ongoing fever; I showed it to Mr. the Abbé d'Orbais who had come to see me, he took copy of that which regards the game of Her, & will send it to his second Mr. de Waldegrave, of whom he received response some days after. Her

Mr. the Abbé d'Orbais, who your reasons have not at all shaken, supporting still the judgment of Mr. de Waldegrave, comes to write me this note, of which I am sure that style will please you.

Read, Sir, this Letter of Mr. de Waldegrave, it is excellent. One cannot respond to Mr. Bernoulli with more justice & precision. Oh! but I had done well leaving this illustrious Geometer my ally to speak, I myself would never be so sharply explained; if you write to him, I pray to you to show him that I have nothing at all to add. Solas admirandi plaudendique partes mihi reliquit. *Besides it is time that you took part in this dispute, Mr. de Waldegrave invited you in the Letter which you have shown me. You are too long time balanced by the name of Bernoulli on one side, & by our reasons on the other. There are no means to permit you a longer time in this situation too prudent in my opinion for one so great Master. I greet very humbly the Ladies, & give you the good night.* Her

Here is therefore a dispute in fashions, it is, this seems to me, quite lovely; & I myself know great pleasure in having given birth to it, the question is subtle & entirely of argument. Our Sirs are charmed by your honesty & by your modesty, they find themselves very honored that a great Geometer as you, at last a Bernoulli, wishes to break a lance with some Novices as themselves: They make both a thousand compliments to you.

I have received at the beginning of the month of August the Book of Mr. Moivre, the Author had addressed it for me to Mr. l'Abbé Bignon[26] who has had the kindness to send it to me. Out of that which you have sent me, & out of the manner of which the Author speaks in the Preface, I myself expected entirely another thing; I expected to find Moivre

[26] *Translator's note*: Jean-Paul Bignon (1662-1743) Librarian of the king.

the solution of the four Problems that I proposed at the end of my Book, or at least the solution of some one of the four, & some novelties of this kind proper to understand the routes that I have opened; but you find that his work is limited nearly entirely to resolve in a more general manner than I have done, the simplest & easiest questions which are in my Book; for example, the five Problems of Mr. Huygens what I have treated summarily only because of their extreme facility, in comparison to the greater part of the other Problems which are resolved in my Book. You will find finally that the questions that he treats, which are not at all resolved, are in our Letters; so that I do not believe that he has in this Work, moreover very well, nothing new for us, & nothing which is able to give us pleasure by the uniqueness, if this is not the way to find what is often new, & always good and ingenious. Here are some remarks that I have cast in haste on the paper these days past, when I worked to render account of this Work to Mr. l'Abbé Bignon who had demanded of me his sentiment. You know without doubt that this illustrious Abbé, who is in France the Protector of the Sciences & of the Scholars, has an expanse of knowledge well beyond the ordinary limits, a very great taste for all that which is of the resort of the mind, & much ardor to contribute to the perfection of the Sciences.

The first Problem is a particular case of the general formula

$$\overline{m-1}^{p-q} \times p \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \frac{p-3}{4} \cdot \frac{p-4}{5} \cdot \text{\&c.}$$

of which I have made part to Mr. your Uncle in my Letter of 15 November 1710. This formula gives the number of chances which there are to bring forth precisely a certain number q of six, with a certain number p of dice, of which the number of faces is m. In the case resolved by Mr. Moivre the question is to find how many chances there are to bring forth no six with 8 dice, or to bring forth one only of them (because it is the same thing to cast eight times in sequence one die, or to cast eight of them at one time,) one has therefore by my formula by taking the denominations of the Author, who calls $a+b$ that which I call m, & n that which I name p,

$$\frac{b^n + nb^{n-1}}{\overline{a+b}^n}$$

for the lot of the one who holds that part &

$$\frac{1}{\overline{a+b}^n} \times \frac{n.n-1}{1\cdot 2}b^{n-2} + \frac{n.n-1.n-2}{1\cdot 2\cdot 3}b^{n-3} + \frac{n.n-1.n-2.n-3}{1\cdot 2\cdot 3\cdot 4}b^{n-4}$$
$$+\frac{n.n-1.n-2.n-3.n-4}{1\cdot 2\cdot 3\cdot 4\cdot 5}b^{n-5} + \frac{n.n-1.n-2.n-3.n-4.n-5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}b^{n-6}+$$
$$+\frac{n.n-1.n-2.n-3.n-4.n-5.n-6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}b^{n-7} + \frac{n.n-1.n-2.n-3.n-4.n-5.n-6.n-7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}b^{n-8}$$
$$=\frac{\overline{a+b}^n - b^n - nb^{n-1}}{\overline{a+b}^n}$$

for the one who holds the contrary part, thus as the Author has found.

Problem of points

The second Problem is not different from the one which one finds resolved on page 177[27] of my Book, but in this that one makes $a = b$; I have supposed this formula in the solution in many other problems.

[27]See page 244 of the second edition. *Translator's note*: This and all other page references are to the 1st edition.

The third Problem is found resolved word for word in my Letter[28] of the first March 1712.

The fourth Problem is a kind parallel to the preceding, & as I have already remarked in my Letter of the first March 1712, all the similar questions where the question is, in supposing equal the lots of the Players, to determine what is the facility that each of the Players has to win in a game, where one will have more points than the other, have no difficulty than that which one finds in the resolution of the equalities; because it is always the same method to suppose the expression of the lot of each of the Players $= \frac{1}{2}$ when there are two Players, $= \frac{1}{3}$ when there are three, $= \frac{1}{4}$ when there are four.

The 5th Problem is resolved on page 144[29] of my Book, & the formula is the same in it.

The Lemma which follows is quite curious, but it is drawn from page 141[30] of my Book, & I have sent the solution of it to Mr. your Uncle in a very general formula. My Letter is from 15 November 1710.

Problems 6 & 7 are a very ingenious extension of the 5th Problem. I do not know if the limits marked by the Author are perfectly correct. I would well wish to know if one could at all have by another way the solution of this Problem.

Problem of points

Problem 8 is resolved in the same manner as on page 175[31] of my Book, & next in my Letter of the eighth of June 1710; but I swear that I am not at all content with these solutions; it is a great fault, it seems to me, to be obliged to examine in detail what are in one same term the arrangements of letters favorable to Pierre, Paul & Jacques, an inconvenience which is not found at all in the case of two Players, & which it would be necessary to try to surmount in the case of many.

Huygens Problem 5

Problem 9 is the last of the five proposed by Mr. Huygens. I myself have noticed in reading it that Mr. Moivre had observed the fault that I have made by putting a false enunciation at the head of this Problem. I have indicated to Mr. your Uncle that which has given place to this mistake. The Problem which follows is quite well resolved, Mr. your Uncle has given the same solution of it in the Letter which he has given me the honor to write me, dated on 17 March 1710.

Huygens Problem 2

The 11th which is the 2nd of the five proposed by Mr. Huygens is resolved otherly than in my Book page 158;[32] this comes from that we have differently understood the enunciation: I do not know who of we two has taken the true sense of the Author. I have found that the number of black casts being b, & the number of white tickets a, the number of Players q, the numbers interposed from q to q of order a, of the figurate numbers, page 80,[33] will give the lots of the Players. This remark which has appeared to me curious gives the facility in order to find the particular formulas, proper to shorten the calculation, & without which it would be impossible to find the lots of the Players, when a & b are large numbers, I have found that the number of Players, q, being 3, as in the Problem of Mr. Huygens, the

[28] *Translator's note*: The problems are introduced in the letter of 10 November 1711 (pp. 332-334) and resolved in the letter of 1 March 1712 (pp. 341-344).

[29] Page 231 of the 2nd Edition.

[30] Page 46 of the 2nd edition.

[31] Page 242 of the 2nd edition.

[32] Page 219 of the 2nd edition.

[33] Page 2 of the 2nd edition.

formula

$$\frac{1}{6} \times n \times p^3 - 3npp - \frac{n \times \overline{n-1}}{1.2} \times 9pp + 2np + \frac{n.n-1}{1.2} \times 45p$$
$$+ \frac{n.n-1.n-2}{1.2.3} \times 54p - 1 \times \overline{n-1} \times 10 + \frac{n-1.n-2}{1.2} \times 46$$
$$+ \frac{n-1.n-2.n-3}{1.2.3} \times 63 + \frac{n-1.n-2.n-3.n-4}{1.2.3.4} \times 27,$$

divided by

$$\frac{p.p-1.p-2.p-3}{1.2.3.4},$$

gave the lots of the Players, I suppose $p = b + q$, & $n = \frac{p}{q}$.

It is necessary to remark that in order to find by this formula the lot of Paul, one must, 1 °, understand by n a quantity equal to the quotient of $p - 1$ divided by q. 2 ° That it is necessary to substitute everywhere $p - 1$, & its powers in the place of p, & of its powers, & for the lot of Jacques, 1 °, understand by n a quantity equal to the quotient of $p - 2$ divided by q. 2 ° To substitute everywhere $p - 2$, & its powers in the place of p, & of its powers. Thus supposing, for example, three Players, 58 black tokens and four white, one will have all in one coup the lots of Pierre, of Paul & of Jacques as these three numbers 198345, 185745, 173755. I myself is served in order to find this formula from the method that I have given in order to find the sum of the figurate numbers interposed as one will wish, & raised to any power: this method will furnish easily some formulas for all the similar cases.

Huygens Problems

I know not why the Author has given the labor to resolve in Propositions 12, 13, & 14 of his Book the Problems posed by Mr. Huygens which are already resolved in mine; because besides these Problems are too facile in order to stop anew; the Author has well seen by the Corollary on page 157[34] that the way of infinite series was not at all unknown to me, & that it was used. If this Author had wished to push this matter, & to teach us some new things, he had been able to seek the sum of the infinite series that one finds in this Corollary: this is in what consists all the difficulty of these sorts of questions.

I have observed by design that this research was not easy; & as there is found an infinity of series in which the exponents have their 2nd, 3rd, 4th, &c. difference constant; this discovery would be a great extension, & one of extreme utility.

Huygens Problem 4

The Author has given the 4th Problem of Mr. Huygens a sense different from the one that I give to it, also it is found differently; for me I believe to have taken the true, & it would be necessary, it seems to me, that the word minimum must be found in the enunciation, in order that the one of Mr. Moivre was preferable. Although it may be, nothing is more indifferent, each solution is only a particular example of my Proposition 13, page 94.[35]

Pool

Problem 15 is our Problem of the Pool of which I have sent you the solution in my Letter of 10 April 1711, I have been quite surprised to find this very risky Corollary by the Author: *Si plures sint collusores, ratio sortium eadem ratiocinatione invenietur*. You have made me understand, Sir, that the application of this Problem to the case of four & of five & of six Players was infinitely more difficult than that which is limited to three Players. The way of the infinite series that Mr. Moivre employs, & which is also employed in my Letter, is easy for three Players, but absolutely impractical for many Players.

Problems 16 & 17 are only two very simple cases of one same Problem, it is nearly the only which has escaped me of all those which I find in this Book. Although the Author

[34] Page 217 of the 2nd edition.

[35] Page 26 of the 2nd edition.

may make profession in the Preface to generalize all, it seems to me that he would have been able to render the Problem more curious & of a greater extent, by supposing that the number of bowls of Pierre is n, & the number of bowls of Paul m, here is that which I have found by reading the Problem of the Author. Let A be the lot of Paul when one point is lacking to him, & one point to Pierre; B his lot when two points are lacking to him & one point to Pierre; C his lot when three points are lacking to him & one point to Pierre, &c. one has

$$A = \frac{m}{m+n},$$
$$B = \frac{m.m-1+m\times n\times A}{m+n.m+n-1},$$
$$C = \frac{m.m-1.m-2+m.m-1\times n\times A+m\times n\times m+n-2\times B}{m+n.m+n-1.m+n-2}$$
$$D = \frac{m.m-1.m-2.m-3+m.m-1.m-2\times n\times A+m.m-1\times n\times m+n-3\times B}{m+n.m+n-1.m+n-2.m+n-3}$$
$$+\frac{m\times n\times m+n-3\times m+n-2\times C}{m+n.m+n-1.m+n-2.m+n-3}$$
$$E = \frac{m.m-1.m-2.m-3.m-4+m.m-1.m-2.m-3\times n\times A+}{m+n.m+n-1.m+n-2.m+n-3.m+n-4}$$
$$+\frac{m.m-1.m-2.n\times m+n-4\times B+m.m-1\times n\times m+n-4\times m+n-3\times C}{m+n.m+n-1.m+n-2.m+n-3.m+n-4}$$
$$+\frac{m\times n\times m+n-4\times m+n-3\times m+n-2\times D}{m+n.m+n-1.m+n-2.m+n-3.m+n-4}$$
$$F = \text{\&c.}$$

One could again render the Problem more general, by supposing the forces:: $a:b$, it seems to me that the solution of it would be more difficult.

Problem 18 is a Problem of combinations, & has much in relation with the first; also the solution of each is drawn easily from my formula, & also from Proposition 30, page 136,[36] which is much more general, & gives how many chances in order to bring forth precisely certain faces, this which does not give the formula of the Author; for example, if one wishes to know how many chances to bring forth in eight coups one ace & one deuce only, I find that there is 229376 against 1450240, & generally, p, being the number of dice, q the number of different points that one must bring forth, f the number of faces of each die, the formula is

$$q.q-1.q-2.q-3.q-4.\text{ \&c.}\times \underset{q}{\overset{p}{\square}}\times \overline{f-q}^{\,p-q}.$$

I have again found that if one demands how many different coups which are able with eight dice to give precisely one ace & one deuce, neither more nor less, the number is 84; & generally the number of faces being f, the rank $f-1$ of the figurate numbers will give how many chances to bring forth precisely one ace. The rank $f-2$ of the figurate numbers will give how many chances to bring forth precisely one ace & one deuce. The rank $f-3$ of the figurate numbers will give how many chances in order to bring forth precisely one ace, one 2 & one 3, &c.

[36] Page 44 of the 2nd edition.

Thus one finds, for example, that playing with some dice which would have each twelve faces, there is one way in order to bring forth ace & deuce with two dice, six ways with three dice, 55 with four dice, 220 with five dice, 715 with six dice, 2002 with seven dice, 5005 with eight dice, &c. these numbers 1, 10, 55, 220, 715, 2002, 5005, &c. belong to the order $f-2$, of the figurate numbers which in this case is the 10^{th}.

Problem 19 has much in relation with the 5^{th} Problem; however the Author employs another method, it appears to me quite well invented, although it has perhaps the fault to not give at all a solution exact enough.

Duration of play

The rest of the Book contains seven propositions on a matter extremely curious to which I have the first thought; namely how long must a game endure where one plays always by reducing, this which I explain in my Book page 178,[37] & better yet in my last Corollary page 184[38] where I give this series

$$\frac{1}{4}+\frac{3^1}{4^2}+\frac{3^2}{4^3}+\frac{3^3}{4^4}+\frac{3^4}{4^5}+\text{\&c.}$$

in order to determine the odds that the game will end in less than 3, 5, 7, 9, 11, 13, &c. points to infinity. I end this Corollary with these words: *one will find without much effort some similar formulas for the other cases, & the research of it will appear curious*. The truth is however that this problem is not at all entirely easy, even with the help of the particular formula for the case of three games. I see with pleasure that Mr. Moivre is come at the end of this Problem in whole, & that his solution accords perfectly with ours. I am much in pain to know how this scholarly Geometer is arrived to this method to raise $a+b$ to the power n, to subtract the extreme terms from this product, & to multiply next the rest by the square of $a+b$, & thus in sequence as many times as there are of units in $\frac{1}{2}d$. a solution of this nature surprises me, & the more that the Author who had supposed equal the number of chances for Peter & for Paul coming to suppose it in any ratio, is obliged to take another route; instead that according to you & according to me the method is the same for the general & particular solution; this does not prevent that I regard highly this discovery, & in general all his Work, in which I am pleased to have given the occasion, in opening first the course. I appears to me first quite singular that he has filled it with some things of which we ourselves have conversed in our Letters; but it is natural that having made his Work out of mine, & wishing to push these matters, the same ideas have come to him as to us. I would have only wished, & it seems to me that equity demands it, that he had recognized with frankness that which I had right to claim in his Work. I am obliged to him of some very honest expressions of which he is served in his Preface in speaking of me & of my Book; but I know not in truth on what he is based when he says

> "Huguenius primus, quod sciam, regulas tradidit ad istius generis Problematum solutionem quas nuperrimus Autor Gallus variis exemplis pulchre illustravit, sed non videntur viri clarissimi ea simplicitate ac generalitate usi fuisse, quam natura rei postulabat: etenim dum plures quantitates incognitas usurpant, ut varias collusorum conditiones repraesentent, calculum suum nimis perplexum reddunt; dumque collusorum dexteritatem semper aequalem ponunt, doctrinam hanc ludorum intra limites nimis arctos continent."[39]

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[37] Page 277 of the 2nd edition.

[38] Page 276 of the 2nd edition.

[39] *Quote added b translator*: Huygens first, who I know, has related the rules to the solution of Problems of this kind which in recent times the French author with various examples has illustrated beautifully; but the illustrious men do not appear to have used that simplicity and generality, as the nature of the thing demanded:

I am not able to conjecture for what reasons this Author made these reproaches to me, & what motive carries him to pronounce against me, leaving me only the merit to have applied to some examples the supposed rules of Mr. Huygens, I call from this judgment to the Geometers who will read this that Mr. Huygens & Mr. Pascal, of whom the Author speaks not at all, have given on this matter.

The hope that you give me, Sir, to give me the honor of coming to see me here, gives me an infinite pleasure; I flatter myself that you yourself are not bored, you will find some persons who love much the people of spirit, & who honor you perfectly; you see also one of the rareties of France, a Princess daughter-in-law of Charles IX King of France, dead 140 years.

The formula that you have sent for the lots of the Players, when one supposes that the degrees of facility which they have to win changes alternatively is extremely good, & I have well extended it. I have also seen with much pleasure the application that you make of your method for the pool in the case of four Players; I do not know yet the demonstration, I hope to understand it from you. The Problems enunciated in the Thesis of your late Uncle are all extremely curious, I have already resolved the first, it seems to me that the last demand very great work; if I have the pleasure to possess from you this winter, it will be for me exercise. Pool

The Research of the truth is for sale, but the new Memoires de l'Académie are not yet printed, when they are I will make these Books part by the way that you have indicated to me. I exhort you strongly, Sir, to furnish us some novelties that you will find in England. Beyond the new edition of the Book of Mr. Newton, that the Geometers & the Philosophers await with so much impatience, one has spoken to me of a new Treatise on the integral calculus by Mr. Ditton; of a new system of Music, &. Inform you, I pray you, if one will print a Commentary of Mr. Gregory on the Book of Mr. Newton, that he showed me at Oxford some years ago. I flatter myself that you wish well to give to me your news when will have the leisure, & that you give me the justice to believe that one is not able to honor you more perfectly, nor to be more truthful than I am.

SIR,

Your very humble & very
obedient Servant R. de M...

Letter of M. (Nicolas) Bernoulli to M. de Montmort (p. 371–375)

At London this 11 October 1712.

SIR,

I am very comfortable, Sir, that you have noticed with me in your letter of 1 March that it is necessary to write $a^7 + 7a^6b + 13a^5bb - 21a^4b^3 - 35a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0$; in place of the equality $a^7 + 7a^6b + a^5bb - 15a^4b^3 - 29a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0$. The way that you follow in order to solve some problems similar to the ones for which you have found the preceding equation, seems to be good, you arrive always to the solution that I find myself, & it seems to me that your are mistaken to believe that one must await a better method; the solution of algebraic equalities is inevitable in these sorts of problems; & when these solutions are very difficult, it is necessary to contend with approximations. I have found the same sorts for the three Players Pierre, Paul & Jacques, in which each lacks Tennis

indeed while they use many unknown quantities, in order that they may represent the various conditions of the players, they render their calculation exceedingly intricate; and while they always set the skill of the players equal, they keep this theory of games within exceedingly narrow limits.

respectively 3, 2 1 points: you have forgotten in the quantity $b^4 + 4ab^3 + 2b^3c + 4abbc + 6aabb + bbcc$, which expresses the strength of Paul, the third term $2b^3c$.

The Remarks that you have made on the occasion of the formulas for the determinate and indeterminate cases of Proposition 30[40] of your Book, are assuredly very good, & the rule that you have given for finding the number of terms of any polynomial raised to any power is very correct. The manner that one draws from the Problem of the dice for the formulation of the terms of these polynomials, is certainly preferable to those that one draws of the general formula of the binomial.

[Here follows a discussion of a paper by Parent on a logarithmic curve.]

Births

...Here is that which I have found on the occasion of your discoveries. I myself am going to make you part of one that I have made lately on the occasion of an argument for Divine Providence, which one has inserted into the Philosophical Transactions. One has already spoken to me of this argument in Holland without saying to me that one had printed some part. This is an argument drawing on the regularity which one observes between the infants of one & and the other sex who are born each year in London. One asserts that if chance should govern the world, it would be impossible that the numbers of males & females approach so near one another during many years in succession, as they have done during 80 years, & one gives for reason that in throwing a great number of tokens, for example, 10000 at random, it is very unlikely that half fall heads & half tails, & again much less probable that this happen a great number of times in sequence. As one has reiterated the same thing here to me, & as one has demanded my sentiment above, I have been obliged to refute this argument, & to prove that there is a great probability that the number of males & females happen each year between some limits again smaller than those which one has observed during 80 years in succession. You sense well, Sir, that it would be a ridiculous thing, if one wished to prove that it is more probable that the number of boys will be rightly equal to the number of girls; but that the ratio between the number of the ones & the others will approach more near to the ratio of equality, is that of which I believe that you are persuaded. I have found in examining the List of infants born in London from 1629 to 1710 inclusively, that there are more males than females; & that by taking an average, the ratio of males to females is very near the ratio of 18 to 17, a little greater; whence I conclude that the probability, in order that there is born a boy, is to the probability in order that there is born a girl about as 18 to 17, & that thus among 14000 infants, which is very nearly the number of infants who are born per year in London, there will be about 7200 males & 6800 females. Now the year where there is born the greatest number of males, with respect to the one of the females, has been the year of 1661, in which there is born 4748 males & 4100 females; & the year where there were born the smallest number of males with respect to the number of females, is the year 1703, in which there is born 7765 males & 7683 females. I say that these limits are so great, that one is able to wager at least more than 300 against 1 that among 14000 infants the number of males & females will fall between these limits rather than outside.

Moivre

I have the pleasure to see here often Mr. de Moivre who has made a present to me of his Book *De Mensura Sortis*. He has said to me the he has sent to you also a Copy, & he awaits with impatience your sentiment on this Work. You will be astonished to find many of the Problems which we have resolved, & among others also the one of the duration of games by reduction, which he has resolved in a manner, quite different from ours, nonetheless very good & very curious. He has also resolved the Problem of the Pool for three Players

[40] See page 44.

by the way of infinite series, & he has advanced in a Corollary, that if the number of the Players were greater, one could find their lots by the same reasoning. As I have shown you the impossibility that there is in succeeding by this method of infinite series, I believe that you will make of this Corollary the same judgment that I have made of it. I have communicated to him the two Theorems which I have found, after having made him sense the difficulty to use his method, when the number of Players is greater than three.

I hope soon to pass again into France, & to have the honor to converse with you on these matters more agreeably than we have done until now in our Letters &c.

Letter of M. (Nicolas) Bernoulli to M. de Montmort (p. 375–387)

At Brussels this 30 December 1712.

SIR,

I have received your last Letter of 5 September only the 27th of the month of November; it had been sent to London in the time of my return from England to Holland, & resent from London to Holland; this long delay has also delayed my response which is here. I begin with that which regards our dispute on Her. I have not the same subject, Sir, to admire & to applaud in the responses of Mr. de Waldegrave, as has done Mr. the Abbé de Monsoury. I know very well that all the reasonings that Paul can make in order to be determined in a choice, Pierre can also make them in order to turn this game to the disadvantage of Paul; but notwithstanding this, I say that Paul does not do so well in following the maxim to guard the seven, as in following the maxim to change at the seven, here are the reasons of it: If it were impossible to decide this Problem, Paul having a seven would not know what choice to take; & in order to rid himself he would commit himself to pure chance, for example, he would put into a sack an equal number of white chips & of black chips, in the design of holding himself at the seven if he draws a white, & to change at the seven if he draws a black; he would put, I say, an equal number of whites & of blacks; because if he would put an unequal number, he would be more carried for one choice than for the other, that which is against the hypothesis. Pierre having an eight would do the same thing in order to see if he must change or not. Now each Player Paul & Pierre having the maxim of being committed thus to chance, the lot of Paul will be $\frac{774}{50.51}$ which is less than $\frac{780}{50.51}$, which is the lot of Paul when he changes at the seven; whence it follows that Paul takes a bad choice when he commits himself to chance, & that he has a better lot in changing at the seven. Therefore it is decided that Paul must change at the seven: Here is another demonstration of it which is based on the same circle as one opposes me. As we can always demonstrate whatever choice that Paul takes, that it is a bad choice, & that he would do better if he had taken the contrary choice, it is worth more to take the choice where one risks the least; now Paul changing at the seven risks only $\frac{36}{50.51}$, instead by guarding the seven he risks $\frac{60}{50.51}$; therefore it is worth more to change at the seven. The reasoning which has lead me to the solution which I have sent to you in my Letter of 10 November 1711, is scarcely different from this one that I just made. Before exposing it to you I wish that you accord me that which follows: I hold in order to demonstrate that if Paul holds to himself or the maxim of holding to himself at the seven, Pierre must hold to himself at the eight; likewise if Paul holds to himself or to the maxim of holding himself at the eight alone, Pierre must change at the eight. Therefore if one has demonstrated that Paul must either hold to himself or change at the seven, one has also demonstrated that Pierre must either hold to himself or change at the eight; & if one is convinced that it is true that Paul must hold to himself at the seven, for example, one is also convinced that it is true that Pierre must hold to himself at the eight; but one cannot make the return of Pierre to Paul;

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& by taking for foundation that Pierre holds to himself at the eight, we cannot, I say, in any manner draw the conclusion from it; therefore it is true that Paul must hold to himself at the seven; because it is certain that Paul must change at the seven, I must seek no more by another way that which he must do, because I know it already; & a man is ridiculous who knowing that he must make a choice wishes again to doubt & seek by other ways if he must make it or not. This put, here is how I have reasoned: Paul having the design to follow the maxim to guard the seven, examines that which can happen to him at worst with respect to the choice which Pierre will take; & he finds that the worst is when Pierre holds to himself at the eight; & that then his lot will be to the lot of Pierre as 2828 to 2697, or that by naming the money of the game 1, his expectation will be $\frac{2828}{5525}$. Next he examines also that which can happen to him at worst when he follows the maxim to change at the seven, & he finds that the worst is when Pierre follows the maxim to change at the eight, & that then his lot will be also $\frac{2828}{5525}$, Therefore the lot of Paul, by following the maxim to guard the seven, is at least $\frac{2828}{5525}$, & it is something more when Pierre follows the maxim to hold to himself at another card than at the eight. Likewise the lot of Paul, by following the maxim to change at the seven is more than $\frac{2828}{5525}$, when Pierre has the maxim of holding himself at another card than at the nine: Therefore the lot of Paul is always at least $\frac{2828}{5525}$, & that which must carry him to follow one maxim rather than the other, is the risk which Pierre courts by not encountering it correct. Now this risk is greater when Paul changes at the seven, than when he holds to himself at the seven: Therefore Paul must change at seven rather than hold to himself; & consequently Pierre must have the maxim to change at the eight; but one cannot return from Pierre to Paul, & to repeat; therefore Paul must have the maxim to guard the seven. This circle offends Logic, & it is impossible that it is a good reasoning: a proposition from which one draws an absurdity is false; now to draw its contradictory is to draw an absurdity; therefore the proposition is false, & consequently its contradictory is true. Now in our case its contradictory knows how to be true no more, because of the same circle; therefore this circle is ridiculous & a false reasoning. One deduces from the proposition A is contradictory B; therefore the proposition A is false, & the proposition B is true. Likewise one deduces from proposition B its contradictory A; therefore B is false & A is true; therefore this circle demonstrates that two contradictory propositions are both true & both false, that which is impossible; therefore there had been a fault in the reasoning, & I have not at all been wrong, when I say that our Sirs suppose two contradictory things at the time in making the circle. In a word when even it would not be true that Paul does better to change than to hold to himself, it would not be their circle which would demonstrate it. I believe that you will fall in accord with this, & you will hold this response sufficient to the objections which Mr. de Waldegrave & Mr. the Abbé de Monsoury have made to me. I salute very humbly these Sirs, & I thank them for the good opinion which they have of me.

Moivre

I am very glad that you have received the Book of Mr. de Moivre De Mensura Sortis. It is true that nearly all the Problems which are proposed there are resolved either in your Book or in our Letters. As I know that Mr. de Moivre would await with impatience the judgment which you would make on his Book, I have taken the liberty to send to him the principals of your remarks, I will touch here some.

Problem of points

The 1st Problem is a particular case of the 2nd which is resolved at page 177 of your Book[41] in the case of $a = b$; and I would like better to deduce the solution of the 1st problem from the general formula which expresses all the terms of the binomial $a + b$

[41]See page 244.

raised to any power, than from the formula

$$\overline{m-1}^{p-q} \times \frac{p.p-1.p-2}{1.2.3} \text{ \&c.}$$

The limits which the Author gives for Problems 5, 6, 7 are correct enough when q is a large number; we know besides that we can wager with advantage to bring forth sonnez in 25 trials, & that he would have disadvantage to endeavor 24 trials; now we multiply 35 which is the value of q by 0.693 which is the first limit indicated by the Author, & we will find 24.255.

I have at no point examined the formulas which you give on the occasion of the 11th Problem, the remark which you make that we can find the general solution of this Problem, by seeking the sums of the figurate numbers interposed as we would wish is quite correct; I would have resolved this Problem in the same manner, or by that which has served me to find the Problems on Pharaon or Bassette which are only a particular case of the one considered generally here. I do not believe that one can find the sum of the series parallel to those which you have given in the Corollary on page 157.[42]

You have quite well resolved the Problems of the bowls for the cases where there lacks one point to one of the Players, & to the other any number of points. I myself remember that Mr. de Moivre has said to me when I was at London, that he had the general solution of this Problem.[43] Corollary 3, *Si desteritates, &c.* is not more difficult than when one supposes equal forces; I have found that if the number of bowls of Paul is m, the number of bowls of Pierre n, their forces as a to b, the probability that Paul will win in a single round a given number of points q precisely neither more nor less, will be

$$\frac{nb}{\overline{n-q}\cdot a+nb} \times \frac{ma}{ma+nb} \times \frac{\overline{m-1}\cdot a}{\overline{m-1}\cdot a+nb} \times \frac{\overline{m-2}\cdot a}{\overline{m-2}\cdot a+nb} \times \frac{\overline{m-3}\cdot a}{\overline{m-3}\cdot a+nb} \times \text{\&c.}$$

it is necessary to take as many products as there are units in $q+1$.

Your formula $a.a-1.q-2$ &c. $\times \underset{q}{\overset{p}{\square}} \times \overline{f-q}^{p-q}$, or more simply $\overline{f-q}^{p-q}$ multiplied by as many products $p.p-1.p-2.p-3$ &c. as there are units in q, in order to express the number of cases in order to bring forth with any number of dice, a determined number of different faces neither more nor less is quite correct; but this Problem is not the same as the 18th Problem of Mr. de Moivre; because when one is proposed to bring forth, for example in eight trials, an ace & a deuce, one has also won when one brings forth many times an ace & a deuce; now these cases are excluded in your Problem.

The propriety which you observed in the horizontal bands, which serve to express all the different trials which there are in order to bring forth a determined number of different faces neither more nor less is very good; one deduces it easily from proposition 32[44] of your Book.

The method of M. de Moivre for the duration of the games that we play by reducing is very natural & founded on this that it is always necessary to subtract the cases for which it can happen that one of the Players win the écus of the other; the method when the number of écus of both is equal, is not different from what he uses when the number of their écus Duration of play

[42]See page 218.
[43]I have sent it to Mr. J. Bernoulli in a Letter of 20 September 1712. See here page 248.
[44]See page 35.

is unequal, what one makes in the first two operations all at once, because of the equality that there is on all sides.

Pool

As you greatly wish to see my method & my demonstrations for the pool, I am going to report to you here all at length. I have differed this response a little from what I would have sent you otherwise from Holland, in order to give me the leisure to recall my ideas, & to put me in a state to content you entirely. This is that solution of the three Problems which you have proposed on the pool which I prefer to everything which I have found until now in these matters. Here are the reasonings which I have made in order to succeed at these three Problems; I offer them to you methodically & all at length in the two Tables following, in order to render me more intelligible.

PROBLEM I.

Many players of whom the number is $n+1$ play a pool,we demand what is the probability that each has to win the pool.

SOLUTION. Let t be called the expectation to win that one of the two who enter first into the game has, u the expectation that the one who enters second into the game has, x the expectation of the third, y that of the fourth, z that of the fifth, &c. Let moreover a be called the expectation to win the pool which a Player who enters into the game has, & who plays against one who has not yet won some trials; b the expectation of the one who enters into the game & who plays against one who just won one trial; c the expectation of the one who plays against one who just won two trials; d the expectation of the one who plays against one who just won three trials, &c. Let further p be called the lot or expectation of the one who exits from the game leaving a Player who has won one match; q the lot of the one who exits from he game leaving a Player who has two trials; r the lot of the one who exits from the game leaving a Player who has three trials; &c. Thus put we will have the following equations marked No. 1, No. 2, No. 3, &c. up to No. 10 in Table I.[45]

$$\begin{array}{lccccccccccccl} & t & + & t & + & v & + & x & + & y & + & z & +\&c. & = 1 \quad \text{No. 1} \\ \text{No. 2} & \| & & \| & & \| & & \| & & \| & & \| & & \\ & a & & a & & b & & \frac{1}{2}c+\frac{1}{2}b & & \frac{1}{4}d+\frac{1}{4}c+\frac{1}{2}b & & \frac{1}{8}e+\frac{1}{8}d+\frac{1}{4}c+\frac{1}{2}b & \&c. & \end{array}$$

The equation marked No. 1, is evident; because the lots or the expectations of all the Players taken together must make 1 or an entire certitude: the other equations are found in the manner that I just explained. Among the equations marked No. 2, we find, for example, $z=\frac{1}{8}e+\frac{1}{8}d+\frac{1}{4}c+\frac{1}{2}b$; because the one who enters the fifth into the game will play against one who will have won either 4, or 3, or 2, or 1 match; now there are odds $\frac{2}{16}$ or $\frac{1}{8}$ that one or the other of the first two Players win four games in sequence; & $\frac{1}{8}$ of probability that he will win against one who has won three matches, $\frac{1}{4}$ that he will win against one who has won tow matches, & $\frac{1}{2}$ that he will win against one who has won one match; therefore his lot or $z=\frac{1}{8}e+\frac{1}{8}d+\frac{1}{4}c+\frac{1}{2}b$.

Among the equations No. 3, we find, for example $c=\frac{1}{2}r+\overline{\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\frac{1}{2^n}}\times p+\frac{1}{2^n}\times 1$; because there are odds of $\frac{1}{2}$ that a Player who newly enters into the game will not win any match, $\frac{1}{4}$ that he will win only one, $\frac{1}{8}$ that he will win only two, $\frac{1}{16}$ that he will win only three, &c. $\frac{1}{2^n}$ that he will win all the games which he must less 1, and $\frac{1}{2^n}$

[45]*Translator's note*: I have divided Table I and inserted the various Equations as appropriate into the text. Table II, however, is intact.

that he will win all the games which he must, if he wins none of them, it leaves a Player who has won three trials, since we suppose in this example that he plays against a Player who has already won two trials; if he wins some of them, but not all that he must for himself, he exits the game, leaving a Player who has won one match; therefore his lot or c is $\frac{1}{2}r + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1$.

$$\left.\begin{array}{c|c|l}
\multicolumn{3}{l}{\text{Enter}} \\
0 & a & a = \overline{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1 \\
1 & b & b = \overline{\frac{1}{2} \times q + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1 \\
2 & c & c = \overline{\frac{1}{2} \times r + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1 \\
3 & d & d = \overline{\frac{1}{2} \times s + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1 \\
4 & e & \\
\vdots & \vdots &
\end{array}\right\} \text{No. 3}$$

Equations, No. 4, are found by a parallel reasoning; because a Player who exits from the game leaving, for example, a Player who has won one match, acquires the expectation either of the one who enters second into the game, or of the one who enters third, or of the one who enters fourth, &c. according as the Player who has left in the game won either one, or two, or three, &c. trials less than it is necessary for him to win the pool.

$$\left.\begin{array}{cll}
\multicolumn{3}{l}{\text{Exit}} \\
1 & p & p = \frac{1}{2^{n-1}} \times v + \frac{1}{2^{n-2}} \times x + \frac{1}{2^{n-3}} \times y + \frac{1}{2^{n-4}} \times z + \cdots \\
2 & q & q = \frac{1}{2^{n-2}} \times x + \frac{1}{2^{n-3}} \times y + \frac{1}{2^{n-4}} \times z + \cdots \\
3 & r & r = \frac{1}{2^{n-3}} \times y + \frac{1}{2^{n-4}} \times z + \cdots \\
4 & s & s = \frac{1}{2^{n-4}} \times z + \cdots \\
\vdots & \vdots &
\end{array}\right\} \text{No. 4}$$

Equations No. 5, are found by the subtraction of equations No. 3; & those No. 6, by the subtraction of equations No. 4. Equations, No. 7, are found by substituting into equations, No. 5, the values found in equations No. 6.

$$\begin{array}{ccccccc|l}
a - b & = & \frac{1}{2}p - \frac{1}{2}q & = & \frac{1}{2^n} \times v & = & t - v & p - q = \frac{1}{2^{n-1}} \times v \\
b - c & = & \frac{1}{2}q - \frac{1}{2}r & = & \frac{1}{2^{n-1}} \times x & = & 2v - 2x & q - r = \frac{1}{2^{n-2}} \times x \\
c - d & = & \frac{1}{2}r - \frac{1}{2}s & = & \frac{1}{2^{n-2}} \times y & = & 4x - 4y & r - s = \frac{1}{2^{n-3}} \times y \\
\multicolumn{3}{c}{\text{No. 5}} & \multicolumn{2}{c}{\text{No. 7}} & \multicolumn{2}{c|}{\text{No. 9}} & \text{No. 6}
\end{array}$$

Equations, No. 8, are found by seeking the values of a, b, c, d, &c. by equations No. 2; & these values being substituted into equations No. 5, we will have equations No. 9, which were compared with equations No. 7, giving equations No. 10, & these last equations furnish my first Theorem.

$$\left.\begin{array}{l|l}
\multicolumn{1}{c}{\text{No. 8}} & \\
a = t & v = t \times \frac{2^n}{1+2^n} \\
b = v & x = v \times \frac{2^n}{1+2^n} \\
c = 2x - b = 2x - u & y = x \times \frac{2^n}{1+2^n} \\
d = 4y - c - 2b = 4y - 2x - v &
\end{array}\right\} \text{No. 10}$$

PROBLEM II.

Being posed that which in the preceding Problem we demand what is the advantage or the disadvantage of each Player.

SOLUTION. As I am being served by the letters a, b, c, d, &c. p, q, r, s, &c. t, u, x, y, z, &c. to express the different probabilities that the Players have to win according to the different states in which they can be found; thus I will be served by similar capital letters A, B, C, D, &c. P, Q, R, S, &c. T, U, X, Y, Z, &c. to express the portion that each Player can claim in these different states; I suppose also that one puts into the game only when one comes to lose against a Player, & I call this stake 1. By following some reasonings similar to those which we have made in the preceding Problem, we will have the equations marked No. 1, No. 2, &c. up to No. 13 of the 2nd Table. In the equations No. 2, we have, for example $Y = \frac{1}{4}D + \frac{1}{4} \times \overline{C + c} + \frac{1}{2} \times \overline{B + 2b}$; because if the one who enters fourth into the game is obliged to play against one who has won three matches, his expectation is D; if he is obliged to play against one who has won two matches, his expectation is $C + c$; I add c to C, because the one who enters fourth finds three écus put into the game, instead that C is the expectation of the one who players against a Player who has two matches under the supposition that he has only two écus into the game; because the letters A, B, C, D, E, &c. P, Q, R, S, &c. signify the lots of the Players in the first entrances and exits; it is necessary therefore to add to C the portion which he can claim of that écu of surplus; now as in this state the probability to win the pool or to win this écu, is c, this portion will be $c \times 1$. Thus if he plays against a Player who has won only one match, it is necessary to add to the expectation B still $2b$, because he finds two écus more in the game than the one finds who plays against a Player who has won one match, & the probability of winning these two écus is b, & consequently the portion which he can claim is $b \times 2$. The reasoning which we make for equations No. 3 & No. 4 is immediately parallel to the one which we just made for No. 2, & which we have made for equations No. 3 & No. 4 of the preceding Problem. Equations, No. 5 and No. 6, are found as in the preceding Problem. Equations, No. 7, are found by substituting the first equation of No. 3, Table I, into equations No. 5. Equations, No. 8, are found by substituting the 1st equation of No. 4, Table I, into equations No. 6. Equations No. 9, are found by substituting equations No. 8 into equations No. 7. Equations No. 10, are found by seeking the values of A, B, C, D, &c. through equations No. 2 of Table I & 2, or No. 2 of the 2nd, & No. 8 of the 1st Table; & these values being substituted into the equations No. 5, we have equations No. 11, which compared with equations No. 9, form equations No. 12; & these equations No. 12, compared with equations No. 10 of Table I, give equations No. 13, which furnishes my second Theorem.

PROBLEM III.

Being put that which above, we demand what is the probability that the pool will be won precisely after a number of given trials.

SOLUTION. Let be expressed by this sequence of letters a, b, c, d, e, f, &c. the probabilities that the pool will be ended precisely at n, $n + 1$, $n + 2$, $n + 3$, &c. trials; it is evident that there must be at least n trials, since it is necessary to win n matches in sequence, & that the probability so that one of the first two Players wins first the n matches is $\frac{1}{2^{n-1}}$; because there are odds for each of the first two Players of $\frac{1}{2^n}$; therefore a will be $= \frac{1}{2^{n-1}}$. The values of the other letters are found always in a like manner; for example, the sixth term f is equal to $\frac{1}{2}e + \frac{1}{4}d + \frac{1}{8}c + \frac{1}{16}b$+&c. where it is necessary always to take as many preceding terms as there are units in $n - 1$; whence it follows that the first term being given, we will have all the following.

I will give the demonstration of it in a particular example, because this will be the same for all the other cases. Let, for example, the number of Players be $= 5$, we demand what is the probability that the game will end in precisely ten trials. It is evident that the one who must win the pool at the 10th match, must enter into the game after the 6th match, & that he must win four trials in sequence. Now he can enter into the game finding a Player who has won either 1, or 2, or 3 matches; if he finds a Player who has won one match, there are as much odds that the pool will be decided at the 9th trial, as there are odds after he has vanquished his adversary as he will win it himself at the 10th trial; therefore before he has beat his adversary the probability that he will win the pool will be the half of this expectation, that is to say $\frac{1}{2}f$ (I call $g,\ f,\ e,\ d,\ c,$ &c. the probability that one will win the pool at the 10th, 9th, 8th, 7th, &c. trial).

If he finds by entering into the game a Player who has two trials, there are as much odds that his adversary will win at the 8th match, as there are odds after he will have vanquished two of his adversaries, that he will win himself at the 10th match; now as the probability that he beats two Players in sequence is $\frac{1}{4}$, the probability that he will win the pool at the 10th match, will be in this case $= \frac{1}{4}e$. If by entering into the game he finds a Player who has won 3 matches, there is as much odds that his adversary will win at the 7th trial as he will have odds after having beat his first three adversaries, he will win yet the 4th; now the odds are $\frac{1}{8}$ that he he will beat three adversaries in sequence; therefore his probability to win the pool in the 10th trial in this case, will be $\frac{1}{8}d$; therefore all these three probabilities taken together are $g = \frac{1}{2}f + \frac{1}{4}e + \frac{1}{8}d$; *That which is was necessary to demonstrate.*

We demonstrate in the same manner that $f = \frac{1}{2}e + \frac{1}{4}d + \frac{1}{8}c$; $e = \frac{1}{2}d + \frac{1}{4}c + \frac{1}{8}b$; $d = \frac{1}{2}c + \frac{1}{4}b + \frac{1}{8}a$; $c = \frac{1}{2}b + \frac{1}{4}a$; $b = \frac{1}{2}a$. Now a in this case of five Players is $= \frac{1}{2^{n-1}} = \frac{1}{8}$; therefore $b = \frac{1}{16}$, $c = \frac{2}{32} = \frac{1}{16}$, $d = \frac{4}{64} = \frac{1}{16}$, $e = \frac{7}{128}$, $f = \frac{13}{256}$, $g = \frac{24}{512} = \frac{3}{64}$, &c. *That which it was necessary to find.* The sum of as many of these terms $a,\ b,\ c,\ d,\ e,\ f,$ &c. as there are units in p, will express the probability that the pool will be finite in at least $n + p - 1$ trials. If we wish to have a formula to express this sum, we will have by putting p for the number of terms

$$\frac{p+1}{1.2^n} - \frac{p-n.p-n+3}{1.2.2^{2n}} + \frac{p-2n.p-2n+1.p-2n+5}{1.2.3.2^{3n}} - \frac{p-3n.p-3n+1.p-3n+2.p-3n+7}{1.2.3.4.2^{4n}} + \text{\&c.}$$

for the expression of this formula; & the formula to express any term of this series $a,\ b,\ c,\ d,\ e,$ &c. of which the quantity is p, will be

$$\frac{1}{2^n} - \frac{p-n+1}{1.2^{2n}} + \frac{p-2n.p-2n+3}{1.2.2^{3n}} - \frac{p-3n.p-3n+1.p-3n+5}{1.2.3.2^{34n}} + \text{\&c.}$$

We will find easily the demonstration of these formulas, by supposing that the numerator of each term of this series $a,\ b,\ c,\ d,$ &c. is the sum of all the preceding, instead that it is only the sum of as many preceding as it is necessary to win of matches less one; & by subtracting next that which by this consideration we will have taken too much: I believe you ought be cautioned here in passing that in my Letter of 10 November 1711, I have called p that which I call here $n - p + 1$. Here is, Sir, all that which I have to communicate Moivre
to you on my method for the pool, I hope that you will be content with it. I have at no point communicated this method to Mr. de Moivre, I believe that if he had seen it he would have recognized that that which he has employed in his Book for the case of three Players, is completely useless for the case of a greater number of Players, & that thus his methods do not always have the advantage to be so general as he thinks.

I do not know if Mr. de Moivre has had plan in his Preface[46] to bring so much reproach on you as you believe; for me I hold the methods which you have given in your Book sufficient enough to resolve all the general Problems of Mr. Moivre, most of which differ from yours only in the generality of the algebraic expressions, & I am persuaded that Mr. Moivre himself will do the justice to admit to you that you have pushed this material much further than Mr. Huygens & Mr. Pascal have done, who have given only the first elements of the science of chance, & that after them you have been the first who has published some general methods for this calculus.

Caramuel

A Jesuit[47] named Caramuel, who I have cited in my Thesis, has wished to push these matters, & even critique Mr. Huygens in the Treatise which he names KYBEIA, & which he has inserted in his great Works of Mathematics; but as all that which he gives is only a mass of paralogisms, I count it for nothing.

[46]From the preface to *De mensura sortis*: "Huygens was the first that I know who presented rules for the solution of this sort of problems, which a French author has very recently well illustrated with various example; but these distinguished gentlemen do not seem to have employed that simplicity and generality which the nature of the matter demands: moreover, while they take up many unknown quantities, to represent the various conditions of gamesters, they make their calculation too complex; and while they suppose that the skills of gamesters is always equal, they confine this doctrine of games within limits too narrow."

[47]*Translator's note*: Juan Caramuel was a Cistercian. He wrote a large number of works including *Mathesis Biceps*.

TABLE II

No. 1

$$\begin{array}{lccccccccccccl}
 & T & + & T & + & V & + & X & + & Y & + & X & +\&c. & = 0 \\
\text{No. 2} & \| & & \| & & \| & & \| & & \| & & \| & & \\
 & A & & A & & B & & \frac{1}{2}C + \frac{1}{2}\times\overline{B+b} & & \frac{1}{4}D + \frac{1}{4}\times\overline{C+c} + \frac{1}{2}\times\overline{B+2b} & & \frac{1}{8}E + \frac{1}{8}\times\overline{D+d} + \frac{1}{4}\times\overline{C+2c} + \frac{1}{2}\times\overline{B+3b} & \&c. &
\end{array}$$

Enter

$$\left.\begin{array}{c|c|l}
0 & A & A = \frac{1}{2}\times\overline{P-1} + \frac{1}{4}\times\overline{P-1+p} + \frac{1}{8}\times\overline{P-1+2p} + \frac{1}{16}\times\overline{P-1+3p}\cdots\frac{1}{2^n}\times\overline{P-1+np-p} + \frac{1}{2^n}\times n \\
1 & B & B = \frac{1}{2}\times\overline{Q-1} + \frac{1}{4}\times\overline{P-1+2p} + \frac{1}{8}\times\overline{P-1+3p} + \frac{1}{16}\times\overline{P-1+4p}\cdots\frac{1}{2^n}\times\overline{P-1+np} + \frac{1}{2^n}\times\overline{n+1} \\
2 & C & C = \frac{1}{2}\times\overline{R-1} + \frac{1}{4}\times\overline{P-1+3p} + \frac{1}{8}\times\overline{P-1+4p} + \frac{1}{16}\times\overline{P-1+5p}\cdots\frac{1}{2^n}\times\overline{P-1+np+p} + \frac{1}{2^n}\times\overline{n+2} \\
3 & D & D = \frac{1}{2}\times\overline{S-1} + \frac{1}{4}\times\overline{P-1+4p} + \frac{1}{8}\times\overline{P-1+5p} + \frac{1}{16}\times\overline{P-1+6p}\cdots\frac{1}{2^n}\times\overline{P-1+np+2p} + \frac{1}{2^n}\times\overline{n+3} \\
4 & E & \\
\vdots & \vdots &
\end{array}\right\}\text{No. 3}$$

Exit

$$\left.\begin{array}{cll}
1 & P & P = \frac{1}{2^{n-1}}\times\overline{V+\overline{n-1}\times v} + \frac{1}{2^{n-2}}\times\overline{X+\overline{n-2}\times x} + \frac{1}{2^{n-3}}\times\overline{Y+\overline{n-3}\times y} + \frac{1}{2^{n-4}}\times\overline{Z+\overline{n-4}\times z} + \cdots \\
2 & Q & Q = \frac{1}{2^{n-2}}\times\overline{X+\overline{n-1}\times x} + \frac{1}{2^{n-3}}\times\overline{Y+\overline{n-2}\times y} + \frac{1}{2^{n-4}}\times\overline{Z+\overline{n-3}\times z} + \cdots \\
3 & R & R = \frac{1}{2^{n-3}}\times\overline{Y+\overline{n-1}\times y} + \frac{1}{2^{n-4}}\times\overline{Z+\overline{n-2}\times z} + \cdots \\
4 & S & S = \frac{1}{2^{n-4}}\times\overline{Z+\overline{n-1}\times z} + \cdots \\
\vdots & \vdots &
\end{array}\right\}\text{No. 4}$$

$$\begin{array}{lclclclcl}
B-A & = & \frac{1}{2}Q - \frac{1}{2}P + \frac{1}{4}p + \frac{1}{8}p + \frac{1}{16}p\cdots\frac{1}{2^n}\times p + \frac{1}{2^n} & = & \frac{1}{2}Q - \frac{1}{2}P + t - \frac{1}{2}p & = & -\frac{1}{2^n}\times V - \frac{n}{2^n}\times v + t\cdots & = & V - T \\
C-B & = & \frac{1}{2}R - \frac{1}{2}Q + \frac{1}{4}p + \frac{1}{8}p + \frac{1}{16}p\cdots\frac{1}{2^n}\times p + \frac{1}{2^n} & = & \frac{1}{2}R - \frac{1}{2}Q + t - \frac{1}{2}p & = & -\frac{1}{2^{n-1}}\times X - \frac{n}{2^{n-1}}\times x - \frac{1}{2^n}\times v + t\cdots & = & 2X - 2V - v \\
D-C & = & \frac{1}{2}S - \frac{1}{2}R + \frac{1}{4}p + \frac{1}{8}p + \frac{1}{16}p\cdots\frac{1}{2^n}\times p + \frac{1}{2^n} & = & \frac{1}{2}S - \frac{1}{2}R + t - \frac{1}{2}p & = & -\frac{1}{2^{n-2}}\times Y - \frac{n}{2^{n-2}}\times y - \frac{1}{2^{n-1}}\times x - \frac{1}{2^n}\times v + t\cdots & = & 4Y - 4X - 2x - v
\end{array}$$

No. 5 No. 7 No. 9 No. 11

$$
\begin{aligned}
Q-P &= -\tfrac{1}{2^{n-1}}\times\overline{V+\overline{n-1}\times v}+\tfrac{1}{2^{n-2}}\times x+\tfrac{1}{2^{n-3}}\times y+\tfrac{1}{2^{n-4}}\times z+\cdots &&= -\tfrac{1}{2^{n-1}}\times V-\tfrac{n}{2^{n-1}}\times v+p\\
R-Q &= -\tfrac{1}{2^{n-2}}\times\overline{X+\overline{n-1}\times x}+\tfrac{1}{2^{n-3}}\times y+\tfrac{1}{2^{n-4}}\times z+\cdots &&= -\tfrac{1}{2^{n-2}}\times X-\tfrac{n}{2^{n-2}}\times x-\tfrac{1}{2^{n-1}}\times v+p\\
S-R &= -\tfrac{1}{2^{n-3}}\times\overline{Y+\overline{n-1}\times y}+\tfrac{1}{2^{n-4}}\times z+\cdots &&= -\tfrac{1}{2^{n-3}}\times Y-\tfrac{n}{2^{n-3}}\times y+\tfrac{1}{2^{n-2}}\times x-\tfrac{1}{2^{n-1}}\times v+p
\end{aligned}
$$

No. 6 No. 8

$$
\begin{aligned}
A &= T\\
B &= V\\
C &= 2X-V-v\\
D &= 4Y-2X-V-2x-2v
\end{aligned}
$$

No. 10

$$
\begin{aligned}
V &= \tfrac{T\times 2^n+t\times 2^n-nu}{1+2^n} &&= \tfrac{\overline{T+t}\times 2^n-nu}{1+2^n}\\
X &= \tfrac{V\times 2^n+v\times 2^{n-1}-\frac{1}{2}+t\times 2^{n-1}-nx}{1+2^n} &&= \tfrac{\overline{V+v}\times 2^n-nx}{1+2^n}\\
Y &= \tfrac{X\times 2^n+x\times 2^{n-1}-\frac{1}{2}+v\times 2^{n-2}-\frac{1}{4}+t\times 2^{n-2}-ny}{1+2^n} &&= \tfrac{\overline{X+x}\times 2^n-ny}{1+2^n}
\end{aligned}
$$

No. 12 No. 13.

Letter from Nicholas Bernoulli to Pierre de Montmort (pg. 388–393)

At Paris this 23 January 1713.

I send you the list of children of each sex born in London from 1629 until 1710, with my demonstrations of that which I have written to you touching the argument by which one wishes to prove that it is a miracle that the numbers of children of each sex born in London are not more distant from one another during 82 years in succession, & that by chance it will be impossible during so long a time they should be always contained between the limits as small as those which are observed in the list of 82 years. I claim that there is no reason to be astonished, and that there is a large probability in order that the number of males & females fall between the limits again smaller than those one has observed. In order to prove this, I suppose that the number of all the infants who are born each year in London is 14000, among whom there will be born 7200 males & 6800 females, so the number of children of each sex should follow exactly the ratio 18 to 17, which expresses the ratio between the easiness of the birth of a boy and that of the birth of a girl; or as the number of boys is sometimes greater, sometimes smaller than 7200, I take the limit: For example, in the year 1703, when the number of girls was the nearest to that of the boys, there are born in this year 7765 males and 7683 females, that is which in reducing the sum to 14000, makes 7037 males and 6963 females; the number of females has therefore surpassed the number 6800 by 163, & the number of males has been as much smaller than 7200. Now I will prove that there are great odds that among 14000 infants, the number of males will be neither greater nor lesser than 7200 by 163; that is to say, that the ratio of the males to the females will not be greater than 7363 to 6637, nor lesser than that of 7047 to 6963. To this end we imagine 14000 dice with 35 faces each of which 18 are white and 17 black. You know that in the terms of the binomial $18+17$, raised to 14000, we will give all the possible events with these 14000 dice to lead up to as many white faces as one would wish; namely, the first term of all of these events to lead up to all white faces; the second, to lead up to a black face & 13999 white; the 3rd, to lead up to two black faces & 13998 white, &c. Altogether the 6801st term will express all the events to lead up to precisely 6800 black faces & 7200 white; the 6638th term the events to lead up to 6637 black faces & 7363 white; and the 6964th term the events to lead up to 6963 black faces and 7037 white. The question is therefore to find what ratio there is between the sum of all the terms from the 6638th to the 6964th taken inclusively, & between the sum of all the other terms which are on this side of the 6638th, & beyond the 6964th. Now as these terms are seriously great, a singular artifice is necessary to find this ratio: here is how I myself take it. Generally let instead of 14000 the number of all the infants $= n$, the easiness of the birth of a male & of a female as m to f, instead of the ratio 18 to 17; & instead of the limit 163, take a limit some l; let also $p = \frac{n}{m+f}$, where $n = mp + fp$; in our example $mp = 7200$, & $fp = 6800$. I seek firstly for an approximation very near the ratio of the term of which the index is $fp+1$, to the term of which the index is $fp - l + 1$. By the law of progression of these terms, the term $fp + 1 = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \cdots \frac{n-fp+1}{fp} \times m^{n-fp} f^{fp}$, & the term $fp - l + 1 = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \cdots \frac{n-fp+l+1}{fp-l} \times m^{n-fp+1} f^{fp-l}$, therefore the ratio of the former to the latter is as $\frac{n-fp+l}{fp-l+1} \times \frac{n-fp+l-1}{fp-l+2} \times \frac{n-fp+l-2}{fp-l+3} \times \cdots \frac{n-fp+1}{fp} \times \frac{f}{m}\Big]^l$ to 1, or in putting np[48] in the place of $n - fp$ this ratio is as $\frac{mp+l}{fp-l+1} \times \frac{mp+l-1}{fp-l+2} \times \frac{mp+l-2}{fp-l+3} \times \cdots \frac{mp+1}{fp} \times \frac{f}{m}\Big]^l$ to 1; I suppose that the factors of the first term of this ratio except the last $\frac{f}{m}\Big]^l$; are in geometric

[48]Should be mp.

progression and their logarithms in arithmetic progression; this supposition is very nearly the truth, especially when n is a large number; the sum therefore of all their logarithms will be $\frac{1}{2}l \times \overline{\log \frac{mp+l}{fp-l+1} + \log \frac{mp+1}{fp}}$, that is to say the sum of the logarithms of the first & last factor, multiplied by the mean of the number of all the terms, to which if one adds the logarithm of $\left.\frac{f}{m}\right\rceil^{l}$, that is to say $l \times \log \frac{f}{m}$, one will have $\frac{1}{2}l \times \overline{\log \frac{mp+l}{fp-l+1} + \log \frac{mp+1}{fp}} + l \times \log \frac{f}{m}$, or $\frac{1}{2}l \times \overline{\log \frac{mp+l}{fp-l+1} + \log \frac{mp+1}{mp} + \log \frac{fp}{mp}}$ for the logarithm of the ratio sought. And by consequence the ratio will be the same as $\left.\overline{\frac{mp+1}{fp-l+1} \times \frac{mp+1}{mp} \times \frac{fp}{mp}}\right\rceil^{\frac{1}{2}l}$ to 1.

If one wishes to approach nearer the true value, one should divide this sequence of factors $\frac{mp+l}{fp-l+1} \times \frac{mp+l-1}{fp-l+2} \times \frac{mp+l-2}{fp-l+3} \times$ &c into many parts, & to suppose that the factors of each part are in geometric progression; but one has no need to do this; because all the values that one will find for these different assumptions will make very little difference one from the other; & all the same for this first assumption I would make this ratio a little greater than it is not, this excess would make very little significance with regard to this that I will disregard in the following.

If one takes at this time the terms which precede immediately the terms of index $fp+1$ & $fp-l+1$; namely those of which the index is fp & $fp-l$, the ratio of the former to the latter will be as $\left.\frac{mp+l+1}{fp-l} \times \frac{mp+l}{fp-l+1} \times \frac{mp+l-1}{fp-l+2} \times \cdots \frac{mp+2}{fp-1} \times \frac{f}{m}\right\rceil^{l}$ to 1; & consequently greater than $\left.\frac{mp+l}{fp-l+1} \times \frac{mp+l-1}{fp-l+2} \times \frac{mp+l-2}{fp-l+3} \times \cdots \frac{mp+1}{fp} \times \frac{f}{m}\right\rceil^{l}$ or $\left.\overline{\frac{mp+1}{fp-l+1} \times \frac{mp+p}{mp} \times \frac{fp}{mp}}\right\rceil^{\frac{1}{2}l}$ to 1, since each factor of the first sequence is greater than that to which the factor corresponds of the second. For the same reason the term of which the index is $fp-1$ will be to the term, of which the index is $fp-l-1$, a greater ratio than the term fp to the term $fp-l$; & the term $fp-2$ will be to the term $fp-l-2$ a greater ratio than the term $fp-1$ to the term $fp-l-1$, & also in reverse sequence always from one term to the first. This is why if one divides all the terms which precede the term $fp+1$ into some classes, of which each contains a number equal to the terms expressed by l, in commencing to compute at the term of which the index is fp; the first term of the first class will be to the first term of the second class a greater ratio than the term $fp+1$ to term $fp-l+1$; & the second term of the first class will be to the second of the second class a ratio greater again; & the third of the first class to the third of the second a ratio greater again, & also in sequence; therefore also all the terms of the first class taken together will be to all the terms of the second class taken together a greater ratio than the term $fp+1$ to the term $fp-l+1$. And by the same reason all the terms of the second class will have to all the terms of the third class; item, all the terms of the third to those of the fourth, &c. a greater ratio than the term $fp+1$ to the term $fp-l+1$; that is to say $\left.\overline{\frac{mp+1}{fp-l+1} \times \frac{mp+1}{mp} \times \frac{fp}{mp}}\right\rceil^{\frac{1}{2}l}$ to 1. Therefore if one names $\left.\overline{\frac{mp+1}{fp-l+1} \times \frac{mp+1}{mp} \times \frac{fp}{mp}}\right\rceil^{\frac{1}{2}l} = q$; & the sum of the terms of the first class $= s$, the sum of the terms of the second class will be smaller than $\frac{s}{q}$; & the sum of the third class smaller than $\frac{s}{qq}$, and that of the terms of the fourth class smaller than $\frac{s}{q^3}$, &c. Therefore the sum of all the classes, excepting the first, not withstanding the number of classes would make infinity, will be smaller than this $\frac{s}{q} + \frac{s}{qq} + \frac{s}{q^3} + \frac{s}{q^4}$, continued to infinity, that is to say smaller than $\frac{s}{q-1}$; whence it follows that the sum of the first class, that is to say of all the terms which are between the term $fp+1$ & the term $fp-l+1$, comprehending also the term $fp-l+1$, will be to the sum of all the preceding a ratio greater than $q-1$ to 1, or in putting for q its value, that $\overline{\frac{mp+1}{fp-l+1} \times \frac{mp+1}{mp} \times \frac{fp}{mp}}\Big]^{\frac{1}{2}l} - 1$ to 1; consequently in placing m instead of f, & f instead of m; the sum of all the terms which are between the terms $fp+1$ & the term $fp+l+1$, in comprehending the term $fp+l+1$, will be to the sum of all the others following to the last a greater ratio than $\overline{\frac{fp+l}{mp-l+1} \times \frac{fp+1}{fp} \times \frac{mp}{fp}}\Big]^{\frac{1}{2}l} - 1$ to 1. Therefore finally the sum of all the terms from the term $fp-l+1$ to the term $fp+l+1$, taken inclusively, without computing even the term $fp+1$ which is in the middle, will be to the sum of all the other terms to the less a greater ratio than the smaller of the two quantities

$$\overline{\frac{mp+l}{fp-l+1} \times \frac{mp+1}{mp} \times \frac{fp}{mp}}\Bigg]^{\frac{1}{2}l}$$

& $\overline{\frac{fp+l}{mp-l+1} \times \frac{fp+1}{fp} \times \frac{mp}{fp}}\Big]^{\frac{1}{2}l}$ less unit by unit; that which it is necessary to find.

We apply this at this time to our example, where

$$n = 14000, mp = 7200, fp = 6800, l = 163,$$

& we will find

$$\begin{aligned} &\frac{1}{2}l \times \overline{\log\frac{mp+l}{fp-l+1} + \log\frac{mp+1}{mp} + \log\frac{fp}{mp}} \\ &= \frac{163}{2} \times \overline{\log.\frac{7363}{6638} + \log.\frac{7201}{7200} + \log.\frac{6800}{7200}} \\ &= \frac{163}{2} \times \overline{0.0450176 + 0.0000603 - 0.0248236} = 1.6507254; \end{aligned}$$

the number of this logarithm is $44\frac{58}{100}$. In putting fp instead of mp, & mp instead of fp, we will find

$$\begin{aligned} &l \times \overline{\log}\frac{fp+l}{mp-l+1} + \log\frac{fp+1}{fp} + \log\frac{mp}{fp} \qquad \frac{1}{2} \\ &= \frac{163}{2} \times \overline{\log}.\frac{6963}{7038} + \log.\frac{6801}{6800} + \log.\frac{7200}{6800} \\ &= \frac{163}{2} \times \overline{-0.0046529 + 0.0000639 + 0.0248236} \\ &= 1.6491199; \end{aligned}$$

the number of this logarithm is $44\frac{58}{100}$; whence I conclude that the probability that among 14000 infants the number of males will neither be greater than 7363, nor smaller than 7037, will be to the probability that the number of males falls outside of these limits of a ratio greater to the less than $43\frac{58}{100}$ to 1.[49] Therefore one is able already to wager with advantage that in 82 times the number of males will not fall three times outside of these limits. Now in examining the list of infants born during the 82 years in London, you will find that the number of males were 11 times greater than 1763; namely in 1629, 39, 42, 46, 49, 51, 59, 60, 61, 69, 76; you will find also easily that one is able to wager more than 226 against 1 that the number of males will not fall in 82 years 11 times outside of

[49] *Translator's note*. If X is the number of times the number of males falls within these limits over 82 trials, then X has a binomial distribution with parameters $n = 82$ and $\pi = 43.58/44.58.97756 = .97756$.

A routine computation shows $P(X \geq 81) = .448231$ and $P(X > 80) = .72037$.

these limits.[50] You must remark also that if I have taken another limit greater than 163, but yet smaller than the greatest that one finds in this list, I would have found a probability much greater than 43 to 1, that the number of infants of each sex will fall each year instead between these limits than outside. Therefore there is no reason at all to be astonished that the numbers of infants of each sex are not more distant from one another, this that I wished to demonstrate. I myself remember that my late uncle demonstrated a similar thing in his tract *De Arte conjectandi*, which is being printed at present in Basel, namely, that if one wishes to discover by experiences often repeated the number of cases by which a certain event is able to happen or not, one is able to increase the observations in such a manner that finally the probability that we would find the true ratio that there is between the number of cases, would be greater than a given probability. When this book is published we will see if in these types of matters I have found an approximation as correct as it. I have the honor to be with perfect esteem,

Ars Conjectandi

SIR,

your very humble & very
obedient servant,
N. BERNOULLI

[50]*Translator's note*. I am unable to confirm this calculation. Taking X as in the previous footnote, $\mathrm{P}(X \geq 71) > .9999$.

List of male and female Infants born at London from 1629 to 1710.

	males	*females*		*males*	*females*
1629	5218	4683	1670	6278	5719
30	4858	4457	71	6449	6061
31	4422	4102	72	6443	6120
32	4994	4590	73	6073	5822
33	5158	4839	74	6113	5738
34	5035	4820	75	6058	5717
35	5106	4928	76	6552	5847
36	4917	4605	77	6423	6203
37	4703	4457	78	6568	6033
38	5359	4952	79	6247	6041
39	5366	4784	80	6548	6299
40	5518	5332	81	6822	6533
41	5470	5200	82	6909	6744
42	5460	4910	83	7577	7158
43	4793	4617	84	7575	7127
44	4107	3997	85	7484	7246
45	4047	3919	86	7575	7119
46	3798	3395	87	7737	7114
47	3796	3536	88	7487	7101
48	3363	3181	89	7604	7167
49	3079	2746	90	7909	7302
50	2890	2722	91	7662	7392
51	3231	2840	92	7602	7316
52	3220	2908	93	7676	7483
53	3196	2959	94	6985	6647
54	3441	3179	95	7263	6713
55	3655	3349	96	7632	7229
56	3668	3382	97	8062	7767
57	3396	3289	98	8426	7626
58	3157	3013	99	7911	7452
59	3209	2781	1700	7578	7061
60	3724	3247	1	8102	7514
61	4748	4107	2	8031	7656
62	5216	4803	3	7765	7683
63	5411	4881	4	6113	5738
64	6041	5681	5	8366	7779
65	5114	4858	6	7952	7417
66	4678	4319	7	8379	7687
67	5616	5322	8	8239	7623
68	6073	5560	9	7840	7380
69	6506	5829	10	7640	7288

Extract of a Letter of Mr. de Montmort to Mr. N. Bernoulli (pages 395–400)

At Paris this 20 August 1713

Madame the Duchess of Angoulême died at Montmort the 12 of this month; although this Princess was, as you know, in an extremely advanced age, she has conserved her reason pure & firm until the last moment. I do not doubt, Sir, that the remembrance of the virtues of this good Princess & of the affection that she bore you, renders her loss very sensible to you.

Her death, beyond the sorrow that it has caused me, gives me some cares & infinite pains; it is necessary for me to pass all my time to call the Ministers to action: what occupation for a Philosopher.

I myself will not be able to undertake today with you of you geometric matters; I have for that neither enough leisure, nor enough tranquility of mind. I will limit myself to fill this Letter, to teach you the little that I know of news of Literature. . .

Although this letter is already very long, & much more without doubt than would be necessary, I am able to resolve myself to end without saying to you something on the subject of your two Letters, the one of 11 October 1712, the other of 23 January 1713, to which I have not yet made response, having not had the time until today to examine them & to understand them.

Moivre

I agree with the remarks that you have made in your Letter of 5 September 1712 on the subject of those that I had sent you on the Book of Mr. Moivre.

Her

Your reasonings on Her have converted our Sirs not at all; they find them very delicate & very subtle, but they assure not to be convinced; & as I know their straightforwardness & their frankness, I can be their guarantee that they say that which they think.

I am charmed by your two Problems, the one on the pool, the other for comparing in any one perpendicular band of the arithmetic triangle: *Extremos terminos cum intermediis quibuscumque.* All this was in truth quite difficult & of great labor. You are a terrible man; I believe that in order to have the set out before I would not be so soon related, but I see well that I myself am deceived: I am at present well behind you; & forced to put all my ambition to follow you from afar. I I were of jealous humor, in order to overrate you, I would love you less, but no, Sir, & your superiority & your great talents have only increased my attachment, & if I dare to serve myself of this term, my sincere friendship for you,

R. D. M.

Extract of a Letter of Mr. N. Bernoulli to Mr. de Montmort
of 9 September 1713

The Book *Ars Conjectandi* of my late Uncle just exited from the press, the Library has said to me that it has sent a Copy of it by Post to Mr. Koenig; if you are curious to see it, you can make a withdraw by someone of house Mr. Koenig, to which I will give notice, by awaiting that I can sent to him some other Copies for you & for my other friends in Paris. I will have recovered nothing new for you. I have had prevented from some time to make some new researches on the matter of chance, this is why I can communicate nothing to you; however in return of the Problems which you have proposed to me, & of which I will examine the solutions when I will have the leisure, I propose to you some others which merit your application. *First Problem.* A & B play alternately with one die with four faces marked with $0,\ 1, 2, 3,$ A puts a certain sum of écus into the game, & begins to play; & after having brought forth either 0, or 1, or 2, or 3 points, he takes back as many écus

5 Problems

from the game as he has brought forth points, & cedes the dice box to B, who takes also besides as many écus as he has brought forth points; but if he brings forth the face marked 0, he pays an écu A; & if he brings forth a greater number of points than there remains of écus in the game, not only he takes nothing, but he puts as many écus into the game as he has brought forth of points too much, & they continue thus until that which there remains nothing in the game; I demand what is the sum that A must put into the game in order that their lots are equal. *Second Problem.* If B instead of paying an écu to A when he brings forth nothing, puts an écu into the game, to find that which then A must put into the game. *Third Problem.* Two Players A & B play alternately with an ordinary die, A puts an écu into the game, B begins to play; if he brings forth an even number, he takes this écu; if he brings forth an odd number, he puts an écu into the game, next this is A who plays, which by bringing forth an even number takes an écu into the game as B; but he puts nothing into the game when he brings forth an odd number, & they continue until there remains nothing more in the game, always with this condition, which they both take an écu from the game when they bring forth an even number; but that B alone puts an écu into the game when he brings forth an odd number, we demand their lots. *Fourth Problem.* A promises to give an écu to B, if with an ordinary die he brings forth at the first throw six points, two écus if he brings forth the six at the second, three écus if he brings this point at the third cast, four écus if he brings it forth at the fourth, & thus in sequence; we demand what is the expectation of B. *Fifth Problem.* We demand the same thing if A promises to B to give him some écus in this progression $1,\ 2,\ 4,\ 9,\ 16,\ 25,$ &c. or $1,\ 8,\ 27,\ 64,$ &c. instead of $1,\ 2,\ 3,\ 4,\ 5,$ &c. as above. Although these Problems are for the most part not difficult, you will find however something quite curious: I have already proposed to you the first in the last Letter. You will give pleasure to me to communicate finally your solution of Her, so that I can give you the explication of my Anagram. Besides, Sir, I replay to myself of that which you sent is best; but I pity you of that which you have lost your Princess. I have the honor to be with an inviolable attachment.

Petersburg Problem

Her

SIR,

Your very humble & very
obedient Servant
N. BERNOULLY.

Letter of Mr. de Montmort to Mr. N. Bernoulli (pages 403–414)

At Paris this 15 November 1713

Since you wish, Sir, that I declare to you finally that which I think in this famous dispute on Her, I am going to obey you.

Her

It seems that our Sirs having claimed at the beginning of the dispute that it was indifferent to Paul to change or to be held at the seven, & to Pierre to change or to be held at the eight, by that they would have too much according to me: you demonstrate quite well that this maxim is false; but you know that in the conversations which they have had with you they themselves are explained in supporting that it was impossible to establish any maxim for the one & the other Player, & in this I believe that they have reason. The two arguments which you produce against this assertion in your Letter of 30 December 1712 is not able to convince me; I am to the contrary persuaded that the solution of the Problem is impossible, that is to say that one is not able to prescribe to Paul the conduct which he must keep when he has a seven, & to Pierre when he has an 8. It is very true that it is worth more for Paul to take the maxim to change at the 7 than to take from them any other fixed & determined.

By this reason that whatever other maxim that we wish to determine for Paul, Pierre who will be instructed in it will take one of them which will render the lot of Paul less than $\frac{780}{50.51}$; but it does not follow that Paul must for this renounce the expectation to render his lot better by holding himself at the 7 than by changing from it. I have believed sometime that a certain composition of tokens for Pierre & for Paul would save the circle; but I have found that we always fall back to it. Suppose that we have prescribed to Paul the maxim of putting a white tokens & b black tokens into a pouch, by intending to change at the 7, if drawing a token it is found white, & to be held if drawing a token it is found black. Pierre who will know the maxim of Paul, which maxim will he follow? He will observe that if he puts into a sack c white tokens & d black tokens, for next drawing a token among all is determined to change at the 8 or to be held, according as he will draw a white or black token; he will observe, I say, by examining this general expression of the lot of Paul:

$$\frac{2828ac + 2834bc + 2838ad + 2828bd}{13.17.25.a + b.c + d}$$

1°. That the lot of Paul is the same when a is infinite with respect to b, & c infinite with respect to d; or when b being infinite with respect to a, d is infinite with respect to c. 2°. That a being infinite with respect to b the lot of Paul will be so much greater as c will be small with respect to d. 3°. That the lot of Paul is never better than when d being very great with respect to c, a is quite great with respect to b, &c. must all this contains the choice that Paul must take only conditionally to the one of Pierre, & the one of Pierre only conditionally to the one of Paul, that which makes a circle. It appears to me that all the reasons which you bring in order to prove that this circle does not take place, & that the return which amends it is not vicious; it seems to me, I say, that these reasons do not prove that which you wish to prove, but only that that which falls in a like circle is impossible: because finally the supposition that Paul must make himself the maxim to change at the 7 necessarily drives for Pierre the maxim to change at the 8; & this maxim thus established for Pierre, carries away a perfect demonstration that Paul will necessarily take the maxim of being held at the 7: this contradiction is deduced legitimately & without notice to the rules of Logic, since one must suppose that one & the other Player is equally subtle, & will take his choice only on the knowledge that he will have of the choice which the other will take. Now as there is not at all here some fixed point, the maxim of one depending on the maxim of the other Player which is not yet known, as soon as one wishes to establish one of them, one deduces from this assumption a contradiction which demonstrates that one had not ought establish it.

The demonstration which you based on the same circle which one opposes you is very subtle, but it is easy to respond to it.

As one can (you say) *demonstrate whatever choice that Paul takes, &c.*[51]

There is here equivocation in the beginning, one does not say at all that one can always demonstrate whatever choice that Paul takes that it is a bad choice, one claims only that one can not establish some maxim. Besides, Sir, it does not suffice to know that Paul can increase his lot by 10 in changing, & only by 6, in being held, when in the two cases Pierre will take a bad choice. It would be necessary in order that your demonstration be complete, that you can at the same time demonstrate that the probability in order to increase his lot by 10 is to the probability in order to increase his lot by 6 in a greater ratio than 6 to 10,

[51] See the letter of 30 December 1712, page 376.

but this is that which you don't know how to prove: it is likewise in your third argument which begins thus[52] *The lot of Paul is always* $\frac{2828}{5525}$, &c.[53]

For me, Sir, of all the reasons which must appear to engage Paul to take the maxim to change at the 7, I conclude from it only that he will be well in the practice to make himself a law to change more often at the seven than to be held; but how much more often he must change than to be held, & in particular that which he must make (*hic & nunc*) because this is principally there the question: the calculus teaches nothing on this subject, & I hold the decision impossible.

By this evident reason that if you establish that Paul must change at the seven, Pierre must make himself the maxim to change at the eight, in which case Paul who is as skilled as Pierre will know that he will change at the eight, & that thus he agrees with him to take the maxim of holding himself at the seven, &c. we will meet again in the circle.

In a word, Sir, if I know that you are the counsel of Pierre, it is evident that I myself Paul must be held at the seven; & likewise if I am Pierre, & if I know that you are the counsel of Paul, I must change at the eight, in which case you have given a bad counsel to Paul.

The more I think, & the more I am forced to think on this subject as our Sirs. It does not follow for this that you have erred, the consequence would be worth nothing; suppose therefore, Sir, that I am myself in error, you will much oblige me to hold me to it, by giving me the explication & the demonstration of your anagram.

As this matter is quite curious & as you have much meditated, I would be very glad if you wished well to instruct me at the same time on another question which is completely of the same kind.

Even-odd

A father wishes to give the gifts to his son & he says to him, I am going to put in my hand a number of chips even or odd as I will judge it à propos: this done, if you name even, & if it is even in my hand, I will give to you four écus; if you name odd, & if there is even in my hand, you will have nothing. If you name odd, & if there is odd in my hand, you will have one écu; if you name even, & if there is odd in my hand, you will have nothing. I demand, 1°, what rule it is necessary to prescribe to the father in order that he economizes his money the best that it is possible. 2°. What rule it is necessary to prescribe to the son in order that he takes the better choice. 3°. That one determines what advantage the father makes to his son, & to how much one can evaluate his gifts, by supposing that each of the two will take the conduct which is the most advantageous to him. These questions are very simple, but I believe them insolubles; if this is, it is great pity, because this difficulty is encountered in many things of civil life: when two persons, for example, having business together, each wishes his rule on the conduct of the other; it has place also in many games, mainly in Brelan, a game which makes now the delights of the Ladies of Paris: here is a kind of it. When the turns are large, as the turns being for example to the two or to the four cards, the pass is triple or quadruple; the one who is last believes able to say from the game & to steal the pass, with a nearly entire assurance, by reason that he is to believe that if the first or the second had had of the game, they would have opened the pass, wishing not at all to be exposed to lack it. On the other side the more the motives are strong in order to engage the last to go from the game or even to make a gross vade, the more the first & the second are tempted to pass with good play, in the hope to catch the last who would wish to steal: what rule to give that there? It is impossible, it seems to me, to prescribe anything assured. All the ability of the more refined Players is reduced to give to those with which they play a false idea of their manner of play, to affect a certain conduct in some coups

[52] See page 377.

[53] See the end of this letter.

of small value, in order to change apropos in the gross coups, & to profit skilfully from an error or prevention in which they will have given in design occasion. To this game, as throughout besides, the skillful are sometimes caught; but it is certain that in this game it is good to be, by reason that one has often business with some persons who are not at all or who are less; because among equally skillful & clairvoyent Players, such as we suppose Pierre & Paul in the game of Her: it would be absolutely impossible to prescribe any rule in the case of Brelan, no more than for our case of Her.

5 problems

The last two of your five Problems have no difficulty, the question is only to find the sums of the series of which the numerators being in progression of squares cubes, &c. the denominators being in geometric progression: your late Uncle has given the method to find the sum of these series.

For that which is of the third Problem it is much more difficult. I have been a longtime to assure myself that in this choice there is neither advantage nor disadvantage for Pierre; this is nevertheless that which I have found as well as Mr. de Waldegrave with whom I have worked at this Problem *conjunctis viribus*. I would wish to have the general solution of this Problem by supposing, for example, that A puts m into the game, that he takes n écus when he brings forth even, & that he puts r écus when he brings forth odd; that Paul takes p écus when he brings forth even, & that he sets q ecus into the game when he brings forth odd, to find, 1°, the lot of Pierre & of Paul, 2°. What must be the value of m, or of n, or of p, or of q, the others being given, & under the supposition that the lots are equal. 3°. How much the odds are that the game will be ended in so many coups. I am persuaded that there is no person as capable as you to surmount similar difficulties, for me, beyond that I believe that this passes me, I swear to you that I am tired to seek, & that I am disposed to taste during some times the gentle pleasure of doing nothing.

I have not thought at all on your two Problems, here is one of them by which I am amused, since it is not difficult, & that it is, it seems to me, rather curious.

One demands who of Pierre or of Paul plays a greater game, & give more to chance than Pierre, who during a month of thirty-one days sets regularly every day a pistole at merriment, or play it with heads-tails; or of Paul who proposes to set three times alone in the month, three pistoles each time, I have found $\frac{300540195}{67108864}a = 44.15.8\frac{344929}{2097152}$ for the value of that which Pierre risks, & $4\frac{1}{2}a = \frac{301989888}{67108864}$ for that which Paul risks; in a way that Paul risks more than Pierre, but very little more; this has not been easy to fathom; but that which there is, it seems to me, curious, is that Paul would not risk advantage by playing four times at heads-tails than three times; by playing one time a pistole than by playing two times one pistole each time, & generally that one does not hasard more to play m times at heads-tails than to play $m + 1$ times, if m is an odd number; I have found this number $\frac{300540193}{67108864}$, by multiplying the 1st term of the 32nd perpendicular band of the arithmetic triangle by 31, the second by 29, the third by 27, &c. you see all at once the ratio.

I have further found that one is able to wager with advantage that playing 31 games of Piquet one écu the game, without doubt there will not be at the end of the month more than three écus of loss; which would be yet the advantage to wager it for 37 games; but that it would be with disadvantage for 39. That which is singular & a true paradox, is that there would be advantage for 40, 42, & even for 44 & perhaps 46 games, (I have not made a calculation of it at all,) you will have no difficulty to discover the ratio.

You see well, Sir, that this last Problem is a particular kind of which the solution depends on method that you have communicated to me in your Letter of 15 January; but in regard of the preceding one will have difficulty to reduce it & to find a general solution by logarithms:

I see nonetheless that this is possible, if I was not overwhelmed with affairs I would have attempted, I keep myself this pleasure for another time where I will be less occupied.

In the time that I write this Letter, Sir, I received one from Mr. de Waldegrave; I wish you part of it, because it exhausts, it seems to me, all that which we can say on this matter. I had sent to him what I worked on to explain to you that which I think on Her; he has wished to make a last effort in order to assure his right & to put it in evidence. His Letter is from Chateau de Berviande & of 13 November: here is the extract of it. Her

"a being $= 3$, & $b = 5$, we see by formula[54] that if Pierre has the maxim to change the 8, the lot of Paul is

$$\frac{8484 + 14170}{13.17.25 \times 8} = \frac{11327}{5525 \times 4} = \frac{2831}{5525} + \frac{3}{5525 \times 4}.$$

"And if Pierre has the maxim to guard his eight

$$= \frac{8514 + 14140}{13.17.25 \times 8} = \frac{2831}{5525} + \frac{3}{5525 \times 4}.$$

"Thus it is equal in this supposition to Pierre to change at the eight or not; & generally the lot of Paul is always $\frac{2831}{5525} + \frac{3}{5525\times 4}$, when $a = 3$ & $b = 5$, whatever value which we can give to the letters c & d, that is to say that his lot will always be $\frac{2831}{5525} + \frac{3}{5525\times 4}$, whatever choice that Pierre takes when he will have an 8, either to change or to guard his 8 determinedly, or to be committed to an equal or unequal number of tokens.

"It follows thence that the lot of Paul is at least $\frac{2831}{5525} + \frac{3}{5525\times 4}$, when he holds only to him to take three white tokens & five black; & if Paul holds another conduct, it is that he hopes to render his lot yet better.

"Therefore Mr. Bernoulli has been wrong, & you also, Sir, not to displease you, to say once that the lot of Paul was to the lot of Pierre :: 2828 to 2697. Mr. Bernoulli has not thought apparently of this way to cast, which effectively seems to be not of the ordinary rules of the game; but he appears to have noticed since our dispute that he has not had reason to say that the lot of Paul was $\frac{2828}{5525}$, since in one of his last Letters he put that the worst which can happen to Paul is to have $\frac{2828}{5525}$.

"It is therefore certain & demonstrated that I have had reason to sustain that under the assumption that each of the Players plays the most advantageously which is possible to him, the lots of Paul & of Pierre are not those which Mr. Bernoulli has given, & that you have held true once; since Paul can render his lot greater than $\frac{2828}{5525}$, & have $\frac{2831}{5525} + \frac{3}{4} \times \frac{1}{5525}$, when he will wish to be contented in it, & (that which it is necessary to note) by putting more tokens to be held at the seven than for changing.

"Here is therefore one of the points of the question decided; I am sure that you will agree & Mr. Bernoulli also. In regard to that which I have also sustained that one could not establish some maxim; although it is impossible to me to demonstrate it with the same evidence, I believe not to be less well-founded to sustain it. Mr. Bernoulli says that the worst which can happen to Paul is to have $\frac{2828}{5525}$, & adds *that that which must determine him to change rather than to be held, is that if Pierre takes a bad choice, the lot of Paul will be greater when he will change at the seven, than when he will be held.*

It is true that under any other assumption of the values of a & b, than that of $a = 3$ & $b = 5$, Paul can render his lot better than in this one, if Pierre takes the bad choice; but also he will render it worse if Pierre takes the good choice; & what way is there to discover the ratio of probability which there is that Pierre will take the good choice to the probability that he will take the bad, this would seem to me absolutely impossible, & would make fall

[54] The previous formula $\frac{2828ac+2834bc+2838ad+2828bd}{13.17.25.a+b.c+d}$

into the circle? It is true also that the more Paul will increase the value of a, with respect to b, the more he can approach his lot of $\frac{2838}{5525}$, which is all that which can happen to him more advantageous if Pierre takes a bad choice, & if in augmenting the value of b, with respect to a, his lot can never pass $\frac{2834}{5525}$, & the more as in the one & the other case, his lot can never be less than $\frac{2828}{5525}$, as Mr. Bernoulli has quite well remarked; but we can not conclude thence that Paul must render a infinite with respect to b; or that which is the same thing, that he must change always at the seven; because if this consequence were good, Pierre who is supposed as capable as Paul could also conclude simultaneously that Paul would be held, if he has a card above the seven, & consequently would change infallibly at the eight, & by this means the lot of Paul would be only $\frac{2828}{5525}$, which is that which can happen to him the more bad; thus he would have done bad to have taken the maxim to change always at the seven, & here we are in the circle.

"I have forgotten to make you observe that Pierre has one way in order to bound the lot of Paul to $\frac{2831}{5525} + \frac{3}{4} \times \frac{1}{5525}$, by making $c = 5$ & $d = 3$, that which we will see yet with evidence, by substituting into the formula the values c & d. We could believe that as it holds only to Pierre to limit the lot of Paul to $\frac{2831}{5525} + \frac{3}{4} \times \frac{1}{5525}$, likewise that it holds to Paul only to be assured this same lot, Paul must make always $a = 3$ & $b = 5$, & Pierre $c = 5$ & $d = 3$, that which would make a constant maxim for the one & the other Player; but it seems to me that it would be bad to establish by the reasons which we have said so many times, & that this must not prevent Pierre & Paul to finesse, in the expectation of rendering each his lot better."

One prints actually your Letter of 15 January, there remains no more than three of them, the one of yours that I will give only by extract from this one, I am sure that your will approve them; one of mine of 20 August, & that which is the last. I was quite led to believe that neither the one nor the other would be worth the pain to be printed, especially the preceding in which I am perhaps too naturally that which I think, where there is found no algebra in order to serve passport to it. Mister de Waldegrave who by an infinite bounty takes care of the impression of my Book, wishes to set all & I leave him to do it. After the Letters we will print the Preface, I do not touch the older; but I add a Foreward excessively long. The Authors do not finish, & have always a thousand things to say that they believe very useful, but of which often one could pass quite well. In order to not incur the risk of falling again into this error, I finish by ensuring you that I honor you perfectly, and am in all my heart,

SIR,

Your very humble & very
obedient Servant R. D. M.

My very humble compliments, if you please, to Mr. your Uncle, to whom I pray you to show this Letter, Mr. de Waldegrave recommends to me always to give you his.

Her

As in these Letters we have spoken of Her, I have judged that it was à propos to put here the calculations, in order to spare to the Reader the pain of making them.[55]

I will add here on the occasion of that which I have said on the subject of Brelan in this Letter some calculations which I myself am amused to make at the plea of a Player of my friends: one finds by the article 25 that one can wager at each coup without advantage or disadvantage.

1°. 1 against 38, that he will find himself a Brelan at least.

[55] See the Table on page 61.

	Lot of Paul when he has the maxim to change at the seven, & Pierre the one to change at the eight.	Pierre has the maxim to hold himself at the eight.	Lot of Paul when he has the maxim to be held at the seven, & Pierre the one to hold himself at the eight.	Pierre the one to change at the eight.
King,	1200	1200	1200	1200
Queen,	1052	1058	1058	1052
Jack,	888	902	902	888
Ten,	724	746	746	724
Nine,	560	590	590	560
Eight,	476	434	434	476
Seven,	390	390	360	408
Six,	444	444	444	444
Five,	490	490	490	490
Four,	528	528	528	528
Three,	558	558	558	558
Two,	580	580	580	580
Ace,	594	594	594	594
	$\frac{8484}{13.51.25} = \frac{2828}{5525}$	$\frac{8514}{13.51.25} = \frac{2838}{5525}$	$\frac{8484}{13.51.25} = \frac{2828}{5525}$	$\frac{8502}{13.51.25} = \frac{2834}{5525}$

TABLE 1. Table for Her

2°. 2 against 7473, that he will find himself at least two of them.
3°. 2 against 1726723, that he will find himself three of them.
4°. 1 against 974, that he will find himself the fourth Brelan.
5°. 1 against 272, the he will find himself the favorite Brelan.

Whence it follows by article 186 that one can wager with advantage that

the first case will arrive in 27 coups
The second in 2591 coups
The third in 604354 coups
The fourth in 676 coups
The fifth in 189 coups
} at least.

That is to say, for example, that there will be advantage to wager that in 27 coups there will be some Brelan, & that there will be disadvantage to wager for 26 coups. It is likewise of the other cases, if this is not in regard to the third where there can be found error in the last two digits. A long work would be necessary in order to render them more exact, & this would not be worth the effort.