Sur la Probabilité des résultats moyens des Observations*

Mr. Poisson

Connaissance des tems for the year 1827, pp. 273-303.

The question that I myself propose to treat in this Memoir has already been the object of the works of many geometers, and particularly of Mr. Laplace, of whom the researches on this interesting matter are reunited in the *Théorie analytique des Probabilités* (Book II, Chapter IV), and in the three supplements to this great work. The generality of the analysis of Mr. Laplace, the variety and the importance of the objects to which he has made application of it, leave without doubt nothing to desire; but it has seemed to me that some points of this theory were able further to be developed; and I have thought that the remarks that I have had the occasion to make in studying it, would be proper to clarify the difficulties of it, and would be able also to not be without use in practice.

(1) Let s be the number of observations that one considers; we designate by i a whole and positive number, and we suppose that each of these observations are susceptible to $2i+1$ errors expressed by

$$-i,\ -i+1,\ \ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots,\ i-1,\ i;$$

we suppose moreover that the probability of a similar error is the same in all this series of observations; let n be one of these numbers, positive, negative or null, and we represent by N the probability of the error n: the sum of the probabilities of all the possible errors being certitude, one will have

$$\sum N = 1;$$

the sum $\sum$ extending to all the values of n from $n=-i$ to $n=i$. We name M the probability that the sum of the errors of the s observations will be equal to m: this probability is the same as that of bringing forth a sum m in projecting a number s of perfectly similar *dice*, of which each would have $2i+1$ faces, marked with the numbers $-i, \ldots, +i$, and having different degrees of probability, N being the probability of the face which carries the number n; consequently the value of M will be the coefficient of the power m of t in the development of the power s of the polynomial

$$\sum (Nt^n),$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. January 17, 2010 composed of $2i+1$ terms, or, that which is the same thing, the term independent of t in the development of

$$t^{-m}\left[\sum(Nt^n)\right],$$

according to the powers of this variable. It is this which results from the first rules of the calculus of the probabilities and of the theory of combinations.

In order to obtain this term, we designate, as ordinarily, by e the base of the Naperian logarithms, and by π the ratio of the circumference to the diameter; we observe that one has

$$\int_{-\pi}^{\pi} e^{n'\theta\sqrt{-1}}d\theta = 0, \quad \text{or} \quad = 2\pi;$$

the first value taking place when n' is a whole number, positive or negative, and the second when $n'=0$: one will conclude from it easily that by putting $e^{\theta\sqrt{-1}}$ in the place of t in the preceding quantity, the demanded term, or the value of M will be

$$M = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum Ne^{n\theta\sqrt{-1}}\right)^s e^{-m\theta\sqrt{-1}}d\theta.$$

Let now p be the probability that the sum of the s errors will be contained between two numbers μ and μ'; it is evident that the value of p will be the sum of the values of M, taken from $m=\mu$ to $m=\mu'$; but between these limits, the sum of the values of $e^{-m\theta\sqrt{-1}}$ is

$$\frac{e^{-(\mu-\frac{1}{2})\theta\sqrt{-1}} - e^{-(\mu'+\frac{1}{2})\theta\sqrt{-1}}}{2\sqrt{-1}\sin\frac{1}{2}\theta};$$

one will have therefore

$$p = \frac{1}{4\pi\sqrt{-1}}\int_{-\pi}^{\pi}\left(\sum Ne^{n\theta\sqrt{-1}}\right)^s\left(\frac{e^{-(\mu-\frac{1}{4})\theta\sqrt{-1}} - e^{-(\mu'+\frac{1}{2})\theta\sqrt{-1}}}{\sin\frac{1}{2}\theta}\right)d\theta.$$

As the mean error is the sum of the errors divided by their number, this probability p is also the one that the mean error is contained between $\frac{\mu}{s}$ and $\frac{\mu'}{s}$.

(2) In order to give to this last expression a form a little different which will permit next to make the errors increase by insensible degrees, we designate by $2a$ the interval in which they are all contained, or the positive excess of the greatest over the smallest; we partition this interval into a number $2i+1$ equal parts; let ω be one of these parts, so that one has $2a=(2i+1)\omega$; we make at the same time

$$n\omega = x, \quad \mu\omega = b-c, \quad \mu'\omega = b+c, \quad \frac{(2i+1)\theta}{2a} = \alpha:$$

N will be a function of x, which we will be able to represent by ωfx; and the value of p will become

$$p = \frac{a}{\pi}\int\left(\sum\omega fxe^{x\alpha\sqrt{-1}}\right)^s e^{-b\alpha\sqrt{-1}}\frac{\sin\left(c+\frac{a}{2i+1}\right)\alpha}{\sin\frac{a\alpha}{2i+1}}\frac{d\alpha}{2i+1};$$

the values of x to which the sum $\sum$ relates increasing by some differences equal to α, and extending from $x=-a$ to $x=a$, and the integral relative to α being taken from

$\alpha = -\frac{(2i+1)\pi}{2a}$ to $\alpha = \frac{(2i+1)\pi}{2a}$. The errors of the observations will no longer be expressed by some whole numbers; and p will be the probability that the sum of the errors x of the s observations will fall between the given quantities $b-c$ and $b+c$. Unless a, b and c cease to be finite and given quantities, we imagine that i becomes infinite, in which case we will have $(2i+1)\sin\frac{a\alpha}{2i+1} = ax$; the limits relative to α will become $\pm\infty$, and the difference α will be infinitely small: by taking it for the differential of x, and changing the sum $\sum$ into a definite integral, the value of p will take the form

$$p = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(fxe^{x\alpha\sqrt{-1}}dx\right)^s e^{-b\alpha\sqrt{-1}}\sin c\alpha\frac{d\alpha}{\alpha}. \tag{1}$$

This probability is related now to the case where the errors of the observations are able to be all the quantities contained between $-a$ and $+a$; their number being infinite, the probability $fxdx$ of any error x is infinitely small. The function fx will have such form as one will wish: it will be able to continue or discontinue, provided that all its values from $x=-a$ to $x=+a$ are positive and do not surpass unity at all, and that their sum, or the integral $\int_{-a}^{a} fxdx$, is equal to unity, a condition which expresses that each error falls certainly between $\pm a$. When this function will be given, one will have by two successive integrations, the value corresponding to p.

(3) By making $s=1$ in equation (1), one will have the probability that the error of a single observation is contained between $b-c$ and $b+c$, which will be

$$p = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\int_{-a}^{a} fxe^{x\alpha\sqrt{-1}}dx\right)e^{-b\alpha\sqrt{-1}}\sin c\alpha\frac{d\alpha}{\alpha}.$$

Now, when the interval contained from $b-c$ to $b+c$ will fall outside the limits $\pm a$ of the possible errors, that is to say when $b-c$ and $b+c$ will be one and the other, or $> a$, or $< a$, setting aside the sign, it is evident that this value of p will have to be null; on the contrary, this probability will be certitude, or equal to unity, when this interval of $b-c$ to $b+c$ will contain the whole entire interval of $-a$ to $+a$; and generally, if we regard fx as null, for all the values of x not contained between the limits $\pm a$, we will have to have

$$p = \int_{b-c}^{b+c} fxdx.$$

In order to verify on this point our analysis, we observe that by changing the order of integrations relative to x and α, the value of p is able to be written thus:

$$p = \frac{1}{\pi}\int_{-a}^{a}\left(\int_0^{\infty}\frac{\sin(b+c-x)\alpha}{\alpha}d\alpha - \int_0^{\infty}\frac{\sin(b-c-x)\alpha}{\alpha}d\alpha\right)fxdx.$$

But one has

$$\int_0^{\infty}\frac{\sin k\alpha}{\alpha}d\alpha = \frac{1}{2}\pi, \quad \text{or} \quad = -\frac{1}{2}\pi,$$

according as the constant k is positive or negative; the difference of the two integrals relative to α will be therefore null or equal to π, according as the two quantities $b+c-x$ and $b-c-x$ will be of like signs or of contrary signs; consequently the integral relative to x will be null for all the values of this variable, which will be, either $> b+c$ and

$> b-c$, or $< b+c$ and $< b-c$, and will have to be extended only to the values of x contained at the same time between $\pm a$ and between $b-c$ and $b+c$. Therefore, by regarding fx as null for the values of x which fall beyond the limits $\pm a$, one will have

$$p = \int_{b-c}^{b+c} fxdx;$$

this which it was the concern to verify.

(4) Before going further, it will not be useless to apply formula (1) to some particular examples.

The simplest case is the one where all the errors contained between $\pm a$ are equally possible; the function fx is then a constant equal to $\frac{1}{2a}$; there results

$$\int_{-a}^{a} fxe^{x\alpha\sqrt{-1}}dx = \frac{\sin a\alpha}{a\alpha},$$

and consequently

$$p = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin a\alpha}{a\alpha}\right)^{s}\frac{\sin c\alpha}{\alpha}\cos b\alpha\, d\alpha;$$

an integral that one will obtain under finite form by the known formulas, for all the values of s in whole numbers.

Suppose in second place that one has

$$fx = \frac{1}{\sqrt{\pi}}e^{-x^{s}},$$

and that the limits $\pm a$ are $\pm\infty$. The condition $\int_{-a}^{a} fxdx = 1$ will be satisfied; one will have

$$\int_{-a}^{a} fxe^{x\alpha\sqrt{-1}}dx = e^{-\frac{\alpha^2}{4}},$$

and hence

$$p = \frac{2}{\pi}\int_{0}^{\infty} e^{-\frac{\alpha^2 s}{4}}\cos b\alpha \sin c\alpha\,\frac{d\alpha}{\alpha};$$

an expression that one is able to write under this other form:

$$p = \frac{2}{\pi}\int\left(\int_{0}^{\infty} e^{-\frac{\alpha^2 s}{4}}\cos b\alpha \sin c\alpha\,\frac{d\alpha}{\alpha}\right)dc,$$

the integral relative to c being taken in a manner that it vanishes when $c = 0$. The integral relative to α is obtained by the known formulas; and the integration made, one has

$$p = \frac{1}{\sqrt{\pi s}}\int\left(e^{-\frac{(b-c)^2}{s}} + e^{-\frac{(b+c)^2}{s}}\right)dc.$$

If one makes $b = b'\sqrt{s}$, $c = c'\sqrt{s}$, one will have

$$p = \frac{1}{\sqrt{\pi}}\int_{-c'}^{c'} e^{-(b'+z)^2}dz;$$

where one sees that when the limits $b \pm c$ between which the sum of the errors must fall, are proportionals to the square root of the number s of the observations, the probability p, for which that takes place, is independent of this number, under the hypothesis that we have made on the form of the function f. Under this same hypothesis, the *maximum* of p, with respect to b, corresponds to $b = 0$, this which was evident *a priori*.

For last example, we take

$$fx = \frac{1}{\pi(1+x^2)}, \quad a = \infty.$$

The condition $\int_{-a}^{a} fxdx = 1$ will be fulfilled. Moreover one will have

$$\int_{-a}^{a} fxe^{x\alpha\sqrt{-1}}dx = e^{-\alpha}, \quad = e^{\alpha},$$

according as the quantity α will be positive or negative; whence one concludes

$$p = \frac{2}{\pi}\int_0^\infty e^{-\alpha z}\cos b\alpha \frac{\sin c\alpha}{\alpha}d\alpha,$$

or else

$$p = \frac{2}{\pi}\int \left(\int_0^\infty e^{-\alpha s}\cos b\alpha \sin c\alpha \, d\alpha\right) dc$$

the integral relative to c vanishing when $c = 0$. In doing, by the ordinary rules, the integration relative to α, there comes

$$p = \frac{1}{\pi}\int \left(\frac{s}{s^2+(b-c)^2} + \frac{s}{s^2+(b+c)^2}\right) dc,$$

and finally

$$p = \frac{1}{\pi}\text{arc}\left(\tan = \frac{2cs}{s^2+b^2-c^2}\right).$$

By making $b = b's$, $c = c's$, the mean error will be contained between the limits $b' \pm c'$, and the probability p which corresponds to them will be found independently from the number s of the observations; whence there results that in this particular example, in measure as this number would increase, the mean error would not converge toward zero or another fixed term; and, however great that the number of observations were, there would always be the same probability that the mean error to fear would be contained between the given limits.

(5) The imaginaries enter only in appearance in the second member of equation (1), and one is able easily to make them disappear.

Let first

$$\left(\int_{-a}^{a} fx\cos\alpha x\,dx\right)^2 + \left(\int_{-a}^{a} fx\sin\alpha x\,dx\right)^2 = \rho^2$$

and next

$$\frac{1}{\rho}\int_{-a}^{a} fx\cos\alpha x\,dx = \cos\phi,$$
$$\frac{1}{\rho}\int_{-a}^{a} fx\sin\alpha x\,dx = \sin\phi:$$

by reuniting in formula (1) the elements of the integral relative to α, which correspond to the values of this variable, equals and of contrary signs, this formula will become

$$p = \frac{2}{\pi}\int_0^\infty \rho^s \cos(s\phi - b\alpha)\frac{\sin c\alpha}{\alpha}d\alpha. \tag{2}$$

The quantity ρ is equal to unity when $\alpha = 0$; for each other value of α, it is less than 1. In fact, the expression ρ^2 is able to be written thus:

$$\rho^2 = \int_{-a}^{a} fx\cos\alpha x\,dx\int_{-a}^{a} fx'\cos\alpha x'\,dx' + \int_{-a}^{a} fx\sin\alpha x\,dx\int_{-a}^{a} fx'\sin\alpha x'\,dx';$$

changing each of these two products of simple integrals into one double integral, next the sum of the two double integrals into one alone, one will have

$$\rho^2 = \int_{-a}^{a}\int_{-a}^{a} fxfx'\cos(x-x')\alpha\,dxdx',$$

a quantity smaller than

$$\int_{-a}^{a}\int_{-a}^{a} fxfx'dxdx', \quad \text{or} \quad < 1;$$

because by subtracting the first double integral from the second, there comes

$$\int_{-a}^{a}\int_{-a}^{a}[1-\cos(x-x')\alpha]fxfx'\,dxdx';$$

an integral of which all the elements are positive by hypothesis, and which is consequently a positive quantity.

This remark is important, and goes to serve us by reducing the value of p to a simpler form, in the case where the observations are in very great number.

(6) We will treat the number s as infinite, so that the following formulas will be rigorous to this limit, and so much the more approximate as s will be more considerable. Now, the quantity ρ being < 1 when the variable a is not null, it follows that in the limit $s = \infty$, the power ρ^s will have finite values only for some infinitely small values of this variable, and will become infinitely small as soon as α will have a finite value. Therefore, by developing the quantity ρ according to the powers of α, one will be able to limit its value to the first two terms of this series; and if one makes

$$\int_{-a}^{a} fxxdx = k, \quad \int_{-a}^{a} fxx^2dx = k',$$

one will have, in this manner,

$$\rho = 1 - \frac{1}{2}(k'-k^2)\alpha^2.$$

This form of the value ρ admits however an exception in the case where the limits $\pm a$ are infinities: it is possible then that the second term of the development of ρ, according to the powers of α, contain only the first power of this variable, which power would be subject to not change sign with α, or would represent, if one wishes, $+\sqrt{\alpha^2}$: it is that which arrives effectively when one has

$$fx = \frac{1}{\pi(1+x^2)},$$

thus as one has seen it in the last example of n° 4. But you do not take account of this particular case, which it will suffice us to have remarked because of its singularity, and which is encountered without doubt in practice.

One would be able thus to fear that the coefficient $k' - k^2$ of the second term of ρ being null, it was not necessary to conserve the term following its development, which would contain a power of α superior to the second; but it is easy to prove that this quantity $k' - k^2$ is always positive, this which is necessary in order that one has $\rho < 1$, and, moreover, that it is never able to be equal to zero. In fact, because $\int_{-a}^{a} fx'x'dx = 1$, one has

$$k' - k^2 = \int_{-a}^{a} fxx^2dx \int_{-a}^{a} fx'x'dx - \int_{-a}^{a} fxxdx \int_{-a}^{a} fx'x'dx,$$

or, that which is the same thing,

$$k' - k^2 = \int_{-a}^{a}\int_{-a}^{a} (x^2 - xx')fxfx'\,dxdx';$$

one is able also to write

$$k' - k^2 = \int_{-a}^{a}\int_{-a}^{a} (x'^2 - xx')fxfx'\,dxdx';$$

and by taking for $k' - k^2$ the half sum of these two values, there comes

$$k' - k^2 = \frac{1}{2}\int_{-a}^{a}\int_{-a}^{a} (x - x')^2 fxfx'\,dxdx',$$

a positive quantity, and which will never be null, since all the elements of this double integral are necessarily positive.

This put, we make, for brevity,

$$\frac{1}{2}(k' - k^2) = h^2,$$

and we set $\frac{y}{\sqrt{s}}$ in place of α; we will have

$$\rho^t = \left(1 - \frac{h^2y^2}{s}\right)^{\prime}:$$

the new variable y will be able to receive some finite values; but, whatever these values be, one will have always

$$\rho^s = e^{-h^2y^2},$$

in the limit $s=\infty$. After the value of $\sin\phi$, we have, at the same time, $\phi = k\alpha$; equation (2) will become therefore

$$p = \frac{2}{\pi}\int_0^\infty e^{-h^2y^2}\cos(ks-b)\frac{y}{\sqrt{s}}\sin\frac{cy}{\sqrt{s}}\frac{dy}{y};$$

or, that which is the same thing,

$$p = \frac{1}{\pi\sqrt{s}}\int\left(\int_0^\infty e^{-h^2y^2}\cos(ks-b+z)\frac{y}{\sqrt{s}}dy\right)dz;$$

the integral relative to z being taken from $z=-c$ to $z=+c$. One must give to the variable y only some finite values; but one extends without fear of sensible error, the integral which is carried back to infinity, because of the factor $e^{-h^2y^2}$, which becomes insensible for the very great values of y. This integral is obtained then by the known formulas, and one has definitely

$$p = \frac{1}{2h\sqrt{\pi s}}\int_{-c}^{c} e^{-\frac{(ks-b+z)^2}{4h^2s}}dz. \tag{3}$$

In the case where fx is constant and equal to $\frac{1}{2a}$, one will have

$$k=0,\quad k'=\frac{a^2}{3},\quad h^2=\frac{a^2}{6},$$

and consequently

$$p=\frac{\sqrt{3}}{a\sqrt{2\pi s}}\int_{-c}^{c} e^{-\frac{3(b-z)^2}{2a^2s}}dz.$$

The limits $\pm a$ being infinite, in the case of

$$fx=\frac{1}{\sqrt{\pi}}e^{-x^2},$$

one will have

$$k=0,\quad h^2=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}x^2dx=\frac{1}{4};$$

whence there will result

$$p=\frac{1}{\sqrt{\pi s}}\int_{-c}^{c} e^{-\frac{(b-z)^2}{s}}dz,$$

this which coincides with the second value of p from n$^{\text{o}}$ 3, which subsists for all the values of s.

(7) For one same value of c, the *maximum* of p with respect to b, will be given by the equation

$$\int_{-c}^{c} e^{-\frac{(ks-b+z)^2}{4h^2s}}(ks-b+z)dz=0;$$

or else, by doing the integration, by this here:

$$e^{-\frac{(ks-b+c)^2}{4h^2s}}-e^{-\frac{(ks-b-c)^2}{4h^2s}}=0,$$

which is reduced to

$$e^{-\frac{c(ks-b)}{2h^2s}} = e^{\frac{c(ks-b)}{2h^2s}},$$

and give $b = ks$. If one makes at the same time $c = 2hr\sqrt{s}$, formula (3) will become

$$p = \frac{2}{\sqrt{\pi}} \int e^{-r^2} dr,$$

the integral being taken in a manner that it vanishes when $r = 0$. This will be the probability that the sum of the errors of a very great number s of observations will have for limits $ks \pm 2hr\sqrt{s}$, or that the mean error will fall between $k - \frac{2hr}{\sqrt{s}}$ and en $k + \frac{2hr}{\sqrt{s}}$; so that the quantity r and the probability which depends on it, remain the same, these limits will be tightened indefinitely in measure as s will increase, and one will be able always to take this number large enough for which it has a given probability, that the mean error will differ also as little as one will wish from the quantity k. Departing from its *maximum*, the value of p given by equation (3) will diminish very rapidly; and for little that b differs from ks, a quantity which is not of the order of smallness of $\frac{1}{\sqrt{s}}$, this value of p will be insensible; the number s being always supposed very great.

All the time that the positive errors and the negative errors will be equally possible, that is to say when the function fx will be the same for the values of x equal and of contrary sign, the quantity k will be null, and the mean error will hold continually to become it in measure as the number of observations will become greater. But when some constant cause will render the errors preponderant, in one sense or in the another, the quantity k will no longer be null, and it will be necessary that its value be known, in order that one be able to assign the fixed term toward which the mean error converges indefinitely. It is evident that k, setting aside its sign, is not able to be$> a$; because the mean error would not know how to pass the limit of the possible errors. It is necessary, for that, that one has $k^2 < a^2$; and in fact, one has

$$\begin{aligned} a^2 - k^2 &= a^2 \int_{-a}^{a} fx dx \int_{-a}^{a} fx' dx' - \int_{-a}^{a} x fx dx \int_{-a}^{a} x' fx' dx' \\ &= \int_{-a}^{a} \int_{-a}^{a} (a^2 - xx') fx fx' dx dx'; \end{aligned}$$

a positive quantity, since all the elements of this double integral are positive.

(8) The preceding analysis is applied without difficulty to the solution of the following problem, which comprehends, as particular case, the one that we come to resolve.

Let E be the sum of the errors of the s observations, each multiplied by a given coefficient; we represent by ε, ε_1, $\varepsilon_2, \dots, \varepsilon_{s-1}$ the errors of the 1st, 2nd, 3rd, …, $(s-1)$st observation, and by γ, γ_1, $\gamma_2, \dots, \gamma_{s-1}$, the coefficients which must multiply them respectively; so that one has

$$E = \gamma\varepsilon + \gamma_1\varepsilon_1 + \gamma_2\varepsilon_2 + \cdots + \gamma_{s-1}\varepsilon_{s-1}:$$

the concern is to find the probability that the sum E is contained between the given limits.

We suppose first, as in n° 1, that all the possible errors are expressed by the whole numbers or zero, comprehended from $-i$ to $+i$; but in order to give to the question all the generality of which it is susceptible, the probability of one same error will not be supposed the same in all the observations; we will designate therefore the probability of any error n, by N in the first observation, by N_1 in the second, ..., by N_{s-1} in the last. Let moreover β be a factor such, that all the products $\beta\gamma$, $\beta\gamma_1, \beta\gamma_2, \beta\gamma_3, \ldots \beta\gamma_{s-1}$ are some whole numbers; this which will be always possible exactly, or to such degree of approximation as one will wish. We make finally, for brevity,

$$\left(\sum N t^{\beta\gamma n}\right)\left(\sum N_1 t^{\beta\gamma_1 n}\right)\left(\sum N_2 t^{\beta\gamma_2 n}\right)\cdots\left(\sum N_{s-1} t^{\beta\gamma_{s-1} n}\right) = T;$$

the sums $\sum$ which enter into this product extending to all the values of n, from $n = -i$ to $n+i$. The probability that βE is equal to a whole number m, will be the coefficient of t^m in the development of T according to the powers of t, or the term independent of t in the product Tt^{-m}. In designating by M this probability, and by P that which T becomes when one puts $e^{\theta\sqrt{-1}}$ in the place of t, we will have

$$M = \frac{1}{2\pi}\int_{-\pi}^{\pi} P e^{-m\theta\sqrt{-1}} d\theta.$$

Calling next p the probability that βE is contained between two whole numbers μ and μ', or equal to one of these numbers, p will be the sum of the values of M, from $m = \mu$ to $m = \mu'$, and its value will be

$$p = \frac{1}{4\pi\sqrt{-1}}\int_{-\pi}^{\pi} P\left[\frac{e^{-\left(\mu-\frac{1}{2}\right)\theta\sqrt{-1}} - e^{-\left(\mu'+\frac{1}{2}\right)\theta\sqrt{-1}}}{\sin\frac{1}{2}\theta}\right] d\theta.$$

In order to establish the continuity among the possible errors of each observation, we divide the given interval in which they are all contained, into $2i+1$ equal parts; let $2a$ be this interval and ω one of these parts; we make besides

$$n\omega = x, \quad \mu\omega = (b-c)\beta, \quad \mu'\omega = (b+c)\beta, \quad \frac{(2i+1)\beta\theta}{2a} = \alpha:$$

there will remain no more next but to suppose ω infinitely small and i infinite, so that the errors vary by insensible degrees. To this limit, the integrals relative to α will be taken from $\alpha = -\infty$ to $\alpha = \infty$; the sums $\sum$ will be changed into some definite integrals, taken from $x = -a$ to $x = a$, ω being the differential dx; and if one makes $N = \omega f x$, one will have, for example,

$$\sum N e^{\beta\gamma n\theta\sqrt{-1}} = \int_{-a}^{a} f x e^{\gamma x\alpha\sqrt{-1}} dx.$$

The other sums $\sum$ will be transformed in the same manner, so that by assuming

$$N_1 = \omega f_1 x, \quad N_2 = \omega f_2 x, \ldots, N_{s-1} = f_{s-1} x,$$

the value of P will become

$$\left(\int_{-a}^{a} f x e^{\gamma x\alpha\sqrt{-1}} dx\right)\left(\int_{-a}^{a} f_1 x e^{\gamma_1 x\alpha\sqrt{-1}} dx\right)\cdots\left(\int_{-a}^{a} f_{s-1} x e^{\gamma_{s-1} x\alpha\sqrt{-1}} dx\right)$$

and all the reductions made, that of p will be

$$p = \frac{1}{\pi}\int_{-\infty}^{\infty} P e^{b\alpha\sqrt{-1}} \sin c\alpha \frac{d\alpha}{\alpha}.$$

The quantity β has disappeared in this formula; and, in fact, p is the probability for which βE is contained between $(b-c)\beta$ and $(b+c)\beta$, or that which is the same thing, for which the sum E is contained between $b-c$ and $b+c$, that which no longer depends on β. One will make the imaginaries vanish, by putting

$$\begin{aligned}
&\left(\int_{-a}^{a} fx\cos\gamma x\alpha\,dx\right)^2 + \left(\int_{-a}^{a} fx\sin\gamma x\alpha\,dx\right)^2 = \rho^2,\\
&\frac{1}{\rho}\int_{-a}^{a} fx\cos\gamma x\alpha\,dx = \cos\phi,\\
&\frac{1}{\rho}\int_{-a}^{a} fx\sin\gamma x\alpha\,dx = \sin\phi;
\end{aligned}$$

designating by ρ_1 and ϕ_1, ρ_2 and ϕ_2, etc., that which ρ and ϕ become when one replaces γ and fx successively, by γ_1 and $f_1 x$, γ_2 and $f_2 x$, etc.; and making next

$$\begin{aligned}
&\rho\rho_1\rho_2\cdots\rho_{s-1} = R,\\
&\phi + \phi_1 + \phi_2 + \cdots + \phi_{s-1} = \psi.
\end{aligned}$$

The expression p is changed into that here:

$$p = \frac{2}{\pi}\int_0^{\infty} R\cos(\psi - b\alpha)\sin c\alpha\,\frac{d\alpha}{\alpha},$$

or, that which reverts to the same,

$$p = \frac{1}{\pi}\int_{-c}^{c}\left(\int_0^{\infty} R\cos(\psi - b + z\alpha)\,d\alpha\right)dz, \tag{4}$$

All the factors of R are reduced to unity, when $\alpha = 0$; one will prove, as in n° 5, that they are all < 1 for each other value of α.

(9) In order to deduce from this formula, some results useful to practice, we are going to consider especially the case where the number s is very great, and is able to be treated as infinite. In this case, if one designates by r those of the quantities ρ, $\rho_1, \rho_2, \ldots, \rho_{s-1}$ which, for the same value of α, differs least from unity, one will have $R < r^s$, and, consequently, the product R will have finite values only for some infinitely small values of α. However this conclusion will be able to be with defect, in the case where the coefficientsγ, γ_1, γ_2, etc., will form a continually decreasing series; it will be able to arrive then that the factors ρ, ρ_1, ρ_2, etc., converge indefinitely toward unity, so that one is not able to assign among them, the factor r which approaches most to be equal to one; and after that it will be possible that the product of these factors in infinite number, be a finite quantity for all the values of α. We will give in the n° following an example of this singular case; in the one here, we will consider the general case where

the product R, for the limit relative to infinite s, becomes infinitely small, as soon as one gives to α a finite value.

Let there be for any index i,

$$\int_{-a}^{a} x f_i x dx = k_i, \quad \int_{-a}^{a} x^2 f_i x dx = k_i',$$
$$\frac{1}{2}(k_i' - k_i^2) = h_i^2;$$

we observe besides that one has

$$\int_{-a}^{a} f_i x dx = 1,$$

and we develop each of the factors of R according to the powers of α: by conserving only the first two terms of each series, we will have

$$R = (1 - \gamma^2 h^2 \alpha^2)(1 - \gamma_1^2 h_1^2 \alpha^2) \cdots (1 - \gamma_{s-1}^2 h_{s-1}^2 \alpha^2).$$

We make $\alpha = \frac{y}{\sqrt{s}}$, in a manner that the new variable y is able to be a finite quantity. If one develops the logarithm of R according to the powers of this variable, one will have

$$\log R = -y^2 \frac{\sum \gamma_i^2 h_i^2}{s} - \frac{1}{2} y^4 \frac{\sum \gamma_i^4 h_i^4}{s^2} - \frac{1}{3} y^6 \frac{\sum \gamma_i^6 h_i^6}{s^3} - \text{etc.};$$

the sums $\sum$ extending from $i = 0$ to $i = s - 1$. By supposing that the quantities $\gamma^2 h^2$, $\gamma_1^2 h_1^2$, $\gamma_2^2 h_2^2$, etc., do not increase indefinitely, and designating by H^2 the greatest among them, these sums $\sum$ will be respectively less than sH^2, sH^4, sH^6, etc.; consequently all the terms of the development of $\log R$, the first excepted, will vanish at the limit $s = \infty$, and one will have simply

$$\log R = -\frac{1}{s} y^2 \sum \gamma_i^2 h_i^2 \quad \text{and} \quad R = e^{-\frac{1}{s} y^2 \sum \gamma_i^2 h^2}.$$

At the same time the quantities ϕ, ϕ_1, ϕ_2, etc., will be reduced to $\alpha\gamma k$, $\alpha\gamma_1 k_1$, $\alpha\gamma_2 k_2$, etc.; one will have therefore $\psi = \alpha \sum \gamma_i k_i$, and formula (4) will become

$$p = \frac{1}{\pi\sqrt{s}} \int_{-c}^{c} \left[e^{-\frac{1}{s} y^2 \sum \gamma_i^2 h_i^2} \cos(\sum \gamma_i k_i - b + z) \frac{y}{\sqrt{s}} dy \right] dz.$$

Because of the rapid increase of its elements, one will extend, without fear of error, the integral relative to y from $y = 0$ to $y = \infty$, this which will permit to obtain it under finite form, and will give

$$p = \frac{1}{2\sqrt{\pi \sum \gamma_i^2 h_i^2}} \int_{-c}^{c} e^{-\frac{(\sum \gamma_i k_i - b + z)^2}{4\sqrt{\sum \gamma_i^2 h_i^2}}} dz. \tag{5}$$

For one same value of c, the *maximum* of p with respect to b will correspond to $b = \sum \gamma_i k_i$; and this probability will increase very rapidly on both sides of its greater value, so that it will be completely insensible, as soon as b will be separated from

$\sum \gamma_i k_i$, by a quantity comparable to $\frac{1}{\sqrt{\sum \gamma_i^2 h_i^2}}$, which is of the same order as $\frac{1}{\sqrt{s}}$. By making $b = \sum \gamma_i k_i$, $c = 2r\sqrt{\sum \gamma_i^2 h_i^2}$, one will have

$$p = \frac{2}{\sqrt{\pi}} \int e^{-r^2} dr,$$

the integral commencing with r. This will be the probability that the sum E has for limits $\sum \gamma_i k_i \pm 2r\sqrt{\sum \gamma_i^2 h_i^2}$, or that $\frac{1}{s}E$ is contained between

$$\frac{1}{s}\sum \gamma_i k_i - \frac{2r}{s}\sqrt{\sum \gamma_i^2 h_i^2} \text{ and } \frac{1}{s}\sum \gamma_i k_i + \frac{2r}{s}\sqrt{\sum \gamma_i^2 h_i^2}.$$

As one has $\sqrt{\sum \gamma_i^2 h_i^2} < H^2 s$, there results from it that one will be able always to take s great enough so that it has a given probability that $\frac{1}{s}E$ differs as little as one will wish from $\frac{1}{s}\sum \gamma_i k_i$, which would be consequently the value of $\frac{1}{s}E$ resulting from an infinite number of observations.

(10) In order to give an example of the exception that we have indicated at the commencement of the n° preceding, we make $a = \infty$; we suppose that the law of probability is the same in all the observations, and also the same for the errors equal and of contrary signs, this which will render null the angles ϕ, ϕ_1, ϕ_2, etc., of the n° 8; and we take $\gamma = 1$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{1}{3}, \dots$ and generally $\gamma_i = \frac{1}{i+1}$; whence there will result

$$\rho_1 = 2\int_0^\infty fx \cos \frac{\alpha x}{i+1} dx.$$

Let besides

$$fx = e^{\mp 2x};$$

the superior sign having place when the variable x is positive, and the inferior when it is negative. This expression of fx gives

$$\int_0^\infty fx dx = \int_0^{-\infty} fx dx = \frac{1}{2},$$

and satisfies consequently the condition $\int_{-a}^{a} fx dx = 1$. The corresponding value of ρ_i will be

$$\rho_i = 2\int_0^\infty e^{-2x} \cos \frac{\alpha x}{i+1} dx = \frac{1}{1 + \frac{\alpha^2}{4(i+1)^2}};$$

by means of what one will have

$$R = \frac{1}{\left(1 + \frac{\alpha^2}{4}\right)\left(1 + \frac{\alpha^2}{4\cdot 4}\right)\left(1 + \frac{\alpha^2}{4\cdot 9}\right)\cdots\left(1 + \frac{\alpha^2}{4\cdot s^2}\right)};$$

now, the number of the factors of the denominator being infinite, this denominator will be equal to

$$\frac{e^{\frac{\pi\alpha}{2}} - e^{-\frac{\pi\alpha}{2}}}{\pi\alpha},$$

after the known decomposition of the exponentials into products of this nature; one will have therefore, under finite form,

$$R = \frac{\pi\alpha}{e^{\frac{\pi\alpha}{2}} - e^{-\frac{\pi\alpha}{2}}};$$

and if one substitutes this value into formula (4), if one makes $\psi = 0$, and if one effects the integration with respect to z, there will result from it

$$p = \int_0^\infty \frac{\sin(b+c)\alpha}{e^{\frac{1}{2}\pi\alpha} - e^{-\frac{1}{2}\pi\alpha}} d\alpha - \int_0^\infty \frac{\sin(b-c)\alpha}{e^{\frac{1}{2}\pi\alpha} - e^{-\frac{1}{2}\pi\alpha}} d\alpha.$$

The exact values of these integrals is deduced from a known formula,[1] and the value of p becomes finally

$$p = \frac{e^{2(b+c)} - 1}{2(e^{2(b+c)} + 1)} - \frac{e^{2(b-c)} - 1}{2(e^{2(b-c)} + 1)}.$$

Such is therefore the probability that the value of E, or of the series

$$1 + \frac{1}{2}\varepsilon_1 + \frac{1}{3}\varepsilon_2 + \frac{1}{4}\varepsilon_3 + \text{etc.},$$

prolonged to infinity, will be contained between $b-c$ and $b+c$. If one makes $b = 0$, this probability will be reduced to

$$p = \frac{1 - e^{-2c}}{1 + e^{-2c}};$$

whence one concludes that, without being obliged to take for c a very great number, by taking for example $c > 5$, there will be a probability very close to certitude, that the sum E will be contained between $\pm c$. By making $b = c$, one will have

$$p = \frac{1 - e^{-2c}}{2(1 + e^{-2c})},$$

for the probability that E is contained between c and $2c$, which is, as one sees, the mean of the preceding.

The law of probability being the same as in the case that we just took for example, if one takes for the coefficients γ, γ_1, γ_2, etc., the series 1, $\frac{1}{3}$, $\frac{1}{5}$, etc., one will find that formula (4) becomes

$$p = \frac{2}{\pi} \int_{-c}^{c} \left(\int_0^\infty \frac{\cos(b-z)\alpha}{e^{\frac{1}{4}\pi\alpha} + e^{-\frac{1}{4}\pi\alpha}} \right) dz.$$

But one has[2],

$$\int_0^\infty \frac{\cos(b-z)\alpha}{e^{\frac{1}{4}\pi\alpha} + e^{-\frac{1}{4}\pi\alpha}} d\alpha = \frac{2}{e^{2(b-z)} + e^{-2(b-z)}};$$

[1] *Journal de l'École Polytechnique*, 18$^{\text{e}}$ cahier, page 297.

[2] *Journal de l'École Polytechnique*, 18$^{\text{e}}$ cahier, page 298.

this which permits to effect the integration relative to z, and gives

$$p = \frac{2}{\pi}\left[\text{arc}(\tan = e^{-2(b-c)}) - \text{arc}(\tan = e^{-2(b+c)})\right],$$

for the probability that the value of the series

$$1 + \frac{1}{3}\varepsilon_1 + \frac{1}{5}\varepsilon_2 + \frac{1}{7}\varepsilon_3 + \text{etc.},$$

prolonged to infinity, is contained between $b-c$ and $b+c$.

In the case of $b = 0$, this value of p becomes

$$\begin{aligned} p &= \frac{2}{\pi}\left[\text{arc}(\tan = e^{2c}) - \text{arc}(\tan = e^{-2c})\right] \\ &= 1 - \frac{4}{\pi}\text{arc}(\tan = e^{-2c}); \end{aligned}$$

a quantity which will differ very little from unity, when c, without being a very great number, will surpass however five or six units. The value of p will be half of that one, or equal to

$$\frac{1}{2} - \frac{2}{\pi}\text{arc}(\tan = e^{-2c}),$$

in the case of $b = c$.

By comparing these results to the one of the preceding n$^{\text{o}}$, one sees that the probabilities of the values of E are very different according as the coefficients γ, γ_1, γ_2, etc. form a series decreasing to infinity, or that they have all a finite value, as we will suppose it in that which is going to follow.

(11) Most frequently the quantity which is given immediately by the observations, is not the same unknown that one wishes to determine, but a function of this unknown, which changes the value of an observation to another. In order that the calculations are not impractical, especially in the case of a great number of observations, it is necessary that this function be linear, or that the unknown be already rather well determined so that the correction that one must make it subject, is very small, and that one is able to neglect in it the powers superior to the first, this which renders the function linear with respect to this correction, which is then the true unknown of the problem. We will represent it by u; by A_i, the approximate value of the function corresponding to the $(i+1)^{\text{st}}$ observation; by $A_i + uq_i$, its corrected value; by B_i, the value of the same function given by this observation; by ε_i, as previously, the unknown error of which it is susceptible. We will have, in this manner,

$$B_i + \varepsilon_i = A_i + uq_i;$$

and if we make

$$B_i - A_i = \delta_i,$$

so that δ_i is the excess of the value observed on the approximate value, this equation, will become

$$\varepsilon_i = uq_i - \delta_i.$$

One will have similarly of it for each of the s observations that one considers. The coefficients q, q_1, q_2, etc., and the quantities δ, δ_1, δ_2, etc., will be given in each particular case; and the concern is to draw from this system of equations, the most independent value of the errors of the observations.

For that, we make the sum of all these equations, multiplied respectively by the coefficients γ, γ_1, γ_2, etc., we will have

$$E = u\sum \gamma_i q_i - \sum \gamma_i \delta_i;$$

the sums $\sum$ extending, as previously, from $i = 0$ to $i = s-1$. In measure as s increases, the value of $\frac{1}{s}E$ approaches to be equal to $\frac{1}{s}\sum \gamma_i k_i$; the value of which u will approach at the same time will be therefore

$$u = \frac{\sum \gamma_i \delta_i}{\sum \gamma_i q_i} + \frac{\sum \gamma_i k_i}{\sum \gamma_i q_i}; \tag{6}$$

and by taking for u this value, it will have a probability expressed by $\frac{2}{\sqrt{\pi}} \int e^{-r^2} dr$ (n$^{\text{o}}$ 9), that the error to fear, or the difference from u to its true value, will be contained between the limits

$$\pm \frac{2r\sqrt{\sum \gamma_i^2 h_i^2}}{\sum \gamma_i q_i}.$$

The probability remaining the same, the error to fear will be therefore so much less as the coefficient of r in this expression will be smaller; thus one will have to choose the system of factors γ, γ_1, γ_2, etc., for which the value of this coefficient will be a *minimum*; now, by equating to zero its differential with respect to any coefficient, there comes

$$\gamma_i h_i^2 \sum \gamma_i \delta_i - q_i \sum \gamma_i^2 h_i^2 = 0;$$

whence one concludes

$$\gamma_i = \frac{\mu q_i}{h_i^2};$$

μ being a constant coefficient, or common to all the factors γ, γ_1, γ_2, etc., which remain entirely arbitrary, as one sees by substituting this expression of γ_i into the preceding equation. The value of u will become then

$$u = \frac{\sum \frac{q_i \delta_i}{h_i^2}}{\sum \frac{q_i^2}{h_i^2}} + \frac{\sum \frac{q_i k_i}{h_i^2}}{\sum \frac{q_i^2}{h_i^2}}, \tag{7}$$

and the limits of the error to fear will be

$$\pm \frac{2r}{\sqrt{\sum \frac{q_i^2}{h_i^2}}}$$

(12) In the particular case where the probability of the errors is the same in all the observations, and where consequently all the quantities h, h_1, h_2, etc., are equals, in the

same way the quantities k, k_1, k_2, etc., one will have simply

$$u = \frac{\sum q_i \delta_i}{\sum q_i^2} + \frac{k \sum q_i}{\sum q_i^2}, \tag{8}$$

and for the limits of the error to fear

$$\pm \frac{2r}{\sqrt{\sum q_i^2}}$$

If the coefficients q, q_1, q_2, etc., formed a series decreasing to infinity, such as $1, \frac{1}{2}, \frac{1}{3}$, etc., for example, one would have

$$\sum q_i^2 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \text{etc.} = \frac{\pi^2}{6};$$

these limits would have therefore a value finite and equal to $\pm \frac{2rh\sqrt{6}}{\pi}$, instead of being tightened more and more, in measure as one would increase the number of observations. But it is necessary to observe that the coefficients γ, γ_1, γ_2, etc., being proportional to the coefficients q, q_1, q_2, etc., would form also a series decreasing to infinity; so that this case falls in the exception of n° 11, the formulas that we just found are not applicable at all. In fact, by adopting the same law of probability of errors as in this n°, one would have

$$k = 0, \quad h^2 = \frac{1}{2} \int_0^\infty e^{-x} x^2 dx = 1;$$

now, the limits of error of u being $\pm \frac{2r\sqrt{6}}{\pi}$, those of the value of E would be $\pm \frac{2r\sqrt{6}}{\pi} \sum \gamma_i^2$, or $\pm \frac{2r\pi}{\sqrt{6}}$, and the corresponding probability would have for expression

$$\frac{1 - e^{-\frac{4r\pi}{\sqrt{6}}}}{1 + e^{-\frac{4r\pi}{\sqrt{6}}}},$$

while, according to the preceding formulas, it would be equal to $\frac{2}{\pi} \int e^{-r^2} dr$, the integral commencing with r.

By supposing always the same error equally probable in all the observations, if one takes all the coefficients γ, γ_1, γ_2, etc., equal to unity, the value of u, deduced from equation (6), will be

$$u = \frac{\sum \delta_i}{\sum q_i} + \frac{ks}{\sum q_i}; \tag{9}$$

and the limits of the error to fear with the probability $\frac{2}{\pi} \int e^{-r^2} dr$, will have for expression

$$\pm \frac{2rh\sqrt{s}}{\sum q_i}$$

These limits must be less narrow than those which correspond to formula (8), for which their extent is reduced to the *minimum*. It is necessary therefore that the ratio

$$\frac{\sum q_i}{\sqrt{s}\sqrt{\sum q_i^2}},$$

be < 1, setting aside the sign, this which is easy to verify. In fact, by calling Δ^2 the sum of the squares of the differences of the coefficients q, q_1, q_2, etc., taken two by two, and Q the sum of their products two by two, we will have

$$\Delta^2 = (s-1)\sum q_i^2 - 2Q,$$
$$\left(\sum q_i\right)^2 = \sum q_i^2 + 2Q;$$

consequently

$$\Delta^2 = s\sum q_i^2 - \left(\sum q_i\right)^2; \tag{10}$$

whence one draws

$$\frac{\sum q_i}{\sqrt{s}\sqrt{\sum q_i^2}} = \sqrt{1 - \frac{\Delta^2}{s\sum q_i^2}};$$

a quantity evidently < 1, excepting in the case where the coefficients q, q_1, q_2, etc., being all equal, one has $\Delta = 0$.

(13) According to the expression of ε_i of n$^{\text{o}}$ 11, one has

$$\sum(\varepsilon_i - k)^2 = \sum(q_i u - \delta_i - k)^2;$$

and if one determines u by the condition that this sum is a *minimum*, one finds

$$u = \frac{\sum q_i\delta_i}{\sum q_i^2} + \frac{k\sum q_i}{\sum q_i^2},$$

this which coincides with formula (8). There results from it therefore that the most advantageous manner to determine u, consists in rendering a *minimum* the sum of the squares of the errors of all the observations, after having diminished each error by the quantity k; and when one supposes $k = 0$, this method is that of *least squares of errors*, as Mr. Laplace has demonstrated first. But when the positive errors and the negative errors are not equally probable, this method, as well as the ordinary process, where one makes the sum of the errors equal to zero, will give an incomplete value of u; and, in order to complete it, it will be necessary to know the value of the constant k for each particular question immediately, one is able to observe that the coefficient of k is less in formula (8) than in formula (9) which reports to the second method; whence there results that by suppressing the term which contains k, one will risk to commit a greater error, by making use of the ordinary process, than by employing the method of *least squares*, this which is further an advantage of this method.

(14) We suppose that the s observations that one considers, are composed of many groups in each of which the law of probability of the errors is the same. Let in the first group, s' be the number of observations, h and k the values of h_i and k_i; in the second, s'' this number, h' and $k'$the values of these quantities; and thus in sequence: according to formula (7), one will have

$$u = \frac{\frac{1}{h^2}\sum' q_i\delta_i + \frac{1}{h'^2}\sum'' q_i\delta_i + \text{etc.} + \frac{k}{h^2}\sum' q_i + \frac{k'}{h'^2}\sum'' q_i + \text{etc.}}{\frac{1}{h^2}\sum' q_i^2 + \frac{1}{h'^2}\sum'' q_i^2 + \text{etc.}}$$

and the limits of the error to fear with the probability $\frac{2}{\sqrt{\pi}}\int e^{-r^2}dr$ will be

$$\frac{\pm 2r}{\sqrt{\frac{1}{h^2}\sum' q_i^2+\frac{1}{h'^2}\sum'' q_i^2+\text{etc.}}};$$

the sum $\sum'$ extending to the first group of observations, $\sum''$ to the second, etc.

This value of u does not suppose that the numbers s', s'', etc. are all very great: it will suffice, in order that it be applicable, that their sum or the number s of all the observations, is a very great number. Although s', s'', etc. are not necessarily very great, if one has determined the value of u according to the rule of the preceding n$^{\text{o}}$ for each group of observations in particular, and if one names $U, U', U'', \ldots$ the results relative to the 1st, 2nd, 3rd,... group, so that one has

$$U\sum' q_i^2=\sum' q_i\delta_i+k\sum' q_i,$$
$$U'\sum'' q_i^2=\sum'' q_i\delta_i+k'\sum'' q_i,$$
$$\text{etc.}$$

if moreover, one makes, for brevity,

$$\frac{1}{h^2}\sum' q_i^2=g,\quad \frac{1}{h'^2}\sum'' q_i^2=g',\ \text{etc.:}$$

the preceding value of u will become

$$u=\frac{gU+g'U'+g''U''+\text{etc.}}{g+g'+g''+\ \text{etc.}};$$

a formula which will serve to calculate the value of u, resulting from many groups of observations of different kinds, when the values of u, given by the rule of the preceding n$^{\text{o}}$, and the quantities g, g', g'',etc., will be known for all these groups. At the same time, the limits of the error to fear, with the probability here above, will take the form

$$\frac{\pm 2r}{\sqrt{g+g'+g''+\ \text{etc.}}}$$

(15) The usage of the preceding formulas requires that one know the two quantities k and h in each kind of observations: the quantity k in order to form the value of the unknown, and the quantity h in order to evaluate the limits of the error to fear on this value with a determined probability. The most natural assumption that one is able to make on the first of the two quantities, is to consider it as null, this which comes back to regarding the positive and negative errors as equally possible; but if this equality does not hold, one will not have $k=0$; and, in a great number of cases, one will be able to determine the true value of k in the following manner.

We suppose that by employing successively two different systems of coefficients γ, γ_1, γ_2, etc., $\ldots\gamma'$, γ'_1, γ''_2, one has formed the two equations

$$\sum\gamma_i\varepsilon_i=u\sum\gamma_i q_i-\sum\gamma_i\delta_i,$$
$$\sum\gamma'_i\varepsilon_i=u\sum\gamma'_i q_i-\sum\gamma'_i\delta_i,$$

we subtract the one from the other, after having multiplied the first by $\sum \gamma_i' q_i$ and the second by $\sum \gamma_i q_i$; there comes

$$\sum \gamma_i'' \varepsilon_i = \sum \gamma_i q_i \sum \gamma_i' \delta_i - \sum \gamma_i' q_i \sum \gamma_i \delta_i,$$

by making, for brevity,

$$\gamma_i \sum \gamma_i' q_i - \gamma_i' \sum \gamma_i q_i = \gamma_i''.$$

Now, after that which one has seen above (n° 9), there is a probability $\frac{2}{\sqrt{\pi}} \int e^{-r^2} dr$ that the sum $\sum \gamma_i'' \varepsilon_i$ is contained between the limits

$$k \sum \gamma_i'' \pm 2rh \sqrt{\sum \gamma_i''^2};$$

the law of the probability being the same in all the observations, and their number s being supposed very great. If therefore one takes $k \sum \gamma_i''$ for the value of this sum, the value corresponding to k will be

$$k = \frac{\sum \gamma_i q_i \sum \gamma_i' \delta_i - \sum \gamma_i' q_i \sum \gamma_i \delta_i}{k \sum \gamma_i''};$$

and the limits of the error to fear on this value, with the probability above, will be

$$\pm \frac{2rh\sqrt{\sum \gamma_i''^2}}{\sum \gamma_i''}.$$

In order that their extent be a *minimum*, it would be necessary that γ_i'' be constant with respect to i; but it is easy to see that the coefficients γ_i and γ_i' would not know how to be such that that took place. If one takes one of these two coefficients constant, and the other proportional to q_i, one will have

$$\gamma_i'' = \mu \left(q_i \sum q_i - \sum q_i^2 \right);$$

μ being a quantity independent of i. There will result from it

$$\sum \gamma_i''^2 = \mu^2 \left[s \left(\sum q_i^2 \right)^2 - \left(\sum q_i \right)^2 \sum q_i^2 \right] = \mu^2 \Delta^2 \sum q_i^2,$$
$$\sum \gamma_i'' = \mu^2 \left[\left(\sum q_i \right)^2 - s \sum q_i^2 \right] = -\mu \Delta^2;$$

Δ^2 having the same signification as in n° 12. The value of k and the limits of the error to fear will become therefore

$$k = \frac{\sum q_i \sum q_i \delta_i - \sum q_i^2 \sum \delta_i}{\Delta^2} \quad \text{and} \quad \pm \frac{2rh\sqrt{\sum q_i^2}}{\Delta};$$

their probability being always $\frac{2}{\sqrt{\pi}} \int e^{-r^2} dr$. When the sum that Δ^2 represents will be very great with respect to $\sum q_i^2$, this value of k will be determined with the same exactitude as the unknown u; but when the coefficients q, q_1, q_2, etc. will be equals, or only when their differences will be very small, the quantity Δ will be null or very small, this which will render illusory the limits of the error to fear on the value of k, which will no longer be able then to be determined by any means.

If the observations that we will consider have for object to determine the coefficient of a periodic inequality, and if they comprehend the entire extent of a period, the sum of the coefficients q, q_1, q_2, etc. will approach more and more to be null in measure as the period will have been divided into a great number of parts, or that the observations will be more numerous; by neglecting therefore $\sum q_i$, one will have $\Delta^2 = s\sum q_i^2$, and the value of k and the limits of the error to fear will be reduced to

$$k = -\frac{\sum \delta_i}{s} \quad \text{and} \quad \pm\frac{2rh}{\sqrt{s}};$$

by dividing therefore, in this case, the sum of the quantities δ, δ_1, δ_2, etc. by their number, one will knew immediately if the quantity k has a sensible value; and the quotient, taken with a contrary sign, will give quite exactly this value.

(16) Instead of seeking to determine this quantity, one would be able to try to make it vanish from the value of u. For this, we take the general expression of u given by formula (6). By supposing that the quantities k_i and h_i are the same in all the observations, this expression and the limits of error which relates will become

$$u = \frac{\sum \gamma_i \delta_i}{\sum \gamma_i q_i} + \frac{k\sum \gamma_i}{\sum \gamma_i q_i} \quad \text{and} \quad \pm\frac{2rh\sqrt{\sum \gamma_i^2}}{\sum \gamma_i q_i}$$

We make therefore

$$\sum \gamma_i = 0,$$

this which will determine one of the factors γ, γ_1, γ_2, etc. We render next the limits of error a *minimum*, with respect to all the others; we will have the two differential equations

$$\sum d\gamma_i = 0, \quad \sum \gamma_i q_i \sum \gamma_i d\gamma_i - \sum \gamma_i^2 \sum q_i d\gamma_i;$$

multiplying the first by an indeterminate factor θ, adding it next to the second, then equaling to zero the coefficient of each differential, one will have

$$\theta + \gamma_i \sum \gamma_i q_i - q_i \sum \gamma_i^2 = 0.$$

The value of γ_i that one will draw from this equation will be of the form

$$\gamma_i = \mu q_i + \theta',$$

μ and θ' being some constants that the concern is to determine. Now, by substituting this value into the preceding equation, and equating next to zero separately the coefficient of q_i beyond the sign $\sum$, and the constant term with respect to i, there comes

$$\mu\theta' \sum q_i + \theta'^2 s = 0,$$
$$\theta + \theta'\mu \sum q_i^2 + \theta'^2 \sum q_i = 0;$$

whence one draws

$$\theta = -\frac{\mu}{s}\sum q_i, \quad \theta = \frac{\mu^2}{s^2}\left[s\sum q_i^2 - \left(\sum q_i\right)^2\right]\sum q_i;$$

by means of which the value of γ_i will become

$$\gamma_i = \mu\left(q_i - \frac{1}{s}\sum q_i\right),$$

and the factor μ will remain indeterminate. The value of u will be therefore

$$u = \frac{s\sum q_i\delta_i - (\sum q_i)^2}{\Delta^2},$$

and designating by Δ^2 the same quantity as previously; and all reduction made, the limits of the error to fear will have for expression

$$\pm\frac{2rh\sqrt{s}}{\Delta},$$

the probability being always $\frac{2}{\sqrt{\pi}}\int e^{-r^2}dr$.

When Δ^2 will be a very small quantity with respect to s, these limits will be illusory, and one will be not be able to make any use of this value of u. When $\sum q_i$ will be a very small quantity, this value and these limits will differ very little from the value of u given by equation (8) and from the limits of error which are reported there.

(17) We occupy ourselves now in the determination of the quantity h, necessary to know in order to calculate the limits of error of the different preceding formulas. I observe, for that, that instead of considering, in n$^{\text{os}}$ 1 and 2, the sum of the errors of the s observations, one could have considered the sum of the values of any function of these errors; the probability p that this sum would be contained between two given limits $b-c$ and $b+c$ would be determined without new difficulties by the analysis of these two n$^{\text{os}}$; and if one indicates this function by the characteristic ϕ, formula (1) will give further the value of p, by putting ϕx in the place of x in the imaginary exponential that the integral contains relative to x, and conserving all the other notations. If one supposes next the number s very great; if one makes

$$\int_{-a}^{a} fx\phi x dx = K, \quad \int_{-a}^{a} fx(\phi x)^2 dx = K', \quad \frac{1}{2}(K' - K^2) = H^2;$$

if one indicates always by ε, ε_1, ε_2, etc., the errors of the observations: one will find, as in n$^{\text{o}}$ 7,

$$p = \frac{2}{\sqrt{\pi}}\int e^{-r^2}dr,$$

for the probability that the sum

$$\phi\varepsilon + \phi\varepsilon_1 + \phi\varepsilon_2 + \cdots + \phi\varepsilon_{s-1},$$

or $\sum \phi \varepsilon_i$, is contained between the limits

$$Ks \pm 2rH\sqrt{s}.$$

Consequently, one will be able always to take the number s great enough in order that it has a given probability that $\frac{1}{s}\sum \phi \varepsilon_i$ differs from K by as little as one will wish; and by taking

$$\frac{1}{s}\sum \phi \varepsilon_i = K,$$

the limits of the error to fear with the probability p, will be

$$\pm \frac{2rH}{\sqrt{s}}.$$

This put, we suppose $\phi x = x^2$, a case in which K and the quantity k' of n° 6 will be equals, so that one will have, according to this n°,

$$K = k' = 2h^2 + k^2.$$

The preceding equation will become therefore

$$h^2 + \frac{1}{2}k^2 = \frac{1}{2s}\sum \varepsilon_i^2;$$

but one has (n° 11)

$$\sum \varepsilon_i^2 = \sum (uq_i - \delta_i)^2;$$

substituting therefore for u its value given by formula (8), which is the least susceptible to error, one will conclude from it

$$as(h^2 + \frac{1}{2}k^2)\sum q_i^2 = \left(\sum q_i\delta_i + k\sum q_i\right)^2 - 2\left(\sum q_i\delta_i + k\sum q_i\right)\sum q_i\delta_i + \sum q_i^2 \sum \delta_i^2,$$

or else, by reducing,

$$2sh^2\sum q_i^2 + \Delta^2 k^2 + \left(\sum q_i\delta_i\right)^2 - \sum q_i^2 \sum \delta_i^2 = 0;$$

an equation which will make known the value of u when that of k will be known.

This formula coincides with that which Mr. Laplace has given for the same object,[3] when one supposes $k = 0$, and in the case where all the coefficients q, q_1, q_2, etc., are equals among them. In this last case, one has $\Delta = 0$, and the preceding formula gives

$$h^2 = \frac{\Delta'^2}{2s^2}, \quad \text{or} \quad h = \frac{\Delta'}{s\sqrt{2}};$$

Δ'^2 designating, with respect to the quantities $\delta, \delta_1, \delta_2$, etc., that which Δ^2 represents relative to the coefficients q, q_1, q_2, etc., that is to say the sum of the squares of the differences of these quantities $\delta, \delta_1, \delta_2$, etc., taken two by two.

[3] *Théorie analytique des Probabilités*, page 321.

In general, the error that one will commit by taking for h the value resulting from the equation that we just found, will depend on the error of which the value of u that we have employed is susceptible, and on the error of the equation $\frac{1}{s}\sum\phi\varepsilon_i = K$. The limits of that here containing a new unknown H, one will not be able to evaluate them exactly, no more than those of the error to fear on the value of h; but that will not prevent at all using this value of h in the formulas of the preceding numerals, where it is multiplied by some very small quantities of the order of $\frac{1}{\sqrt{s}}$.