Buffon's Experiment

Sections extracted from
Researches sur la probabilité des jugements en matière criminelle et en matière civile *

Simon Denis Poisson

1837

pp. 132–135

§50. One finds in the works of Buffon[1] the numerical results of an experiment on the game of *heads* and *tails*, which furnishes us an example and a verification of the preceding rule.

In this game, the chance to bring forth one or the other of two faces of the coin depends on its physical constitution which is not well known to us; and even when we would know it well, it would be a problem of mechanics that no person could resolve, to conclude the chance of *heads* or *tails*. It is therefore from experience that the approximate value of this chance must be deduced for each coin in particular; so that if in a very great number μ of trials, *heads* is arrived a number m times, the ratio $\frac{m}{\mu}$ must be taken for the chance of *heads*. It would also be the probability or the reason to believe that this face will arrive in a new trial made with the same coin; and, according to the result of this series of trials, one will be able to wager in an equal game, m against $\mu - m$ in order to arrive to *heads*. It is also by means of this probability $\frac{m}{\mu}$ of the simple event that one must calculate the probabilities of the composite events, at least when they will not be very weak by the nature of these events.

This being, we suppose that one has made a very great number m of series of trials, by continuing each series, as in the experiment cited, until this that *heads* has taken place. Let a_1, a_2, a_3, etc., be the numbers of times that *heads* is arrived at the first coup, at the second, at the third, etc. The total number μ of coups or trials, will be

$$\mu = a_1 + 2a_2 + 3a_3 + \text{etc.};$$

the number m of the arrived of *heads* will be, at the same time,

$$m = a_1 + a_2 + a_3 + \text{etc.};$$

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[1] *Arithmétique morale*, article XVIII.

and if one calls p the chance of this face, one will have

$$p = \frac{m}{\mu},$$

with so much more approximation and exactitude as μ ill be a greater number.

The probabilities of *heads* on the first coup, on the second coup without having taken place on the first, on the third coup with being arrived in the first two, etc., will be p, $p(1-p)$, $p(1-p)^2$, etc. Now, the numbers of times that these events have taken place being by hypothesis $a_)1$, a_2, a_3, etc., in a number m of series of trials, one must therefore have, very nearly,

$$p = \frac{a_1}{m}, \quad p(1-p) = \frac{a_2}{m}, \quad p(1-p)^2 = \frac{a_3}{m}, \quad \text{etc.},$$

if this number is very great, and when these probabilities will not have become from very small fractions. By dividing each of these equations by the preceding, one concludes from it different values of $1-p$, and, consequently,

$$p = \frac{a_1}{m}, \quad p = 1 - \frac{a_2}{a_1}, \quad p = 1 - \frac{a_3}{a_2}, \quad \text{etc.}$$

These values of p, or at least of a certain number of the first, will differ so much less among them and from the ratio $\frac{m}{\mu}$, as m and μ will be greater numbers: in order that they were necessarily equals, it would be necessary that these numbers were infinite. By employing for p, the mean of these very slightly unequal fractions, or else by making use of the value $\frac{m}{\mu}$ of p, resultant of the set of trials, one will have

$$a_1 = mp, \quad a_2 = mp(1-p), \quad a_3 = mp(1-p)^2, \quad \text{etc.}$$

for the values calculated from the numbers a_1, a_2, a_3, etc., which must deviate very little from the observed numbers, at least in the first terms of this decreasing geometric progression.

In the experience of Buffon, the number m of the series of trials was

$$m = 2048.$$

One is able to conclude in the manner by which it is reported by the author, that one has had

$$\begin{array}{lllll} a_1 = 1061, & a_2 = 494, & a_3 = 232, & a_4 = 137, & a_5 = 56, \\ a_6 = 29, & a_7 = 25, & a_8 = 8, & a_9 = 6. & \end{array}$$

The numbers a_{10}, a_{11}, etc., have not taken place at all, that is to say that the number m of the series of trials has not been great enough in order that *heads* did not arrive in one or many series. This number is the sum of the values of a_1, a_2, a_3, etc.; one deduces from it also

$$\mu = 4040,$$

and, consequently,

$$p = \frac{m}{\mu} = 0.50693.$$

By means of this value of p, one finds

$$a_1 = 1038, \quad a_2 = 512, \quad a_3 = 252, \quad a_4 = 124, \quad a_5 = 61,$$
$$a_6 = 30, \quad a_7 = 15, \quad a_8 = 7, \quad a_9 = 4. \quad a_{10} = 1,$$

by neglecting the fractions: the following numbers a_{11}, a_{12}, etc., would be below unity. Now, if one compares this series of calculated values, to those of the numbers a_1, a_2, a_3, etc., which result from observation, one sees that they deviate little from one another in their first terms. The deviations are greater in the following terms; for example, the calculated value of a_7 is only three fifths of the observed value; but this number a_1 corresponds to an event of which the probability is below a hundredth. By stopping at the first three terms of the series of observed numbers, one deduces from it

$$p = \frac{a_1}{m} = 0.51806, \quad p = 1 - \frac{a_2}{a_1} = 0.53441, \quad p = 1 - \frac{a_3}{a_2} = 0.53033;$$

quantities which differ very little among them, and of which the mean, or the third of their sum, is

$$p = 0.52760,$$

which differs hardly by 0.02, from the value $\frac{m}{\mu}$ of p, resultant from the set of trials.

I have chosen this experiment because of the name of the author, and because the work where it is found, renders it authentic. Each of them is able to make many others of the same kind, either with a piece of money, or with a *die* of six faces. In this last case, the number of times that each face will arrive, out of a very great number of trials, will be very nearly a sixth of those, at least when the *die* is not false or badly constructed.